Linear Dependence of Canonical Alegbraic Curvature Tensors of Symmetric and Anti-Symmetric Builds

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Abstract

This paper studies linearly independent sets of canonical curvature tensors and strives to understand the consequences. Our goal is to relate the symmetrically built and anti-symmetrically built canonical curvature tensors. By developing an identity for an anti-symmetrically built curvature tensor to be expressed as a sum of symmetrically built tensors, we are able to extend results that were proven for sets of only symmetrically built curvature tensors. We examine situations in which the sets of curvature tenors are of both builds. We also consider cases where any of the operators are allowed to have a nontrivial kernel of a certain type. Finally, we develop methods to reduce the bound on the least number of canonical curvature tenors it takes to express a canonical curvature tensor as a sum.

1 Introduction and Motivation

Definition Let V be a finite dimensional real vector space of dimension n and V^* be the corresponding dual space. An *algebraic curvature tensor*, is $R \in \otimes^4 V^*$ that satisfies

1. $R(x, y, z, w) = -R(y, x, z, w)$,
2. $R(x, y, z, w) = R(z, w, x, y)$,
3. $R(x, y, z, w) + R(z, x, y, w) + R(y, z, x, w) = 0$,

the last of which is the Bianchi Identity.

Let $S^2(V^*)$ denote the set of symmetric bilinear forms and $\Lambda^2(V^*)$ be the set of anti-symmetric bilinear forms. There are two types of canonical algebraic curvature tensors, depending on whether the form used is symmetric or anti-symmetric. The two types differ by the form that is used, as well as the terms that are summed. We refer to the build of the canonical curvature tensor as symmetric or anti-symmetric.

Definition If $\phi \in S^2(V^*)$ and $\tau \in \Lambda^2(V^*)$, then a *canonical algebraic curvature tensor* is

1. $R^S_\phi(x, y, z, w) = \phi(x, w)\phi(y, z) - \phi(x, z)\phi(y, w)$, or
2. $R^\Lambda_\tau(x, y, z, w) = \tau(x, w)\tau(y, z) - \tau(x, z)\tau(y, w) - 2\tau(x, y)\tau(z, w)$.

Definition Let $A, B : V \to V$ be a linear operator, and $(\cdot, \cdot)$ be the inner product. Then,

1. $R^S_A(x, y, z, w) = (Ax, w)(Ay, z) - (Ax, z)(Ay, w)$, or
2. $R^\Lambda_B(x, y, z, w) = (Bx, w)(By, z) - (Bx, z)(By, w) - 2(Bx, y)(Bz, w)$.

In the case where A is the identity, we have only $\phi \in S^2(V^*)$. For the symmetric build, R^S_A is an algebraic curvature tensor if and only if $A = A^*$. Similarly, for the anti-symmetric build, R^Λ_B is an algebraic curvature tensor if and only if $B = -B^*$ [2] . Throughout the paper, if the statement allows for either build of canonical curvature tensor, then we write the canonical curvature tensor without the superscript. We also assume that the operator is self or skew adjoint so that the tensor is an alegbraic curvature tensor. Moreover, any linear map is assumed to be an endomorphism of the vector space, unless otherwise stated.

We shall sometimes find it convenient to consider $\tau(x,y) := (Ax,y)$, so that if $A^* = A$, then $\tau \in S^2(V^*)$, and $A^* = -A$, then $\tau \in \Lambda^2(V^*)$. Subsequently we would refer o the canonical algebraic curvature tensor R_A as R_τ. This distinction will either be explicitly stated, or clear from context. We let ϕ be a positive definite inner product throughout the paper.

The set of canonical algebraic curvature tensors, denoted $\mathcal{A}(V)$, is a vector space and Gilkey and Fiedler [4, 7] have proven that

$$\mathcal{A}(V) = span\{R_B^S| \text{ for } B = B^*\} = span\{R_A^\Lambda| \text{ for } A = -A^*\}.$$

There is an interest in determining linearly dependent sets of canonical curvature tensors in order to determine a basis for the vector space.

For the most part, the previous results regarding linear dependence of sets of curvature tensors have consisted of only a single type, all symmetric or all anti-symmetric builds. Moreover, the results assume that one or more of the operators have full rank.

Our work strives to relate the canonical curvature tensors that are symmetrically built with those that are anti-symmetrically build, as well as introduce cases where any of the operators are allowed to have a non-trivial kernel. We develop an identity for an anti-symmetric build curvature tensor, in terms of symmetric build tensors. The identity allows us to extend results regarding symmetric canonical curvature tensors to similar results for the anti-symmetric canonical curvature tensors. It also allows for a consideration of sets of canonical curvature tensors certain properties, where the build need not be specified.

Theorem 1.1. *Let $A : V \to V$ a linear map and $A = -A^*$ with respect to ϕ. Then,*

$$\begin{aligned} R_A^\Lambda(x,y,z,w) &= 2R_A^S(x,y,z,w) + R_A^S(x,z,y,w) + R_A^S(x,w,z,y) \\ &= 2R_\phi^S(Ax,Ay,z,w) + R_\phi^S(Ax,Az,y,w) + R_\phi^S(Ax,Aw,z,y). \end{aligned}$$

The identity allows for a simple proof that a linear map preserves a curvature tensor of anti-symmetric build if and only if it perserves the curvature tensor's operator, up to a sign. This proves that for $\tau \in \Lambda^2(V^*)$, the structure group of R_τ^Λ is equivalent to the structure group of τ, up to a sign.

Theorem 1.2. *Let $\tau \in \Lambda^2$. Then $G_{R_\tau^\Lambda} = G_\tau^\pm$.*

The sign ambiguity is due to the fact that there exists an A such that precomposing A with τ results in $-\tau$. In the symmetric case, such an A cannot exist unless the signature of the inner product is balanced [**?**].

A necessary condition for the dependence of three symmetric build canonical curvature tensors with certain rank assuptions is that the operators be simultaneously diagonalized [1]. The notion of diagonalization of self-adjoint operators translates to block diagonalization in 2×2 blocks with zeros elsewhere for skew-adjoint operators. We extend these results to a set with one anti-symmetric and two symmetric canonical curvature tensors, and prove that in certain cases, the self-adjoint operators must be simultaneously diagonalized and the skew-adjoint operators must be block-diagonalized in 2×2 blocks.

Theorem 1.3. *Let $A = A^*$, $B = B^*$, and $Rank B \geq 4$. Let $C = -C^*$ and $C^2 = -I$. If $\{R_A, R_B, R_C\}$ is linearly dependent, then A and B must be simultaneously diagonal and C simultaneously block diagonal in 2×2 blocks with zeros elsewhere.*

Moreover, we prove in a more general situation that the set of one anti-symmetric and two symmetric canonical curvature tensors is actually linearly independent.

Theorem 1.4. *Let $A = A^*$, $B = B^*$ and $C = -C^*$. Then for $dimV > 3$, $\{R_A, R_B, R_C\}$ is linearly independent.*

We consider a set of two symmetrically built and two anti-symmetrically built canonical curvature tensors and prove that in for $dimV = 4$, it is linearly dependent. Moreover, we prove that in $dimV = 4$, for the set to be linearly dependent, the self-adjoint operators must be diagonalized and the skew-adjoint operators must be block-diagonalized, all simultaneously.

Theorem 1.5. *If $dimV = 4$, $A = A^*$, $B = B^*$, $C = -C^*$ $D = -D^*$, then there exists A, B, C, D, such that $\epsilon R_A + \delta R_B = R_C + \alpha R_D$ holds, where $\epsilon, \delta, \alpha = \pm 1$. Moreover, if A and B are simultaneously diagonalized, then B, and C must be block-diagonalized in 2×2 blocks simultaneously with the diagonalization of A and B. Also, $|Spec(C)| = |Spec(D)| = 2$, with each eigenspace of multiplicity 2.*

The numbers $\nu(R)$ and $\eta(R)$ provide lower bounds on the dimension of $\mathcal{A}(V)$, as they give the least number of canonical curvature tensors required to express another canonical curvature tensor.

Definition Let R denote a canonical algebraic curvature tensor. Then

$$\nu(R) = min\{k| \text{ there exist } \psi_i \in S^2(V^*), \text{ such that } R = \sum^k a_i R_{\psi_i} \text{ for } a_i \in \mathbb{R}, \text{ for all } R\},$$
$$\eta(R) = min\{k| \text{ there exist } \tau_i \in \Lambda^2(V^*), \text{ such that } R = \sum^k a_i R_{\tau_i} \text{ for } a_i \in \mathbb{R}, \text{ for all } R\}.$$

We take a general approach to estimating $\nu(R)$ and $\eta(R)$, as well as sets of both builds of canonical curvature tensors. For sets of any number of symmetrically built and any number of anti-symmetrically built canonical curvature tensors, we count the number of equations and compare it to the number of unknowns in the matrix array of the operators to determine if it would seem that a solution for the operators should exist. Comparing the estimates with the known results, we find that the estimates are exact.

We use a similar method to give an estimate for the number of operators that can be simultaneously diagonalized as well as simultaneously block-diagonalized. By comparing the number of equations with the number of unknowns that result from evaluating the hypothesis with the basis elements, we determine when the system of equations is overdetermined.

We consider sets of canonical curvature tensors where any of the operators are allowed to have nontrivial kernels, in contrast to the hypothesis of full rank which is usually seen [1]. In particular, we assume the operators forming the curvature tensors form a chain complex. The results hold for each canonical curvature tensors being either build.

Theorem 1.6. *If A, B, and C in the following chain complex and $R_A + \epsilon R_B + \delta R_C = 0$ for $\epsilon, \delta = \pm 1$, then $R_B = 0$. Moreover, if $Rank(A) \geq 3$ or $Rank(C) \geq 3$, then $C = \pm A$ and if $Rank(A) \geq 4$ and $Rank(C) \geq 4$, then R_A and R_C must be the same build. Also, given those rank assumptions, $\delta = -1$. Furthermore, if the chain complex is an exact sequence and $B = -B^*$ then A and C are invertible.*

$$V \xrightarrow{A} V \xrightarrow{B} V \xrightarrow{C} V$$

Another arrangement of the operators $A, B_1, ..., B_k$ is as a set of k chain complexes each of length two, where either $imA \subseteq kerB_i$ for all i, or $kerA \subseteq imB_i$ for all i.

Theorem 1.7. *If $A, B_1, ..., B_k$ are linear maps, and either $imA \subseteq kerB_i$ for all i, or $kerA \subseteq imB_i$ for all i, and $0 = R_A + \sum^k \epsilon_i R_{B_i}$, then $R_A = 0$. Moreover, if $A = -A^*$ then for each sequence that is exact at V, then the corresponding B_i is invertible.*

We also consider sets of four canonical curvature tensors, where each curvature tensor may be either build and where the operators are arranged in two different types of chain complexes. First we consider where the operators are arranged in a cyclic sequence, with each $imA_i \subseteq kerA_{i+1}$ as well as $imA_k \subseteq kerA_1$. Then we consider the operators arranged in a chain complex, so each $imA_i \subseteq kerA_{i+1}$.

We provide motivation for relating $\nu(R)$ and $\eta(R)$ by developing methods for reducing $\nu(R^\Lambda_\psi)$ and $\eta(R^S_\psi)$, given that one of the operators has a nontrivial kernel.

Theorem 1.8. *Consider $R_\psi = \epsilon R_\gamma + \sum^k \epsilon_i R_{\gamma_i}$, where $ker(\gamma) \neq 0$. If $A : V \to ker(\gamma)$ and $A^*\psi = \pm\psi$, then $R_\psi = \sum^k \epsilon_i R_{A^*\gamma_i A}$. Moreover, $R_{A^*\gamma_i A} \in \mathcal{A}(V)$, for both $\gamma_i = \gamma_i^*$ and $\gamma_i = -\gamma_i^*$.*

Then we can re-express that same canonical curvature tensor as a sum of canonical curvature tensors without R_τ, so in one fewer terms. If we apply this method to a curvature tensor of one type, expressed as a sum of another type, then this method reduces $\eta(R^S_\psi)$ or $\nu(R^\Lambda_\tau)$. As a more general case, we do not require A to preserve any of the operators. This provides a method for reducing $\nu(R)$ and $\eta(R)$, given that at least one of the operators has a nontrivial kernel.

Theorem 1.9. *Let $R = \epsilon R_\tau + \sum^k \epsilon_i R_{B_i}$, where $ker(\tau) \neq 0$. Then, for $A : V \to ker(\tau)$, $\bar{R} = A^*R = \sum^k \epsilon_i R_{A^*B_iA}$. Moreover, $R_{A^*B_iA} \in \mathcal{A}(V)$, for $B_i = B_i^*$ or $B_i = -B_i^*$.*

In both cases, the kernels of all of their terms are aligned, as they all contain $kerA$. Both methods extend to sums of both builds of curvature tensors, thereby, providing a motivation for introducing a new bound, $\mu(R)$, which allows the sum to be of both symmetric and anti-symmetric canonical curvature tensors.

As a note through the paper, because we can permute the basis vectors without changing any relations in the operators, we can assume that if there exist an arbitrary $\tau(e_i, e_j)$, then we can permute the basis so that it is $\tau(e_1, e_2)$. Thus, we do not loose generality by referring to the basis vectors by e_1, for example, rather than e_i. Also, for simplicity in notation we refer to $\tau(e_i, e_j)$ by τ_{ij}.

2 Previous Results

Our results include a completion on the classification of the linear independence of three canonical algebraic curvature tensors. We summarize the results on three tensors, where we assume that the operators are linearly independent,

1. For $dim(V) \geq 4$, A positive definite, $Rank(B) = n$ and $Rank(C) \geq 3$, then $\{R^S_A, R^S_B, R^S_C\}$ is linearly dependent if and only if either $|Spec(C)| = |Spec(B)| = 1$ or $|Spec(B)| = \{b_1, b_2, b_2, b_2, ...\}$, and $|Spec(C)| = \{c_1, c_2, c_2, c_2, ...\}$, with $b_1 \neq b_2$, $c_2^2 = \epsilon(\delta b_2^2 - 1)$, and $c_1 = \frac{\epsilon}{\lambda_2}(\delta b_1 b_2 - 1)$, where $\epsilon, \delta \in \{1, -1\}$ [1]
2. For $A_1, A_2, A_3 \in \Lambda^2(V^*)$ and $A_i \neq \lambda A_j$ for $\lambda \in \mathbb{R}$, $\{R_{A_1}, R_{A_2}, R_{A_3}\}$ is linearly independent [2].
3. For $A \in S^2(V^*)$ and $B, C \in \Lambda^2(V^*)$ and $Rank(A), Rank(B) \geq 4$, $\{R_A, R_B, R_C\}$ is linearly independent. [9]

Thus, the linear dependence of $\{R_A, R_B, R_C\}$, where $A, B \in S^2$ and $C \in \Lambda^2$, is the only other case left for a complete classification of the linear dependence of three canonical algebraic curvature tensors in each type of build. We prove that for $A, B \in S^2$, and $C \in \Lambda^2$, $\{R_A, R_B, R_C\}$ is linearly independent.

Some of the previous result that we will refer to rather frequently are summarized below:

Lemma 2.1. *(Gilkey, [7])*

1. *Let $A : V \to V$ be a self-adjoint linear map. Then $R^S_A = 0$ if and only if $Rank(A) \leq 1$.*
2. *Let $A : V \to V$ be a skew-adjoint linear map. Then $R^\Lambda_A = 0$ if and only if $A = 0$.*

Lemma 2.2. *(Gilkey, [7])*

1. *If $A, B : V \to V$, are self-adjoint linear maps with Rank$(A) \geq 3$ and $R^S_A = R^S_B$, then $A = \pm B$.*
2. *If $A, B : V \to V$ are skew-adjoint linear maps and $R^\Lambda_A = R^\Lambda_B$, then $A = \pm B$.*

Lemma 2.3. *(Diaz, Dunn [1] and Treadway [10]),*

1. *If $A : V \to V$ is a self-adjoint linear map with Rank$(A) \geq 3$, then there does not exist a self-adjoint B such that $R^S_A = -R^S_B$.*
2. *If $A : V \to V$ is a non-zero skew-adjoint linear map with Rank$(A) \geq 4$, then there does not exist a skew-adjoint B such that $R^\Lambda_A = -R^\Lambda_B$.*

Lemma 2.4. *(Treadway [10] and Lovell [9])*

1. *If $A \in \Lambda^2(V^*)$, and Rank$(A) \geq 4$, then there does not exist $B \in S^2(V^*)$ so that $R^\Lambda_A = R^S_B$.*
2. *If $A \in \Lambda^2(V^*)$ and Rank$(A) \geq 4$, then there does not exist $B \in S^2(V^*)$ so that $R^\Lambda_A = -R^S_B$.*

3 Identity for Decomposing A Canonical Tensor with Respect to an Anti-Symmetric Form

We develop an identity for the anti-symmetric build curvature tensor in terms of symmetric build tensors. First we include the following lemma proved by Diaz and Dunn [1], which relate the symmetric curvature tensors to their operators.

Lemma 3.1. *Let ϕ be the inner product and $A : V \to V$. Then for all $x, y, z, w \in V$,*

$$R_A(x, y, z, w) = R_\phi(Ax, Ay, z, w) = R_\phi(x, y, A^*z, A^*w).$$

Lemma 3.2. *Let ϕ be the inner product and $A = -A^*$ with respect to the basis. Then*

$$R^\Lambda_A(x, y, z, w) = R^S_A(x, y, z, w) - 2\phi(Ax, y)\phi(Az, w).$$

Proof.

$$\begin{aligned} R^\Lambda_A(x, y, z, w) &= \phi(Ax, w)\phi(Ay, z) - \phi(Ax, z)\phi(Ay, w) - 2\phi(Ax, y)\phi(Az, w) \\ &= R^S_\phi(Ax, Ay, z, w) - 2\phi(Ax, y)\phi(Az, w) \\ &= R^S_A(x, y, z, w) - 2\phi(Ax, y)\phi(Az, w) \end{aligned}$$

□

Theorem 3.3. *Let ϕ be the inner product and $A = -A^*$ with respect to ϕ. Then,*

$$\begin{aligned} R^\Lambda_A(x, y, z, w) &= 2R^S_A(x, y, z, w) + R^S_A(x, z, y, w) + R^S_A(x, w, z, y) \\ &= 2R^S_\phi(Ax, Ay, z, w) + R^S_\phi(Ax, Az, y, w) + R^S_\phi(Ax, Aw, z, y). \end{aligned}$$

Proof. We use the Bianchi Identity and Lemmas 3.2 and 3.1. Then,

$$\begin{aligned} R^\Lambda_A(x, y, z, w) &= -R^\Lambda_A(z, x, y, w) - R^\Lambda_A(y, z, x, w) \\ &= -R^S_A(z, x, y, w) + 2(Az, x)(Ay, w) - R^S_A(y, z, x, w) + 2(Ay, z)(Ax, w) \\ &= 2R^S_\phi(Ax, Ay, z, w) - R^S_A(z, x, y, w) - R^S_A(y, z, x, w) \\ &= 2R^S_A(x, y, z, w) + R^S_A(x, z, y, w) + R^S_A(x, w, z, y) \end{aligned}$$

□

Remark Although $R^S_A(x,y,z,w)$, $R^S_A(x,z,y,w)$, and $R^S_A(x,w,z,y)$ are canonical symmetric build tensors, they are not curvature tensors because they fail to satisfy the Bianchi Identity. Also, for R^Λ_A to be a canonical algebraic curvature tensor, $A = -A^*$. Then when it is expressed as a sum of symmetrically built tensors, A is still skew-symmetric.

4 The Structure Group of R_τ

The identity provides a proof that if a linear map preserves an anti-symmetric build curvature tensor, then it preserves the anti-symmetric form of the curvature tensor. This proves that the structure group of an anti-symmetric curvature tensor is equal to the structure group of its anti-symmetric form, up to a sign.

Let A be a linear map and let A^* denote precomposition, so $A^*R_B = R_B(Ax, Ay, Az, Aw)$. Also $A^*\psi$, for ψ a symmetric or anti-symmetric form, then $A^*\psi = \psi(Ax, Ay)$.

Lemma 4.1. *Let $C = C^*$ and $B = -B^*$, then*

$$R^S_C(Ax,Ay,Az,Aw) = R^S_{A^*CA}(x,y,z,w)$$

$$R^\Lambda_B(Ax,Ay,Az,Aw) = R^\Lambda_{A^*BA}(x,y,z,w).$$

Proof. Let $C = C^*$. Then, by Lemma 3.1

$$\begin{aligned} R^S_C(Ax,Ay,Az,Aw) &= R^S_\phi(CAx,CAy,Az,Aw) \\ &= R^S_\phi(A^*CAx,A^*CAy,z,w) \\ &= R^S_{A^*CA}(x,y,z,w). \end{aligned}$$

The proof just given for the symmetric case is by Diaz and Dunn [1]. For the anti-symmetric case, we use the identity and are then able to use the relations between the symmetric build curvature tensors and their operators.

Let $B = -B^*$. Then,

$$\begin{aligned} A^*R^\Lambda_B &= R^\Lambda_B(Ax,Ay,Az,Aw) \\ &= 2R^S_B(Ax,Ay,Az,Aw) + R^S_\beta(Ax,Az,Ay,Aw) + R^S_\beta(Ax,Aw,Az,Ay) \\ &= 2R^S_\phi(BAx,BAy,Az,Aw) + R^S_\phi(BAx,BAz,Ay,Aw) + R^S_\phi(BAx,BAw,Az,Ay) \\ &= 2R^S_\phi(A^*BAx,A^*BAy,z,w) + R^S_\phi(A^*BAx,A^*BAz,y,w) + R^S_\phi(A^*BAx,A^*BAw,z,y) \\ &= 2R^S_{A^*BA}(x,y,z,w) + R^S_{A^*BA}(x,z,y,w) + R^S_{A^*BA}(x,w,z,y) \\ &= R^\Lambda_{A^*BA}(x,y,z,w). \end{aligned}$$

□

In the rest of the section, we are referring to R^Λ_τ where $\tau \in \Lambda^2(V^*)$.

Definition Let $A : V \to V$, and A^* denote precomposition. The structure group of τ is

$$G_\tau = \{A | A^*\tau = \tau\}$$

and the structure group of R_τ is

$$G_{R_\tau} = \{A | A^*R = R_\tau\}.$$

Theorem 4.2. *Let $\tau \in \Lambda^2$. Then $G_{R^\Lambda_\tau} = G^\pm_\tau = \{A | A^*\tau = \pm\tau\}$.*

Proof. Let A^* denote precomposition. From Lemma 4.1, $A^*R^\Lambda_\tau = R^\Lambda_{A^*\tau}$. Thus, if $A^*\tau = \pm\tau$, then $A^*R_\tau = R_\tau$ and so $G^\pm_\tau \subseteq G_{R_\tau}$. Now, let $A \in G_{R^\Lambda_\tau}$ and so $A^*R^\Lambda_\tau = R^\Lambda_\tau$. Then

$$R^\Lambda_\tau = A^*R^\Lambda_\tau = R^\Lambda_{A^*\tau}.$$

We apply a result of Gilkey [7], that $R^\Lambda_{A^*\tau} = R^\Lambda_\tau$ implies that $\tau = \pm A^*\tau A$, giving the containment $G_{R_\tau} \subseteq R^\pm_\tau$.

□

We compare and contrast this result with the case of $\psi \in S^2(V^*)$. Dunn, Franks and Palmer [3] proved that if $Rank(\psi) \geq 3$, then $G_{R_\psi} = G_\psi$ if the signature of ψ is imbalanced, and $G_{R_\psi} = G^\pm_\psi$ if the signature of ψ is balanced. Thus it is interesting to note that, for the anti-symmetrical forms, the equality takes the same form as the case of the symmetric form, when the symmetric form has a balanced signature. Since there exist $A : V \to V$, such that $A^*\tau = -\tau$, independent of the signature, the sign ambiguity remains, despite the signature of τ.

5 A Necessary Condition for the Dependence of $\{R^S_\phi, R^S_\psi, R^\Lambda_\tau\}$

The linear dependence of three curvature tensors of all symmetric build has been addressed by Diaz and Dunn [1]. They determined that if two of the operators have full rank, among other assumptions, then a necessary condition for the dependence is that the operators be simultaneously diagonalized. We extend the results to the anti-symmetric build curvature tensors, where the notion of diagonalization for self-adjoint operators translates to block diagonalization in 2×2 blocks for skew-adjoint operators. We examine sets of three curvature tensors, where one is of an anti-symmetric build and the other two are of symmetric build. With some restriction on the skew-adjoint operator of the anti-symmetric curvature tensor, we determine that for the set to be linearly dependent, the skew-adjoint operator must be block diagonalized in 2×2 blocks simultaneously with the diagonalization of the other two operators. In a more general setting, we prove that the set is linearly independent, but by other means.

Lemma 5.1. *Let $\tau^* = -\tau$, and $\tau^2 = -I$, then*

1. $R^S_\tau(\tau x, \tau^{-1}y, \tau z, \tau^{-1}w) = R^S_\tau(x, y, z, w)$,
2. $R^S_\tau(\tau x, \tau^{-1}y, \tau^{-1}z, \tau w) = R^S_\tau(x, y, z, w)$,
3. $R^\Lambda_\tau(\tau x, \tau y, \tau^{-1}z, \tau^{-1}w) = R^\Lambda_\tau(x, y, z, w)$,
4. $R^\Lambda_\gamma(\tau x, \tau y, \tau^{-1}z, \tau^{-1}w) = R^\Lambda_\gamma(x, y, z, w)$.

Proof. Since $\tau^2 = -I$, then $\tau^{-1} = -\tau$. For assertion 1,

$$\begin{aligned} R^S_\tau(\tau x, \tau^{-1}y, \tau z, \tau^{-1}w) =& R^S_\phi(\tau^2 x, y, \tau z, \tau^{-1}w) \\ =& R^S_\phi(\tau^*\tau^2 x, (\tau^{-1})^*y, z, w) \\ =& R^S_\phi(\tau x, \tau y, z, w) \\ =& R^S_\tau(x, y, z, w) \end{aligned}$$

For assertion 2,

$$\begin{aligned} R^S_\tau(\tau x, \tau^{-1}y, \tau^{-1}z, \tau w) &= R^S_\phi(\tau^2 x, y, \tau^{-1}z, \tau w) \\ &= R^S_\phi((\tau^{-1})^*\tau^2 x, \tau^* y, z, w) \\ &= R^S_\phi(\tau x, \tau y, z, w) \\ &= R_\tau(x, y, z, w). \end{aligned}$$

For part 3, we invoke the identity 3.3,

$$\begin{aligned}R^{\Lambda}_{\tau}(\tau x,\tau y,\tau^{-1}z,\tau^{-1}w) =&2R^S_{\tau}(\tau x,\tau y,\tau^{-1}z,\tau^{-1}w)+R^S_{\tau}(\tau x,\tau^{-1}z,\tau y,\tau^{-1}w)+R^S_{\tau}(\tau x,\tau^{-1}w,\tau^{-1}z,\tau y)\\ =&2R^S_{\tau}(x,y,z,w)+R^S_{\tau}(x,z,y,w)+R^S_{\tau}(x,w,z,y)\\ =&R^{\Lambda}_{\tau}(x,y,z,w),\end{aligned}$$

where Lemma 3.1 is applied for the first term, and parts 1 and 2 of this lemma are applied for the second and third terms, interchanging y and z for the second term, and y and w for the third term.

Finally,

$$\begin{aligned}R^{\Lambda}_{\gamma}(\tau x,\tau y,\tau^{-1}z,\tau^{-1}w) &= 2R^S_{\gamma}(\tau x,\tau y,\tau^{-1}z,\tau^{-1}w)+R^S_{\gamma}(\tau x,\tau^{-1}z,\tau y,\tau^{-1}w)+R^S_{\gamma}(\tau x,\tau^{-1}w,\tau^{-1}z,\tau y)\\ &= 2R^S_{\phi}(\gamma\tau x,\gamma\tau y,\tau^{-1}z,\tau^{-1}w)+R^S_{\phi}(\gamma\tau x,\gamma\tau^{-1}z,\tau y,\tau^{-1}w)+R^S_{\phi}(\gamma\tau x,\gamma\tau^{-1}w,\tau^{-1}z,\tau y)\\ &= 2R^S_{\phi}(\tau^{-1}\gamma\tau x,\tau^{-1}\gamma\tau y,z,w)+R^S_{\phi}(\tau\gamma\tau x,\tau^{-1}\gamma\tau^{-1}z,y,w)+R^S_{\phi}(\tau\gamma\tau x,\tau^{-1}\gamma\tau^{-1}w,z,y)\\ &= 2R^S_{\phi}(\tau\gamma\tau x,\tau\gamma\tau y,z,w)+R^S_{\phi}(\tau\gamma\tau x,\tau\gamma\tau z,y,w)+R^S_{\phi}(\tau\gamma\tau x,\tau\gamma\tau w,z,y)\\ &= 2R^S_{\tau\gamma\tau}(x,y,z,w)+R^S_{\tau\gamma\tau}(x,z,y,w)+R^S_{\tau\gamma\tau}(x,w,z,y)\\ &= R^{\Lambda}_{\tau\gamma\tau}(x,y,z,w)\end{aligned}$$

□

Theorem 5.2. *Let A, B be self adjoint linear operators, and $Rank B \geq 4$. Let C be skew adjoint linear operator and $C^2 = -I$. If $\{R^S_A, R^S_B, R^{\Lambda}_C\}$ is linearly dependent, then A and B must be simultaneously diagonal and C simultaneously block diagonal in 2×2 blocks.*

Proof. Let $C^2 = -I$, and so C^{-1} exists and $C^{-1} = -C$. We choose a basis so that the linear map A is the identity. By hypothesis, consider $aR^S_{\bar{C}} + bR^S_{\bar{B}} + cR^{\Lambda}_{\bar{A}} = 0$, for nonzero $a, b, c \in \mathbb{R}$. Then, let $C = \sqrt{|a|}\bar{C}$, $B = \sqrt{|b|}\bar{B}$, and $A = \sqrt{|c|}\bar{A}$. Then, multiply by -1 if necessary, so that the first term is positive,

$$R_C = \epsilon R_B + \delta R_A,$$

where ϵ, $\delta \in \{-1, +1\}$.

Consider the case where one of the curvature tensors is zero. Then by [7], it has rank less than 1. Clearly, it is diagonalized or block-diagonalized. Then we are left with $R^S_A = \pm R^S_B$ $R^S_B = \pm R^{\Lambda}_C$, or $R^S_A = \pm R^{\Lambda}_C$, and so $A = \pm B$, since the last two cannot happen. If two of the canonical curvature tensors are zero, then the operators are trivially diagonalized and block-diagonalized simultaneously.

Thus we consider where all of the curvature tensors are nonzero. If any of them are zero, referring to Section 2, we see that the sum reduces to a case that has been previously studied. We can also assume that $A \neq \lambda B$ for $\lambda \in \mathbb{R}$. This is because if $A = \lambda B$, then, $R_C = R_A + \epsilon R_B = R_A + \epsilon\lambda^2 R_A = (1+\epsilon\lambda^2)R_A$. Then clearly A and B are diagonalized if and only if one of them is, and it reduces to the case of diagonalizing D with respect to A.

Then by the hypotheses and Lemma 5.1

$$\begin{aligned}R^{\Lambda}_C(x,y,z,w) &= R^{\Lambda}_C(Cx,Cy,C^{-1}z,C^{-1}w)\\ &= \epsilon R^S_B(Cx,Cy,C^{-1}z,C^{-1}w)+\delta R^S_A(Cx,Cy,C^{-1}z,C^{-1}w)\\ &= \epsilon R^S_{\phi}(C^{-1}BCx,C^{-1}BC^{-1}y,z,w)+\delta R^S_A(C^{-1}Cx,C^{-1}Cy,z,w)\\ &= \epsilon R^S_{CBC^{-1}}(x,y,z,w)+\delta R^S_A(x,y,z,w)\end{aligned}$$

Thus, $R^S_B = R^S_{C^{-1}BC}$. By Gilkey, [7] and since $Rank(B) \geq 3$, this implies that $B = \pm C^{-1}BC$. Now, in order to show that the commutativity follows, we prove that B and C cannot anti-commute.

For contradiction, assume that $\psi\tau = -\tau\psi$. Then let $\{e_1, ..., e_n\}$ be an orthonormal basis with respect to *phi*, such that ψ is diagonalized. Then we consider the matrix array of

$$B = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 & ... \\ 0 & \lambda_2 & 0 & 0 & ... \\ 0 & 0 & \lambda_3 & 0 & ... \\ 0 & 0 & 0 & \lambda_4 & ... \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \qquad C = \begin{pmatrix} 0 & \tau_{12} & \tau_{13} & \tau_{14} & .... \\ -\tau_{12} & 0 & \tau_{23} & \tau_{24} & ... \\ -\tau_{13} & \tau_{23} & 0 & \tau_{34} & ... \\ -\tau_{14} & -\tau_{24} & -\tau_{34} & 0 & ... \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Then, by hypothesis,

$$BC = \begin{pmatrix} 0 & \lambda_1\tau_{12} & \lambda_1\tau_{13} & \lambda_1\tau_{14} & ... \\ -\lambda_2\tau_{12} & 0 & \lambda_2\tau_{23} & \lambda_2\tau_{24} & ... \\ -\lambda_3\tau_{13} & -\lambda_3\tau_{23} & 0 & \lambda_3\tau_{34} & ... \\ -\lambda_4\tau_{14} & -\lambda_4\tau_{24} & -\lambda_4\tau_{34} & 0 & ... \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$= -\begin{pmatrix} 0 & \lambda_2\tau_{12} & \lambda_3\tau_{13} & \lambda_4\tau_{14} & ... \\ -\lambda_1\tau_{12} & 0 & \lambda_3\tau_{23} & \lambda_4\tau_{24} & ... \\ -\lambda_1\tau_{13} & -\lambda_2\tau_{23} & 0 & \lambda_4\tau_{34} & ... \\ -\lambda_1\tau_{14} & -\lambda_2\tau_{24} & -\lambda_3\tau_{34} & 0 & ... \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = -CB.$$

Since $C \neq 0$, there exists i, j such that $C_{ij} \neq 0$. By permuting the basis vectors, we may assume without loss of generality that $C_{ij} = C_{12}$. From the equality of $BC = -CB$, we get $-\lambda_1\tau_{12} = \lambda_2\tau_{12}$. Since $\tau_{12} \neq 0$, then $\lambda_1 + \lambda_2 = 0$. Then

$$\lambda_3\lambda_1 + \lambda_3\lambda_2 = 0. \tag{1}$$

By evaluating the hypothesis with (e_1, e_2, e_3, e_1), we get that $\tau_{12}\tau_{13} = 0$ and since $\tau_{12} \neq 0$, then $\tau_{13} = 0$. Similarly, evaluating (e_2, e_1, e_3, e_2) leads to $\tau_{23} = 0$. Then, from evaluating the hypothesis with (e_1, e_3, e_3, e_1), and (e_2, e_3, e_3, e_2), and since $\tau_{13} = \tau_{23} = 0$, we get that

$$1 + \epsilon\lambda_1\lambda_3 = 0, \text{ and } 1 + \epsilon\lambda_2\lambda_3 = 0.$$

Substituting for $\lambda_3\lambda_1$ and $\lambda_3\lambda_2$, into Equation (1), gives $-\epsilon + -\epsilon = 0$, a contradiction.

Refer to [8] for a proof that commutativity of two operators implies that they are simultaneously diagonalizable and block diagonalizable in 2×2 blocks down the diagonal with zeros elsewhere.

□

Remark The argument only required three distinct basis vectors and so a similar argument can be made in an arbitrary dimension.

6 Linear Independence of 2 Symmetric Canonical Curvature Tensors and 1 Anti-Symmetric Canonical Curvature Tensor

Lemma 6.1. *Let $A = A^*$, $B = B^*$, $C = -C^*$, and $dimV \geq 3$. If $\{R^S_A, R^S_B, R^\Lambda_C\}$ linearly dependent and A and B are simultaneously diagonalized, then C must also be block diagonalized in 2×2 blocks down the diagonal with zeros elsewhere and $Rank(C) = 2$.*

Proof. Recall that ϕ is a positive definite inner product and diagonalize A with respect to the basis so that $A = I$. Then simultaneously diagonalize B with respect to ϕ. Then evaluating (e_1, e_2, e_3, e_1) into $R_C = \epsilon R_A + \delta R_B$, we get

$$-C_{13}C_{21} - 2C_{12}C_{31} = 0,$$

which simplifies to $0 = C_{13}C_{12}$. Similarly, for (e_2, e_1, e_3, e_2), we get $0 = C_{23}C_{12}$. Finally, (e_3, e_1, e_2, e_3) results in $0 = C_{23}C_{13}$. Now, since $C \neq 0$, then $C_{ij} \neq 0$, and we can permute the basis vectors so that $C_{12} \neq 0$, then $C_{13} = 0$ and $C_{23} = 0$. Thus, if the dimension of V is 3, then C must be block diagonalizable with a 2×2 block.

Now we prove that for arbitrary dimension of V, the matrix representation of C with respect to the basis still gives zeros everywhere except for $C_{12} = -C_{21}$. Evaluating (e_1, e_2, e_k, e_1), results in that $C_{12}C_{1k} = 0$. Evaluating (e_2, e_1, e_k, e_2) results in $C_{12}C_{2k} = 0$. Thus, $C_{1k} = C_{2k} = 0$. Then consider evaluating (e_1, e_2, e_i, e_k), which results in

$$C_{1k}C_{2i} - C_{1i}C_{2k} - 2C_{12}C_{ik} = 0$$

and since $C_{1k} = C_{2k} = 0$, then we have $-2C_{12}C_{ik} = 0$ and so $C_{ik} = 0$.

Thus, for any dimension V, the matrix representation of C will be block-diagonalizable with 2×2 blocks and the only non-zero entries being $C_{12} = C_{21}$.

$\square$

Remark Since we only consider evaluating with three distinct indices or more, the symmetric curvature tensors always yield zero, if they are all simultaneously diagonalized. Thus, C must be the same form, for any sum of symmetric curvature tensors, so long as they are simultaneously diagonalized.

Theorem 6.2. *Let $A = A^*$, $B = B^*$, and $C = -C^*$. Then if $A \neq \lambda B$ for $\lambda \in \mathbb{R}$ and $dimV > 3$, $\{R_A^S, R_B^S, R_C^\Lambda\}$ is linearly independent.*

Proof. Let ϕ is a positive definite inner product and let the basis $\{e_1, ..., e_n\}$ be so that $A = I$. Diagonalize B with respect to the basis. Now, for contradiction, assume $R_A + \epsilon R_B = \delta R_C$ for some $A = A^*$, $B = B^*$, and $C = -C^*$, and $\epsilon, \delta \in \{-1, 1\}$. Also, assume that the curvature tensors are all nonzero and that $A \neq \lambda B$ for $\lambda \in \mathbb{R}$.

Since $C \neq 0$, we can permute the basis vectors to assume that $C_{12} \neq 0$. Then, by Lemma 6.1, the only non-zero entry of C is C_{12}.

We consider $dimV = 4$ first, in order to motivate the case where V has dimension n. Since $C_{23} = 0$, plugging in (e_2, e_3, e_3, e_2) to the hypothesis results in the equation $1 + \epsilon\lambda_2\lambda_3 = 0$, where $\lambda_i \in \mathbb{R}$ refers to the ith eigenvalue of B on this basis. Similarly, evaluating (e_1, e_3, e_3, e_1), (e_3, e_4, e_4, e_3), and (e_2, e_4, e_4, e_2), yields in total the following equations:

$$-\epsilon = \lambda_2\lambda_3 \tag{2}$$

$$-\epsilon = \lambda_3\lambda_1 \tag{3}$$

$$-\epsilon = \lambda_3\lambda_4 \tag{4}$$

$$-\epsilon = \lambda_2\lambda_4 \tag{5}$$

These equations imply that $\lambda_i \neq 0$. Then subtracting Equation (3) from Equation (2) yields

$$0 = \lambda_3(\lambda_2 - \lambda_1).$$

Thus, $\lambda_2 = \lambda_1$. Now subtract Equation (4) from Equation (3), yielding

$$0 = \lambda_3(\lambda_1 - \lambda_4),$$

and so $\lambda_1 = \lambda_4$. Finally, subtract Equation (4) from Equation (5) and we have

$$0 = \lambda_4(\lambda_2 - \lambda_3),$$

and so $\lambda_2 = \lambda_3$. Thus we have shown that $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 =: \lambda \neq 0$. Thus, since B is orthogonal with respect to A, in dimension 4 has λ down the diagonals and 0s elsewhere. Now we explain the generalization to higher dimensions. We have proven that $C_{ij} = 0$ for all pairs of $(i, j) \neq (1, 2)$ or $(2, 1)$. Then, for all pairs of $i, j \neq (1, 2)$ or $(2, 1)$, we have

$$-\epsilon = \lambda_i \lambda_j$$

$$-\epsilon = \lambda_i \lambda_k$$

$$-\epsilon = \lambda_j \lambda_k.$$

These equations imply that $\lambda_i \neq 0$ for all i. Subtracting each equation from the other yields $0 = \lambda_i(\lambda_j - \lambda_k)$ and $0 = \lambda_j(\lambda_k - \lambda_i)$. Then since $\lambda_i \neq 0$ for all i, $\lambda_j = \lambda_k$ and $\lambda_k = \lambda_i$). This shows that all of the eigenvalues of B are the same. Thus $B = \lambda A$, a contradiction.

□

7 Dimension 3 Case of $\{R^S_\phi, R^S_\psi, R^\Lambda_\tau\}$

Theorem 7.1. *Let $dimV = 3$, A, B self adjoint linear maps, and C a skew-adjoint linear map. Then there exists A, B, and C and $\delta, \epsilon = \pm 1$ such that $R^S_A + \epsilon R^S_B = \delta R^\Lambda_C$.*

Proof. The equations that need to be satisfied are as follows:

$$C_{ij} = 0, \text{ for all } i, j, \text{ except } C_{12} = -C_{21} \neq 0, \tag{6}$$

$$\lambda_1 \lambda_3 = -\epsilon, \tag{7}$$

$$\lambda_2 \lambda_3 = -\epsilon, \tag{8}$$

$$\lambda_1 \lambda_2 = -\epsilon(\delta 3 C_{12}^2 + 1). \tag{9}$$

Equations 7 and 8 imply that $\lambda_1 = \lambda_2$, which, along with Equation 9, require

$$\epsilon(\delta 3 C_{12}^2 + 1) \leq 0.$$

Based on the sign of δ and ϵ, we determine the conditions on C_{12} so that C is a solution to the sum. If $\epsilon = 1$, then $\delta C_{12}^2 \leq -1$ and so $\delta = -1$. Then $C_{12}^2 \geq \frac{1}{3}$. If $\epsilon = -1$, then δ could be -1 or $+1$. If $\delta = -1$, then $C_{12}^2 \leq \frac{1}{3}$. If $\delta = 1$, then any choice of C_{12} satisfies the constraints.

We include a particular example. Let $\epsilon = 1$ and $\delta = -1$, and so the equation becomes $R_C = R_A - R_B$. We still let $A = I$. Then

$$B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & \end{pmatrix}.$$

We demonstrate this by evaluating (e_i, e_j, e_k, e_l) in the hypothesis. There are two cases, where there are two distinct or three distinct indices (since $dimV = 3$, there cannot be all distinct indices).

For two distinct indices, our choice must satisfy $-3C_{ij}^2 = 1 - \lambda_i \lambda_j$. For $i, j = 1, 2$, we have $-3 \cdot 1^2 = 1 - 2 \cdot 2$. For $i, j = 1, 3$ and $i, j = 2, 3$, we have $0 = 1 - 2 \cdot \frac{1}{2}$.

For three distinct indices, our choice must satisfy $-3C_{ij}C_{ik} = 0$. The only non-zero C_{ij} is when $i, j = 1, 2$, so the left-hand side will always be zero as well.

□

8 Simultaneous Diagonalization of $\{R^S_\phi, R^\Lambda_\tau, R^\Lambda_\gamma\}$

Theorem 8.1. *Let $A = A^*$, $D = -D^*$, $C = -C^*$, and $C^2 = -I$. If $\{R^S_A, R^\Lambda_D, R^\Lambda_C\}$ is linearly dependent, then $CD = \pm DC$.*

Proof. Let $\{e_1, ..., e_n\}$ be an orthonormal basis, so that $A = I$ with respect to the basis. By hypothesis, we consider $R^S_A = \epsilon R^\Lambda_\tau + \delta R^\Lambda_\gamma$. Assume the none of the curvature tensors are zero. Otherwise, that tensor is clearly simultaneously diagonalizable with any other curvature tensor, and so the case is trivial.

Then

$$\begin{aligned} R^S_A &= R^S_A(Cx, Cy, C^{-1}z, C^{-1}w) \\ &= \epsilon R^\Lambda_C(Cx, Cy, C^{-1}z, C^{-1}w) + \delta R^\Lambda_D(Cx, Cy, C^{-1}z, C^{-1}w) \\ &= \epsilon R^\Lambda_C(x, y, z, w) + \delta R^\Lambda_{CDC}(x, y, z, w), \end{aligned}$$

by Lemma 5.1. Thus, $R^\Lambda_D = R^\Lambda_{CDC}$. Then, by [7], $D = \pm CDC$, which implies that $DC = \mp CD$.

Theorem 8.2. *Let $A = A^*$, $C = -D^*$, $D = -D^*$. Then if $dimV \geq 3$, there do not exist A, C and D such that $R^S_A = \epsilon R^\Lambda_C + \delta R^\Lambda_D$ for $\epsilon = \pm 1$.*

Choose the basis so that $A = I$ and C is block-diagonalized C in 2×2 blocks, with zeros elsewhere.

Consider evaluating (e_1, e_3, e_3, e_1), $(e_2, e_3, 3_3, e_2)$, and (e_3, e_1, e_2, e_3) into the hypothesis yields

$$1 = \delta 3 D^2_{13} \text{ and } 1 = \delta 3 D^2_{23}$$

$$0 = \delta 3 D_{13} D_{23}.$$

Since the first two equations imply that $D_{13} \neq 0$ and $D_{23} \neq 0$, contradict the third. An equivalent statement has been proved by Lovell [9]. □

9 Linear Dependence of $\{R^S_A, R^S_B, R^\Lambda_C, R^\Lambda_D\}$

Theorem 9.1. *For $dimV = 4$, and $A =^A *$, $B = -B^*$, and $C = -C^*$, $D = -D^*$, then there exists A, B, C, D, such that $\epsilon R_A + \delta R_B = R_C + \alpha R_D$ holds. Moreover, if A is diagonalizable with respect to B, then C and D must be simultaneously block-diagonalized in 2×2 blocks. Also, $|Spec(C)| = |Spec(D)| = 2$, with each eigenspace of multiplicity 2.*

Proof. By hypothesis, assume that $\epsilon R_A + \delta R_B = R_C + \alpha R_D$, where $\epsilon, \delta, \alpha \in \{-1, 1\}$. If one of the canonical curvature tensors is zero, then refer to previous results. Thus we assume that all of them are nonzero. Moreover, if $b = \lambda a$ or $C = \eta D$, for $\lambda, \eta \in \mathbb{R}$, then this reduces to cases which are linearly independent. Thus, we assume that the operators are linearly independent. We will first show the necessary condition that C and D are simultaneously block diagonalized in 2×2 blocks.

Let ϕ be the inner product and $\{e_1, ..., e_n\}$ be the orthonormal basis so that $A = I$. Then diagonalize B with respect to the basis. Then by computing $R(e_i, e_j, e_k, e_i)$ for distinct i, j, k, we get the following equation

$$0 = C_{ij} C_{ki} + \alpha D_{ij} D_{ki}.$$

Since $dimV = 4$, there are 12 equations.

Then computing $R(e_i, e_j, e_k, e_l)$, with i, j, k, l all distinct. We get

$$C_{il} C_{jk} - C_{ik} C_{jl} - 2 C_{ij} C_{kl} + \alpha D_{il} D_{jk} - \alpha D_{ik} D_{jl} - 2\alpha D_{ij} D_{kl} = 0.$$

Then, in $dimV = 4$, we get 3 equations.

Now we have 15 equations for 12 unknowns. Solving this system in a computer algebra system yields a total of 5 sets of solutions. (Some of the solution sets involve complex numbers, which we disregard.) Each solution set comprised of zeros everywhere and one dependence relation between the C_{ij} and D_{ij}. We list the equation that define the nonzero values in the matrix representation of C and D for each solution set as follows, with each $C_{ij} = D_{ij} = 0$ if omitted.

$$C_{12}C_{14} = \alpha D_{12}D_{14} \tag{10}$$

$$C_{12}C_{23} = \alpha D_{12}D_{23} \tag{11}$$

$$C_{12}C_{24} = \alpha D_{12}D_{24} \tag{12}$$

$$C_{12}C_{13} = \alpha D_{12}D_{13} \tag{13}$$

$$C_{12}C_{34} = \alpha D_{12}D_{34} \tag{14}$$

Now, showing that solution sets (10), (11), (12), and (13) yield a contradiction and that only solution set (14) results in operators that satisfy the sum will finish our proof. First we consider the case where $C_{12}C_{14} = \alpha D_{12}D_{14}$, and then provide a general proof for solution sets (10), (11), (12), and (13).

Let λ_i be the *ith* eigenvalue of B. Evaluating the hypothesis with two distinct basis vectors yields the following equations:

$$1 + \epsilon\lambda_1\lambda_3 = 0$$

$$1 + \epsilon\lambda_2\lambda_3 = 0$$

$$1 + \epsilon\lambda_2\lambda_4 = 0$$

$$1 + \epsilon\lambda_3\lambda_4 = 0.$$

The equations imply that $\lambda_i \neq 0$. Then subtracting them from each other, we have

$$\lambda_3(\lambda_1 - \lambda_2) = 0$$

$$\lambda_2(\lambda_3 - \lambda_4) = 0$$

$$\lambda_4(\lambda_2 - \lambda_3) = 0.$$

Thus $\lambda := \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4$ and so $\psi = \lambda\phi$, which contradicts the linear independence of the operators. Note that we do not acute need to assume that C_{12}, $C_{14} \neq 0$, $D_{12} \neq 0$, or $D_{34} \neq 0$ in order to arrive at the contradiction.

Computing (e_i, e_j, e_j, e_i) for i, j distinct, into the hypothesis give the relationship

$$1 + \epsilon\lambda_i\lambda_j = \delta C_{ij}^2 + \alpha D_{ij}^2.$$

Then, consider solution sets (10), (11), (12), and (13). As a "worst case", we assume that $C_{12} \neq 0$, $D_{12} \neq 0$, $C_{ik} \neq 0$, and $D_{ik} \neq 0$, for $i \in \{1, 2\}$ and $k \in \{3, 4\}$. When we evaluate the basis into the equation, we will not use the equations that include those indefinite values. Because $dimV = 4$, there exists an $l \in \{3, 4\}$, $l \neq k$, such that $C_{lh} = 0$ for all h. Then

$$1 + \epsilon\lambda_1\lambda_l = 0$$

$$1 + \epsilon\lambda_2\lambda_l = 0$$

$$1 + \epsilon\lambda_k\lambda_l = 0.$$

Subtract these equations from each other so that we have $\lambda_1 = \lambda_2 = \lambda_k$. Now consider $j \in \{1, 2\}$, $j \neq i$. Then

$$1 + \epsilon\lambda_j\lambda_k = 0.$$

Then subtract this equation from the one above that has λ_1 if $j = 1$, or subtract it from the one with λ_2, if $j = 2$. This is in order to factor out that value of λ_j, so that we have $\lambda_k = \lambda_l$. Thus, $\lambda_1 B = A$, contradicting the linear independence of B and A.

Now consider the last case, solution set (14). If $C_{34} = 0$, then we can apply the above argument and get a contraction. So assume that $C_{34} \neq 0$, as well as the original assumption that $C_{12} \neq 0$. Then $D_{12} \neq 0$ and $D_{34} \neq 0$. Since C_{12}, D_{12}, C_{34}, and D_{34} are the only nonzero terms, we have the following equations

$$1 + \epsilon\lambda_2\lambda_3 = 0$$
$$1 + \epsilon\lambda_1\lambda_3 = 0$$
$$1 + \epsilon\lambda_1\lambda_4 = 0$$
$$1 + \epsilon\lambda_2\lambda_4 = 0,$$

which together implies that $\lambda_1 = \lambda_2$ and $\lambda_3 = \lambda_4$. Then, in order to have B independent of A, we must also have that $\lambda_1 \neq \lambda_3$. Thus, there is one eigenvalue that corresponds to each $2imes2$ block of C and of C.

This proves that $|Spec(C)| = |Spec(D)| = 2$, with each eigenspace of multiplicity 2.

□

10 Example of The Sum of Two Symmetric and Two Anti-Symmetric Canonical Curvature Tensors

We provide a summary of the necessary conditions of B, C, and D in order for the sum $\epsilon R_A^S + \delta R_B^S = R_C^\Lambda + \alpha R_D^\Lambda$ to be expressed.

1. $\lambda_1 = \lambda_2$, $\lambda_3 = \lambda_4$, and $\lambda_1 \neq \lambda_3$
2. $C_{12}C_{34} = \alpha D_{12}D_{34}$
3. $C_{12} \neq 0$, $C_{34} \neq 0$, $D_{12} \neq 0$, and $D_{34} \neq 0$
4. Either $C_{12} \neq D_{12}$ or $C_{34} \neq D_{34}$
5. $\epsilon + \delta\lambda_3^2 = C_{34}^2 + \alpha D_{34}^2$
6. $\epsilon + \delta\lambda_1^2 = C_{12}^2 + \alpha D_{12}^2$
7. $\lambda_2\lambda_3 = \lambda_1\lambda_3 = \lambda_1\lambda_4 = \lambda_2\lambda_4 = -\epsilon\delta$

We provide an example of B, C, and D that satisfy these conditions and demonstrate that it satisfies $\epsilon R_A + \delta R_B = R_C + \alpha R_D$. Let $\epsilon = \delta = \alpha = 1$. We still choose the basis so that $A = I$, and diagonalize B with respect to the basis. Consider

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

We demonstrate that this choice of B, C, and D satisfies the sum by evaluating the basis elements (e_i, e_j, e_k, e_l), where there are two distinct, three distinct, and all distinct i, j, k, l.

For two distinct, we have

$$1 + \lambda_i\lambda_j = C_{ij}^2 + D_{ij}^2.$$

For $i, j = 1, 2$, this becomes $1 + (1)^2 = (1)^2 + (1)^2$. For $i, j = 3, 4$, this becomes $1 + (-1)^2 = 1^2 + (-1)^2$. And finally, for $i, j = 1, 3$, this becomes $1 + (1)(-1) = 0$. Because $\lambda_1 = \lambda_2$ and $\lambda_3 = \lambda_4$, we have covered all the distinct cases for two distinct indices.

For three distinct and all distinct we have the respective equations:

$$0 = C_{ij}C_{ik} + D_{ij}D_{ik} \tag{15}$$

$$0 = C_{il}C_{jk} - C_{ik}C_{jl} - 2C_{ij}C_{kl} + D_{il}D_{jk} - D_{ik}D_{jl} - 2D_{ij}D_{kl}. \tag{16}$$

Clearly, the only nonzero term will be $C_{12}C_{34}$ and $D_{12}D_{34}$, and so the left hand side of equation 15 will always be zero and for Equation (16), the only distinct case is $C_{12}C_{34} + D_{12}D_{34} = 0$. And for our choices, we have $1 \cdot 1 + 1(-1) = 0$.

11 Dimension Argument for Linear Dependence

The linear dependence of a set of canonical curvature tensors may be examined by the approach of whether there exists a solution for the set of equations that arrises from plugging in the elements of the basis. We take a very general approach to the question, comparing the number of equations with the number of unknowns for three different cases.

For a given dimension of V, we determine the number of tensors that could likely be a part of a dependence relation. We determine the number of equations and the number of unknowns that arise for a given dimension and a given number of canonical curvature tensors. It is interesting that the number of unknowns and equations are different between the anti-symmetric and the symmetric builds. We consider three cases, a "best case" where there is a positive definite inner product, with all the self adjoint operators diagonalized and all the skew adjoint operators block-diagonalized simultaneously; a "worst case", where there is an inner product and none of the operators are diagonalized (which corresponds to when the inner product is only nondegenerate); and a standard case where there is a positive definite inner product, and so one of the self-adjoint operators is diagonalized.

Theorem 11.1. *Let dimension $V=n$, $\tau=-\tau^*$ and $\psi=\psi^*$. Consider $\sum_a \delta_a R^{\Lambda}_{\tau_a}=\sum_b \epsilon_b R^S_{\psi_b}$.*

1. *If ϕ is the inner product, all ψ_i are diagonalized, and τ_i are block diagonalized in 2×2 blocks, then there are more unknowns than equations when $\frac{n}{2}\cdot a+n(b-1)>\frac{n(3n-2)}{4}$.*
2. *If ψ_i is the inner product and none of the operators are diagonalized or block-diagonalized, then there are more unknowns than equations when $\binom{n}{2}\cdot a+\binom{n}{2}\cdot n\cdot(b-1)>\frac{n^2(n^2-1)}{12}$.*
3. *If ϕ is a positive definite inner product so that $\psi_i=I$ and there exists a $j\neq i$, such that ψ_j is diagonalized, then there are more unknowns than equations when $\binom{n}{2}\cdot a+\binom{n}{2}\cdot n\cdot(b-2)+n>\frac{n^2(n^2-1)}{12}$.*

Proof. Consider
$$\sum_a \delta_a R^{\Lambda}_{\tau_a}=\sum_b \epsilon_b R^S_{\psi_b},$$
where $\tau_a\in\Lambda^2(V^*)$ and $\psi_b\in S^2(V^*)$.

We first consider the "best case," so let ϕ_1 be the inner product, $\{e_1,f_1,...e_{n/2},f_{n/2}\}$ be the basis, so that $\psi_1=I$. For all i, let ψ_i be diagonalized with respect to the basis and τ_i simultaneously block diagonalized into 2×2 blocks. There are three cases of equations given by inputting (e_i,e_j,e_k,e_l), depending on whether there are 2, 3, or all distinct indices.

For (e_i,f_i,f_i,e_i), the equation is

$$\sum_a \delta_a(3\tau_{ij}^2)_a=\epsilon_1+\sum_b^{m-1}\epsilon_b(\lambda_i\lambda_j)_b$$

, where $\psi_i(e_i,e_i)=\lambda_i$ and $\psi_i(f_i,f_i)=\lambda_j$. For (e_i,e_k,e_k,e_i) and (f_i,f_k,f_k,f_i), we get

$$0=\epsilon_1+\sum_b^{m-1}(\lambda_i\lambda_k)_b.$$

Thus, the number of equations is the number of choices of two basis elements, where order is not count. Thus, we have $\binom{n}{2}$. For three distinct, all terms on both sides are zero. For all distinct terms, (e_i,f_i,f_j,e_j) is the only nonzero equations, which is

$$\sum_a \delta_a(-2\tau_{ii}\tau_{jj})=0.$$

The number of equations is the number of choices of the first term, and the number of choices of the third term, of which must be the opposite type of basis element. Since order does not matter, we have $n\binom{\frac{n}{2}}{2}$. Thus, the total number of equations is $\frac{(3n-2)n}{4}$. The number of unknowns for this case is one entry in each 2×2 block along the diagonal for $[\tau_a]_{ij}$ and the entries in the diagonal for $[\psi_b]_{ij}$. We subtract n unknowns for ψ_1, the inner product. Thus we have $\frac{n}{2}\cdot a + n(b-1)$ unknowns.

For the "worst case", we chose the basis so that ψ_1 as an inner product, and let $\{e_1, e_2, ..., e_n\}$ be the basis. None of the other operators are diagonalized or block diagonalized. For two distinct, three distinct, and all distinct indices, we get the equations respectively:

$$\sum_a \delta_a(3\tau_{ij}^2)_a = \epsilon_1 + \sum_b^{m-1} \epsilon_b(\psi_{ii}\psi_{jj} - \psi_{ij}^2)_b,$$

$$\sum_a \delta_a(3\tau_{ij}\tau_{ik})_a = \sum_b^{m-1} \epsilon_b(\psi_{ii}\psi_{jk} - \psi_{ik}\psi_{ji})_b,$$

$$\sum_a \delta_a(\tau_{il}\tau_{jk} - \tau_{ik}\tau_{jl} - 2\tau_{ij}\tau_{kl})_a = \sum_b^{m-1} \epsilon_b(\psi_{il}\psi_{jk} - \psi_{ik}\psi_{jl})_b.$$

The number of equations respectively is $\binom{n}{2}$, $n\binom{n}{2}$, and $2\binom{n}{4}$. Then, we sum the three to obtain that the total number of equations is

$$\frac{n^2(n^2-1)}{12},$$

which is the dimension of $\mathcal{A}(V)$. The number of unknowns is the upper triangle of each $[\tau_a]_{ij}$ and the entire matrix $[\psi_b]_{ij}$, so we have $\binom{n}{2}\cdot a + n^2 \cdot b$.

For the "standard case", we have $\psi_1 = I$ and we diagonalize ψ_2 with respect to the basis $\{e_1, e_2, ..., e_n\}$. Let η_i be the ith eigenvalue of ψ_2. Then, for two distinct, three distinct, and all distinct indices, the respective equations are

$$\sum_a \delta_a(3\tau_{ij}^2)_a = \epsilon_1 + \epsilon_2\eta_i\eta_j + \sum_b^{m-2} \epsilon_b(\psi_{ii}\psi_{jj} - \psi_{ij}^2)_b,$$

$$\sum_a \delta_a(\tau_{ij}\tau_{ik})_a = \sum_b^{m-2} \epsilon_b(\psi_{ii}\psi_{jk} - \psi_{ik}\psi_{ji})_b,$$

$$\sum_a \delta_a(3\tau_{il}\tau_{jk} - \tau_{ik}\tau_{jl} - 2\tau_{ij}\tau_{kl})_a = \sum_b^{m-2} \epsilon_b(\psi_{il}\psi_{jk} - \psi_{ik}\psi_{jl})_b.$$

The number of equations is the same as the worst case, $\frac{n^2(n^2-1)}{12}$ and what differs with this case is that there are fewer unknowns for the self-adjoint operators, since now ψ_i is diagonalized, we only need find the diagonal entires. Thus, the number of unknowns is $\binom{n}{2}\cdot a + \binom{n}{2}\cdot n\cdot(b-2) + n$. $\square$

So the number of equations depends purely on the dimension of V. The number of unknowns depends both on the dimension and on the number of symmetric and anti-symmetric canonical curvature tensors. With such an analysis, one can see that trying to sum only anti-symmetric or only symmetric canonical curvature tensors results in less equations and so there is less constraint, and smaller sums can be expressed in a given dimension than with using both forms.

From such an analysis, we can see that sets of one type of canonical curvature tensors are linearly dependent with fewer elements in the set, than those sets of mixed types.

12 Anti-Symmetric Canonical Curvature Tensor Expressed as a Sum of Symmetric Canonical Curvature Tensors

Theorem 12.1. *For $\tau = -\tau^*$, $\psi_i = \psi_i^*$ such that ψ_i are simultaneously diagonalized. Then, $R_\tau^\Lambda = \sum_b \epsilon_b R_{\psi_b}^S$ has more unknowns than equations when $1 + bn \geq \binom{n}{2}$.*

Consider $R_\tau = \sum_b \epsilon_b R_{\psi_b}$, where $\psi_1 = I$ and ψ_i are simultaneously diagonalized. As we have proven in Lemma 6.1, τ must be block diagonal and if $\tau \neq 0$, then there exist only one $\tau_{ij} \neq 0$, which without loss of generality, we said was τ_{12}. Then, the question of whether we may express the anti symmetric canonical curvature tensor in terms of a given number of symmetric canonical curvature tensors is reduced to the dependence of the following equations

$$1 + \sum_b (\epsilon \lambda_i \lambda_j)_b = 0 \qquad \text{and} \qquad \sum_b (\epsilon \lambda_1 \lambda_2)_b = \tau_{12}^2,$$

where λ_i refers to the ith eigenvalue of ψ_b for each b. The number of equations is $\binom{n}{2}$. The total number of unknowns is the number of eigenvalues of each ψ_b plus τ_{12}, so it is $1 + bn$. Then, by comparing when

$$1 + bn \geq \binom{n}{2}$$

determines the highest dimension that such a sum could likely be expressed; or conversely that given a dimension and an anti-symmetric canonical curvature tensor, what number of symmetric canonical curvature tensors one is likely able to express it in.

For example, if we are trying to express the anti-symmetric canonical curvature tensor in terms of two symmetric canonical curvature tensors, the formula gives that for $dimV = 3$, there are 3 equations, with 7 unknowns, and therefore it would appear that a solution exists. In fact, we gave an example of a solution in Section 6. However, in dimension 4, there are 9 unknowns and 12 equations so it would appear that a solution does not exist. In fact, we have proved in Section 5 that we cannot express an anti-symmetric canonical curvature tensor in terms of two symmetric canonical curvature tensors in for $dimV = 4$.

For expressing one anti-symmetric canonical curvature tensor in terms of 3 symmetric canonical curvature tensors, the number of equations does not exceed the number of unknowns until dimension 8 (then there are 25 unknowns and 28 equations). Thus, for $dimV < 8$, it would seem that we can express any anti-symmetric canonical curvature tensor in terms of 3 symmetric canonical curvature tensors.

13 Symmetric Canonical Curvature Tensor Expressed as a Sum of Anti-Symmetric Canonical Curvature Tensors

Theorem 13.1. *Let $\psi = \psi^*$, $\tau_b = -\tau_b^*$. Then*

$$R_\psi = \sum_a \delta_a R_{\tau_a},$$

has more unknowns than equations when $a \geq \frac{n(n+1)}{6}$.

Proof. Evaluating 2, 3, and all distinct indices gives the following equations

- $1 = \sum_a 3\delta_a (\tau_{ij}^2)_a$
- $0 = \sum_a 3\delta_a (\tau_{ij}\tau_{ik})_a$
- $0 = \sum_a \delta_a (\tau_{il}\tau_{jk} - \tau_{ik}\tau_{jl} - 2\tau_{ij}\tau_{kl})_a$

In total, the number of equations is the same as the "standard case", and so there are $\frac{n^2(n^2-1)}{12}$ equations. The number of unknowns is the upper triangle of each τ_a, and so we have $\binom{n}{2}a$. Thus, comparing the number of equations with the number of unknowns, it seems that we would have a way to express the symmetric canonical curvature tensor in terms of an anti-symmetric canonical curvature tensor when

$$\binom{n}{2}a \geq \frac{n^2(n^2-1)}{12}.$$

□

For example, consider $a = 2$. Then for $dimV = 3$, there are an equal number of equations and unknowns. However, once $dimV \geq 3$, there are more equations than unknowns. These indicate what Lovell proved, that $\{R_\tau, R_\gamma, R_\phi\}$ for $\tau, \gamma \in \Lambda^2$ and $\phi \in S^2$ is linearly dependent for $dimV = 3$ and linearly independent for $dimV \geq 4$, [9].

For $a = 3$, there are more unknowns than equations in $dimV \leq 4$, and in $dimV = 5$, there are exactly the same number of unknowns as equations. In $dimV \geq 6$, there are more equations than unknowns.

Conjecture 13.2. *Let $\phi = \phi^*$, τ, γ, ρ skew adjoint. Then $\{R_\tau, R_\gamma, R_\rho, R_\phi\}$ is linearly independent for $dimV \geq 6$ and linearly dependent for $dimV < 6$.*

14 Dimension Argument: Simultaneous Diagonalization

Given a sum of the same type of canonical curvature tensors, we develop an estimate for when the sum implies that the corresponding operators must be simultaneously diagonalized.

Theorem 14.1. *If $R^S_A + \sum^k \epsilon_i R^S_{A_i} = 0$, where A_i are simultaneously diagonalized, then if k satisfies*

$$n(k+1) \leq \frac{n^2(n^2-1)}{12} \leq kn + \frac{n(n+1)}{2},$$

having A diagonalized simultaneously overdetermines the system of equations.

Proof. Consider $R_A + \sum^k \epsilon_i R^S_{A_i} = 0$ where A_i are simultaneously diagonalized. Then, the number of equations is $dim\mathcal{A}(V) = \frac{n^2(n^2-1)}{12}$. Let α refer to the number of unknowns if A is simultaneously diagonalized, so

$$\alpha = n(k+1).$$

Let β refer to the number of unknowns if A is not simultaneously diagonalized. Then

$$\beta = kn + \frac{n(n+1)}{2}.$$

Then we compare when $\alpha \leq \frac{n^2(n^2-1)}{12} \leq \beta$, and so when

$$n(k+1) \leq \frac{n^2(n^2-1)}{12} \leq kn + \frac{n(n+1)}{2}.$$

□

The value of k that satisfies this inequality tells when having the next operator being simultaneously diagonalized overdetermines the system of equations. Then, that value of k would estimate the number of terms that must be simultaneously diagonalized.

We now give a similar argument for anti-symmetric canonical curvature tensors.

Theorem 14.2. *If $R^\Lambda_A + \sum^k \epsilon_i R^\Lambda_{A_i} = 0$, where A_i are simultaneously block-diagonalized in 2×2 blocks down the diagonal with zeros elsewhere, then if k satisfies*

$$\frac{n}{2}(k+1) \leq \frac{n^2(n^2-1)}{12} \leq k\frac{n}{2} + \frac{n(n-1)}{2}$$

overdetermines the system of equations.

Proof. Consider $R^\Lambda_A + \sum^k \epsilon_i R^\Lambda_{A_i} = 0$, where A_i are simultaneously block-diagonalized in 2×2 blocks down the diagonal with zeros elsewhere. Then the number of equations that arise from evaluating basis vectors into the sum is the dimension of $\mathcal{A}(V) = \frac{n^2(n^2-1)}{12}$. Let α refer to the number of unknowns if A is simultaneously block diagonalized in 2×2 blocks down the diagonal, so

$$\alpha = \frac{n}{2}(k+1).$$

Let β refer to the number of unknowns if A is not simultaneously block diagonalized in 2×2 blocks. Then

$$\beta = kn2 + \frac{n(n-1)}{2}.$$

Then we compare when $\alpha \leq \frac{n^2(n^2-1)}{12} \leq \beta$, and so when

$$\frac{n}{2}(k+1) \leq \frac{n^2(n^2-1)}{12} \leq k\frac{n}{2} + \frac{n(n-1)}{2}.$$

□

The value of k that satisfies this inequality tells when having the next being operator simultaneously block diagonalized in 2 blocks down the diagonal overdetermines the system of equations. Then, that value of k would estimate the number of terms that must be simultaneously block diagonalized in 2×2 blocks down the diagonal with zeros elsewhere.

15 Chain Complexes and Linear Dependence

We consider situations where the operators are in a chain complex. This allows any of the operators to have a non-trivial kernel.

Lemma 15.1. *If $imA \subseteq kerB$ or $imB \subseteq kerA$, $B = \pm B^*$, then $B^* R_A = 0$ for symmetrically built and anti-symmetrically built canonical curvature tensors.*

Proof. Let ϕ be the inner product. Then $imA \subseteq kerB$ implies that $BA = 0$ and $imB \subseteq kerA$ implies $AB = 0$. Thus, we have either $BA = 0$ or $AB = 0$. Let $B = \pm B^*$. For the symmetric canonical curvature tensor, and apply Lemma 4.1,

$$\begin{aligned} B^* R^S_A &= R^S_{B^*AB} \\ &= R^S_{\pm BAB} \\ &= 0. \end{aligned}$$

For the anti-symmetric canonical curvature tensor, apply Lemma 4.1,

$$\begin{aligned} B^* R^\Lambda_A &= R^\Lambda_{B^*AB} \\ &= R^\Lambda_{\pm BAB} \\ &= 0. \end{aligned}$$

□

Lemma 15.2. *If $A = \pm A^*$, and $Rank(A^k) = p$, then $Rank(A) = p$.*

Proof. For $A = A^*$, diagonalize A with respect to ϕ. Then

$$A^k = \begin{pmatrix} \lambda_1^k & 0 & 0 & 0 & ... \\ 0 & \lambda_2^k & 0 & 0 & ... \\ \vdots & & \ddots & & \\ 0 & ... & \lambda_p^k & 0 & ... \\ 0 & ... & 0 & 0 & .... \\ \vdots & .... & & \ddots & \end{pmatrix}.$$

Then $Rank(A^k) = p$ if and only if $\lambda_i^k \neq 0$ for $1 \leq i \leq p$. Thus, $\lambda_i \neq 0$ for $1 \leq i \leq p$, and so $Rank(A) = p$.

For $A = -A^*$, block-diagonalize A in 2×2 blocks down the diagonal and zeros elsewhere. Then, each 2×2 block of A is of the form

$$\tilde{A} = \begin{pmatrix} 0 & \lambda_i \\ -\lambda_i & 0 \end{pmatrix}.$$

For k even, the 2×2 blocks of A^k are of the form

$$\tilde{A}^k = \epsilon \begin{pmatrix} \lambda_i^k & 0 \\ 0 & \lambda_i^k \end{pmatrix}$$

where $\epsilon = 1$ if $k = 0 mod 4$, and $\epsilon = -1$ if $k = 2 mod 4$. For k odd, the 2×2 blocks are of the form

$$\tilde{A}^k = \epsilon \begin{pmatrix} 0 & \lambda^k \\ -\lambda^k & 0 \end{pmatrix}.$$

Then $\epsilon = 1$ if $k = 1 mod 4$ and $\epsilon = -1$ if $k = 3 mod 4$.

Then $Rank(A^k) = p$ if and only if $\lambda_i^k \neq 0$ for $1 \leq i \leq p$. Thus, this happens if and only if $\lambda_i \neq 0$ for $1 \leq i \leq p$. Thus $Rank(A) \neq 0$. □

Theorem 15.3. *For A, B, and C in the following chain complex that satisfy* $R_A + \epsilon R_B + \delta R_C = 0$ *for* $\epsilon, \delta = \pm 1$, *then* $R_B = 0$. *Moreover, if* $Rank(A) \geq 3$ *or* $Rank(C) \geq 3$, *then* $C = \pm A$. *If* $Rank(A) \geq 4$ *and* $Rank(C) \geq 4$, *then* R_A *and* R_C *must be the same build. Also, given those rank assumptions,* $\delta = -1$. *Furthermore, if the chain complex is an exact sequence and* $B = -B^*$ *then* A *and* C *are invertible.*

$$V \xrightarrow{A} V \xrightarrow{B} V \xrightarrow{C} V$$

Proof. By hypothesis, $R_A + \epsilon R_B + \delta R_C = 0$ for R_A, R_B, and R_C symmetric or anti-symmetric build and $\epsilon, \delta = \pm 1$. The chain complex implies that $CB = 0$ and $BA = 0$. Then precomposing the sum with B,

$$B^* R_A(x, y, z, w) = \epsilon B^* R_B(x, y, z, w) + \delta B^* R_C(x, y, z, w).$$

By Lemma 15.1 we are left with $0 = B^* R_B(x, y, z, w)$. Lemma 4.1 implies that

$$B^* R_B(x, y, z, w) = R_{B^3}(x, y, z, w).$$

If $B = -B^*$, then $R^{\Lambda}_{B^3} = 0$ if and only if $B^3 = 0$ [7]. By Lemma 15.2, $B = 0$. If $B = B^*$, then $R^S_{B^3} = 0$ if and only if $Rank(B^3) \leq 1$ [7]. Then $Rank(B) \leq 1$ and so $R^S_B = 0$. Thus, for either build, $R_B = 0$.

Then, $R_A + \delta R_C = 0$. If R_A and R_C are both symmetrically built or both anti-symmetrically built, then by [1], [10] $\delta = -1$. If $R^{\Lambda}_A = R^{\Lambda}_C$, then $A = \pm C$ [7]. For the symmetric case, $R^S_A = R^S_B$, we assume that $Rank(A) \geq 3$ or $Rank(C) \geq 3$, to get that $A = \pm C$.

By Lovell's results, if $Rank(A) \geq 4$ and $Rank(C) \geq 3$, then $R_A^{\Lambda} \neq \pm R_C^S$ [9]. And the opposite, we assume $Rank(C) \geq 4$ and $Rank(A) \geq 3$, then $R_C^{\Lambda} \neq \pm R_A^S$. Thus, if $Rank(A) \geq 4$ and $Rank(C) \geq 4$, then R_A and R_C must be the same build.

Consider where $B = -B^*$ and the sequence is exact. Since $B = -B^*$, then $R_B^{\Lambda} = 0$, and so $B = 0$. Since the sequence is exact, $imA = kerB$ and $imB = kerC$. Thus, $imA = V$ and $kerC = 0$. $\square$

Theorem 15.4. *If $A, B_1, ..., B_k$ are linear maps in one of the two following sets of chain complexes such that $0 = R_A + \sum^k \epsilon_i R_{B_i}$ for R symmetric or anti-symmetric build, then $R_A = 0$. Moreover, if $A = -A^*$ then for each sequence that is exact at V, then the corresponding B_i is invertible.*

$$\begin{tikzcd}
 & & V \arrow[d, dotted, no head] \\
V \arrow[r, "A"] & V \arrow[ur, "B_1"] \arrow[r, "B_i"] \arrow[dr, "B_k"'] & V \arrow[d, dotted, no head] \\
 & & V
\end{tikzcd}
\quad or \quad
\begin{tikzcd}
V \arrow[d, dotted, no head] \arrow[dr, "B_1"] & & \\
V \arrow[r, "B_i"] \arrow[d, dotted, no head] & V \arrow[r, "A"] & V \\
V \arrow[ur, "B_k"'] & &
\end{tikzcd}$$

Proof. First, note that both diagrams depict a union of k chain complexes of length 2. Thus, $imA \subseteq kerB_i$ for all i or $imB_i \subseteq kerA$

Consider $0 = R_A + \sum \epsilon_i R_{B_i}$. The set of chain complexes imply that $B_i A = 0$ or all i or $AB_i = 0$ for all i. Then precompose with A so,

$$0 = A^* R_A + \sum \epsilon_i A^* R_{B_i} = R_{A^3} + \sum \epsilon_i R_{AB_iA} = R_{A^3},$$

by Lemmas 15.1 and 4.1 . Thus, $R_{A^3} = 0$. For $A = -A^*$, $R_{A^3}^{\Lambda} = 0$, if and only if $A^3 = 0$ [7]. Then $A = 0$ by Lemma 15.2. If $A = A^*$, then $R_{A^3}^S = 0$ if and only if $Rank(A^3) \leq 1$ [7]. Then $Rank(A) \leq 1$ and so $R_A^S = 0$.

Consider where $A = -A^*$ and a given sequences is exact at V, so $imA = kerB_i$ for some i (or $imB_i = kerA$). Then $A = -A^*$, implies that R_A is anti-symmetric build, and so $R_A^{\Lambda} = 0$ implies that $A = 0$. Then, if exact at V, $0 = imA = kerB_i$. Thus B_i is invertible. $\square$

Remark For example, consider A, B, and C in the set of the two following chain complexes and $R_A + \epsilon_1 R_B + \epsilon_2 R_C = 0$.Then the theorem proves that $R_B = 0$.

1) $\quad V \xrightarrow{A} V \xrightarrow{B} V$

2) $\quad V \xrightarrow{C} V \xrightarrow{B} V$

Moreover, given that 1) is exact at V, then A is invertible. If 2) is exact at V, then C is invertible.

Theorem 15.5. *Let A, B, C, and D be linear maps in the following chain complex such that $R_A + \epsilon_1 R_B + \epsilon_2 R_C + \epsilon_3 R_D = 0$. Then,*

1. *If $Rank(A) \geq 4$ or $Rank(C) \geq 4$, then A and C are both self-adjoint or both skew-adjoint. If $Rank(B) \geq 4$ or $Rank(D) \geq 4$, then B and D are both self-adjoint or both skew-adjoint.*
2. *If R_A and R_C are anti-symmetric build, then $A^3C = \pm C^3 A$. If R_A and R_C are symmetric build and $Rank(A), Rank(C) \geq 3$, then $A^3 C = \pm C^3 A$.*
3. *If R_B and R_D are anti-symmetric build, then $B^3 D = \pm D^3 B$. If R_B and R_D are symmetric build and $Rank(B), Rank(D) \geq 3$, then $B^3 D = \pm D^3 B$.*

4. *If the symmetric terms have Rank ≥ 3, and the anti-symmetric terms have Rank ≥ 4, then $\epsilon_2 = -1$ and $\epsilon_1 = -\epsilon_3$.*

$$\begin{tikzcd} V \arrow[r, "A"] & V \arrow[d, "B"] \\ V \arrow[u, "D"] & V \arrow[l, "C"] \end{tikzcd}$$

Proof. Consider $R_A + \epsilon_1 R_B + \epsilon_2 R_C + \epsilon_3 R_D = 0$. Precompose it with A results in

$$\begin{aligned} 0 &= A^* R_A + \epsilon_1 A^* R_B + \epsilon_2 A^* R_C + \epsilon_3 A^* R_D \\ &= R_{A^3} + \epsilon_2 R_{A^* CA} \\ &= R_{A^3} + \epsilon_2 R_{ACA} \end{aligned}$$

Then precompose with B, C, and D which are done similarly. In total, there are the following equations:

$$R_{A^3} = -\epsilon_2 R_{ACA}, \tag{17}$$

$$\epsilon_1 R_{B^3} = -\epsilon_3 R_{BDB}, \tag{18}$$

$$\epsilon_2 R_{C^3} = -R_{CAC}, \tag{19}$$

$$\epsilon_3 R_{D^3} = -\epsilon_1 R_{DBD}. \tag{20}$$

We refer to previous results about sums of two curvature tensors. If $A \in \Lambda^2(V^*)$, $Rank(A) \geq 4$, then there does not exist $B \in S^2(V^*)$ such that $R_A^\Lambda = \pm R_B^S$ [9, 10]. Thus, for these equations to hold, R_{B^3} must be the same build as R_{BDB}, which implies that R_B is the same build as R_D. Similarly, R_{C^3} must be the same build as R_{CAC}, which implies that R_C is the same build as R_A.

Consider a result of Gilkey's [7]:

1. If $A = A^*$, $Rank(A) \geq 3$, and $R_A^S = R_B^S$, then $A = \pm B$.
2. if $A = -A^*$ and $R_A^\Lambda = R_B^\Lambda$, then $A = \pm B$.

If R_A and R_C are anti-symmetrically built, $R_{A^3}^\Lambda = R_{ACA}^\Lambda$ and $R_{C^3}^\Lambda = R_{CAC}^\Lambda$ imply that $A^3 = \pm ACA$ and $C^3 = \pm CAC$. If R_B and R_D are anti-symmetrically built, then $R_{B^3}^\Lambda = R_{BDB}^\Lambda$ and $R_{D^3}^\Lambda = R_{DBD}^\Lambda$ imply that $B^3 = \pm BDB$ and $D^3 = \pm DBD$.

If R_A and R_C are symmetrically built, then we need the rank assumption, that $Rank(A) \geq 3$, $Rank(C) \geq 3$. These imply that $Rank(A^3), Rank(C^3) \geq 3$ and so $A^3 = \pm ACA$, $C^3 = \pm CAC$. If R_B and R_D are symmetrically built, and that $Rank(B), Rank(D) \geq 3$, then $Rank(C^3) \geq 3$ and $Rank(D^3) \geq 3$. Thus we conclude that $B^3 = \pm BDB$ and $D^3 = \pm DBD$. Finally, $A^3C = \pm ACAC = \pm AC^3$ and $B^3 D = \pm BDBD = \pm BD^3$.

Now by Diaz, Dunn [1] and Treadway [10], we have the two results:

1. Let $A = A^*$ and $Rank(A) \geq 3$. Then there does not exist B such that $R_A^S = -R_B^S$.
2. Let $A = -A^*$, $A \neq 0$, and $Rank(A) \geq 4$. Then there does not exist B, such that $R_A^\Lambda = -R_B^\Lambda$.

We apply these results to the sign choice $\epsilon_1, \epsilon_2, \epsilon_3$. To meet the rank requirements, assume $Rank(A), Rank(C) \geq 3$ if symmetrically built, and so $Rank(A^3), Rank(C^3) \geq 3$. If R_A, R_C are symmetrically built and assume $Rank(A), Rank(C) \geq 4$ if anti-symmetrically built, and so $Rank(A^3), Rank(C^3) \geq 3$. For Equations 17 and 19 to hold, $\epsilon_2 = -1$ and $\epsilon_1 = -\epsilon_3$.

$\square$

Theorem 15.6. *Let A, B, C, and D be in the following chain complex and $R_A + \epsilon_1 R_B + \epsilon_2 R_C + \epsilon_3 R_D = 0$.*

1. *If $Rank(C) \geq 4$, then A and C are both self-adjoint or both skew-adjoint. If $Rank(B) \geq 4$, then B and D are both self-adjoint or both skew-adjoint.*
2. *If R_B and R_D are anti-symmetrically built, then $B^3 = \pm BDB$. If R_B and R_D are symmetrically built and $Rank(B) \geq 3$, then $B^3 = \pm BDB$.*
3. *If R_A and R_C are anti-symmetrically built, then $C^3 = \pm CAC$. If R_A and R_C are symmetrically built and $Rank(C) \geq 3$, then $C^3 = \pm CAC$.*
4. *If $D = 0$, then $B = 0$ and if $A = 0$, then $C = 0$.*
5. *Let $Rank(C) \geq 3$ if R_C symmetrically built or $Rank(C) \geq 4$ if anti-symmetrically built. Then $\epsilon_2 = -1$. Let $Rank(B) \geq 3$ if R_B is symmetrically built or $Rank(B) \geq 4$ if anti-symmetrically built. Then $\epsilon_1 = -\epsilon_3$.*

$$V \xrightarrow{A} V \xrightarrow{B} V \xrightarrow{C} V \xrightarrow{D} V$$

Proof. Precompose $R_A + \epsilon_1 R_B + \epsilon_2 R_C + \epsilon_3 R_D = 0$ with B and then with C, to get that

$$\epsilon_1 R_{B^3} + \epsilon_3 R_{BDB} = 0, \tag{21}$$

$$\epsilon_2 R_{C^3} + R_{CAC} = 0. \tag{22}$$

For part one we use the result that if $A \in \Lambda^2(V^*)$, $Rank(A) \geq 4$, then there does not exist $B \in S^2(V^*)$ such that $R_A^\Lambda = \pm R_B^S$ [9, 10]. Thus, for these equations to hold, R_{B^3} must be the same build as R_{BDB}, which implies that R_B is the same build as R_D. Similarly, R_{C^3} must be the same build as R_{CAC}, which implies that R_C is the same build as R_A.

For part 2 and 3, consider a result of Gilkey's [7]:

1. If $A = A^*$, $Rank(A) \geq 3$, and $R_A^S = R_B^S$, then $A = \pm B$.
2. if $A = -A^*$ and $R_A^\Lambda = R_B^\Lambda$, then $A = \pm B$.

If R_A and R_C are anti-symmetrically built, then $R_{C^3}^\Lambda = R_{CAC}^\Lambda$ imply that $C^3 = \pm CAC$. If R_B and R_D are anti-symmetrically built, then $R_{B^3}^\Lambda = R_{BDB}^\Lambda$ implies that $B^3 = \pm BDB$.

If R_A and R_C are symmetrically built, then we need the rank assumption, that $Rank(C) \geq 3$. Then $Rank(C^3) \geq 3$ and so $C^3 = \pm CAC$. If R_B and R_D are symmetrically built and $Rank(B) \geq 3$, then $Rank(B^3) \geq 3$. Thus we conclude that $B^3 = \pm BDB$.

Now by Diaz, Dunn [1] and Treadway [10], we have the two results:

1. Let $A = A^*$ and $Rank(A) \geq 3$. Then there does not exist B such that $R_A^S = -R_B^S$.
2. Let $A = -A^*$, $A \neq 0$, and $Rank(A) \geq 4$. Then there does not exist B, such that $R_A^\Lambda = -R_B^\Lambda$.

We apply these results to the sign choice $\epsilon_1, \epsilon_2, \epsilon_3$. To meet the rank requirements, assume $Rank(A), Rank(C) \geq 3$ if symmetrically built, and so $Rank(A^3), Rank(C^3) \geq 3$. If R_A, R_C are symmetrically built and assume $Rank(A), Rank(C) \geq 4$ if anti-symmetrically built, and so $Rank(A^3), Rank(C^3) \geq 3$. For Equations 17 and 19 to hold, $\epsilon_2 = -1$ and $\epsilon_1 = -\epsilon_3$.

□

16 Bounds on $\nu(R)$ and $\eta(R)$

We develop a method for reducing the number of terms in a sum of canonical curvature tensors, given that at least one term has an operator with a nontrivial kernel.

Theorem 16.1. *Let $R = \epsilon R_\tau + \sum^k \epsilon_i R_{B_i}$, where $ker(\tau) \neq 0$. Then, for $A : V \to ker(\tau)$, $\bar{R} = A^* R = \sum^k \epsilon_i R_{A^* B_i A}$. Moreover, $R_{A^* B_i A} \in \mathcal{A}(V)$, for $B_i = B_i^*$ or $B_i = -B_i^*$.*

Proof. Consider $R = \epsilon R_\tau + \sum^k \epsilon_i R_{B_i}$, where $ker(\tau) \neq 0$. Let $A : V \to ker(\tau)$. Then,

$$A^* R = \epsilon A^* R_\tau + \sum^k \epsilon_i A^* R_{B_i} = \sum^k \epsilon_i R_{A^* B_i A},$$

by Lemmas 15.1 and 4.1.

Now we prove that $R_{A^* B_i A}$ is still an algebraic curvature tensor of the same type. We have that $B^* = \epsilon B$, for $\epsilon = \pm 1$. Then $(A^* B A)^* = A^* B^* A = \epsilon A^* B A$. Thus for both builds, $R_{A^* B_i A} \in \mathcal{A}(V)$ and remains the same build as R_{B_i}.

□

If the curvature tensors are all of the same build, then this gives a method for reducing $\eta(R)$ or $\nu(R)$.

Theorem 16.2. *Consider $R_\psi = \epsilon R_\gamma + \sum^k \epsilon_i R_{\gamma_i}$, where $ker(\gamma) \neq 0$. If $A : V \to ker(\gamma)$ and $A^* \psi = \pm \psi$, then $R_\psi = \sum^k \epsilon_i R_{A^* \gamma_i A}$. Moreover, $R_{A^* \gamma_i A} \in \mathcal{A}(V)$, for both $\gamma_i = \gamma_i^*$ and $\gamma_i = -\gamma_i^*$.*

Proof. Consider $R_\psi = \epsilon R_\gamma + \sum^k \epsilon_i R_{\gamma_i}$, where $ker(\gamma) \neq 0$. Let $A : V \to ker(\gamma)$, such that $A^* \psi = \pm \psi$. Then

$$R_\psi = A^* R_\psi = \epsilon A^* R_\gamma + \sum^k \epsilon_i A^* R_{\gamma_i} = \sum^k \epsilon_i R_{A^* \gamma_i A},$$

by Lemmas 15.1 and 4.1.

For either build, $\gamma_i^* = \epsilon \gamma_i$, where $\epsilon = \pm 1$. Then $(A^* \gamma_i A)^* = A^* \gamma_i^* A = \epsilon A^* \gamma_i A$. Thus, for both builds, $R_{A^* \gamma_i A} \in \mathcal{A}(V)$ and remains the same build as R_{γ_i}.

□

Remark *In order for A to preserve ψ, A is of the form $A = \begin{bmatrix} \tilde{A} & 0 \\ B & \tilde{B} \end{bmatrix}$, such that $\tilde{B}$ is invertible. Then ψ is preserved because the block next to $\tilde{A}$ is 0. In order for A to map to the kernel of τ, let $\tilde{A}$ be the kernel of τ. Thus we have constructed such an A.*

This motivates a relationship between $\nu(R)$ and $\eta(R)$. In particular, if we have an anti-symmetric build curvature tensor expressed as a sum of symmetric build curvature tensors, then the theorem gives a method for reducing $\nu(R_\psi^\Lambda)$ and for the opposite case, a method for reducing $\eta(R_\psi^S)$. If the sums are combinations of both types of tensors, these theorems motivate the definition of a new number, $\mu(R)$, and how to possibly find better bounds on it.

Definition Let $\mu(R) = min\{k | R = \sum^k R_A$, where $A \in S^2$ or $A \in \Lambda^2\}$.

Clearly $\mu(R) \leq min\{\nu(R), \eta(R)\}$. Then, since $\nu(2) = 1$, then $\mu(2) = 1$. Moreover,

$$\mu(n) \leq min\{\nu(n), \eta(n)\}$$

.

17 Conjectures and Questions

1. How strict is the bound $\mu(R)$? This question is essentially asking if the linearly dependent sets need fewer elements when they are of the same type of build. My conjecture is that $\mu(R) = min\{\nu(R), \eta(R)\}$.
2. Consider other chain complexes, or even exact sequences. For example, the chain complex with four terms yields many equations relating just two of the curvature tensors, or even one that is equal to zero (of the form R_{ABC}, (because maybe you have to feed it A and C in order to kill off the other terms). However, for most of these equations, the "curvature tensor"

is most likely no longer a curvature tensor, because the composition does not preserve the self-adjoint or skew-adjoint property of the operator. For example, $R_B^S(Cx, Cy, Az, Aw) = R_{A^*BC}^S(x, y, z, w)$. This would be a curvature tensor if and only if $(A^*BC)^* = A^*BC$. What can be said further about these equations? Can one necessitate that it be a curvature tensor, and then that (ABC) be self or skew adjoint (so $A^*BC = C^*B^*A$)? Moreover, a lot could be done if we had some result regarding if a four tensor is 0 for all x, y, z, w. If we have this, what can we say about the operator? For example, $R_A(x, y, z, w) = 0$, if it is not a curvature tensor, implies what about A? Is there a limit on its rank? Can one generalize Gilkey's result that says that if R is a canonical curvature tensor equal to zero, then its rank is less than or equal to one for either build? If so, then one may consider more complicated chain complexes that require precomposing with combinations of operators, rather than just one (i.e., $R_A(Bx, By, Cz, Cw)$).

3. How can the chain complex approach be generalized? If the chain complexes are disjoint, is there a way to piece them together?
4. Can the results of the chain complexes be taken backwards? i.e., given the linear dependence and some results about the operators, can we construct a chain complex? Will it ever be a unique construction? It seems more likely to be the case if we have exact sequences, rather than chain complexes. For example consider my result with three curvature tensors, and A, C invertible and $B = 0$. Then if all the operators are arranged in a sequence, they must be arranged in A to B to C or C to B to A. What happens if we relax some of the constraints? or consider other construction?
5. I conjecture that a sum of three anti-symmetric curvature tensors with one symmetric is dependent (based off the estimates that I got). I also conjecture that the three of anti-symmetric will not have to be simultaneously block-diagonalized. Moreover, in dimension $V = 4$, I proved that the set is linearly dependent. What about in higher dimensions? What dimension does it switch to becoming independent? A new method of approach needs to be discovered though, because trying to approach this using similar methods to mine becomes too complicated–to many equations!
6. Can better bounds on $\eta(R)$ be determined by writing an arbitrary anti-symmetrically built canonical curvature tensor as a sum of symmetrically built canonical curvature tensor. Then, finding an upper bound on this number k, we can apply the bound on $\nu(R)$. The real question then becomes finding a bound on k.
7. Consider various relations between the kernels/ranks of the operators. What can be said about the operators or dependence if all the operators have the same size rank? For example in the case of a set of two symmetrically built and two anti-symmetrically built operators, I proved that the set is linearly dependent in $dimV = 4$. If all the operators have rank equal to 4, then this also holds in $dimV = n$. Moreover, can the equations vs. unknowns results be specified more closely by factoring in the kernel of each of the operators?

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