RECOVERING LINK DATA FROM THE B-GRAPH OF AN ALTERNATING DIAGRAM

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ABSTRACT. In this paper we study ways in which link data may be recovered from the B-graph G of an alternating link diagram D of link L. In particular we find explicitly the coefficients of the three-variable backet (Kauffman square bracket) of D in terms of the Tutte polynomial of G. We go on to show how both polynomials capture the twist number of alternating 2-bridge and 3-braid link diagrams. Finally, we discuss the open questions of recovering the signature of L (when L is a knot) or the first homology group of the double cover of S^3 branched over L from G.

1. INTRODUCTION

1.1. Basic Definitions.

For the purposes of this paper, a *knot* is the image of a smooth embedding of S^1 into S^3. A k-component *link* is the image of a smooth embedding of $\sqcup_k S^1$ into S^3. Note that a knot is a 1-component link. Two links are said to be equivalent if there exists an isotopy of S^3 taking one to the other.

A *diagram* of a link is a projection of the link onto an equatorial copy of S^2 of S^3 so that the projection is everywhere locally homeomorphic to $\mathbb{R}$ or to $'+'$ (i.e. each crossing is between two strands). Where the projection is homeomorphic to $'+'$, we say there is a *crossing*. We partially erase the strand of the crossing which is further south so that we may visualize the knot in S^3, as in Figure 1. The partially erased strand is said to be the undercrossing strand, while the solid strand is said to be the overcrossing strand.

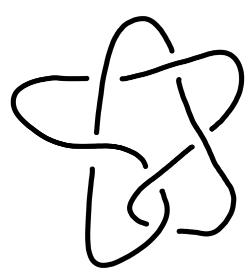

FIGURE 1. An alternating diagram of a 1-component link (i.e. a knot).

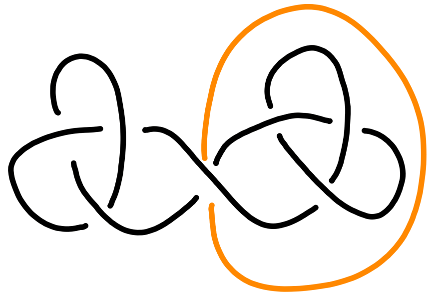

Figure 2. A knot diagram (black) with a nugatory crossing. Shown is a simple closed curve (orange) intersecting the diagram exactly at the nugatory crossing, with D intersecting both components of S^2 excluding the curve.

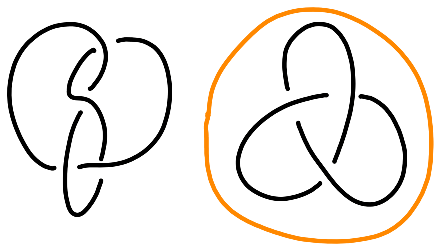

Figure 3. A split link (black). Shown is a simple closed curve (orange) not intersecting the diagram which divides the diagram into two parts.

A diagram is said to be *alternating* if, when tracing the diagram, once comes upon crossings alternatingly on undercrossing and overcrossing strands, as in Figure 1. A link is said to be alternating if it has an alternating diagram.

Given a link diagram, a crossing is said to be *nugatory* if there exists a simple closed curve γ in S^2 so that D intersects γ only at the nugatory crossing and D intersects both components of $S^2 - \gamma$ (see Fig. 2).

Finally, a link L is said to be *split* if for some diagram D of L, there exists a simple closed curve γ in S^2 so that D does not intersect γ and D intersects both components of $S^2 - \gamma$ (see Fig. 3). Clearly a knot cannot be split.

We study only alternating links in this paper.

1.2. **Associating a planar graph with an alternating link.**

We describe a procedure to give a 1-1 correspondence between alternating link diagrams and planar embedded graphs (up to planar isotopy). Given a diagram, we may eliminate a crossing by performing a *smoothing*, in which we erase a small neighborhood of the crossing and pairwise connect the four remaining ends, as in Figure 4. Up to isotopy, there are two choices of smoothings at a crossing, called A-smoothing and B-smoothing (again, see Fig. 4).

A *state* s of a link diagram is a choice of smoothing at every crossing, resulting in $|s|$ disjoint simple closed curves, as in Figure 5.

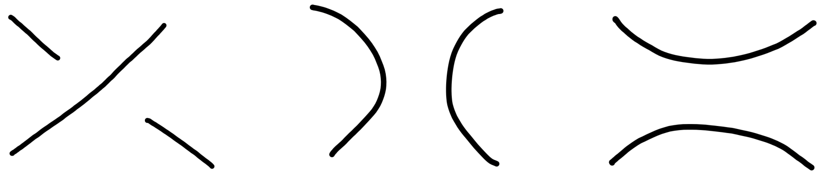

FIGURE 4. Left to right: A crossing, the A-smoothing of that crossing, the B-smoothing of the crossing

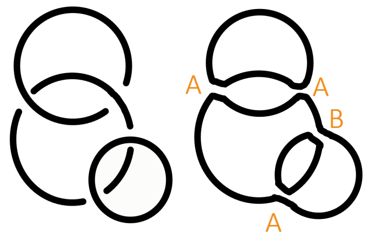

FIGURE 5. Left: a 3-component link. Right: a state of the link.

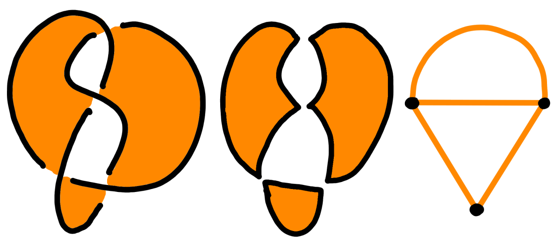

FIGURE 6. Left to right: An alternating link (checkerboard shaded), its all-B-smoothing state, its B-graph.

Given a link diagram D, the B-*graph* G of the diagram is defined as follows: for each loop in the all-B-smoothing state, G has one corresponding vertex. For each crossing in D, two (not necessarily distinct) loops in s meet. G has one edge between the vertices corresponding to these loops. If D is an alternating non-split link, then the loops in s will enclose either the black or white regions in S^2 checkerboard shaded with respect to s, as in Figure 6. This yields a natural embedding of G in the plane; moreover the diagram (naturally viewed as a 4-valent graph) is clearly the medial graph of G. If D is alternating and split, then G is the disjoint union of the B-graphs of the split components.

Given a planar embedded graph G, there exists a unique link diagram (up to planar isotopy) which

- is the medial graph of G,
- is alternating,
- has all-B-smoothing state with loops corresponding to faces enclosing vertices of G.

And of course, G is the B-graph of this diagram. This gives the desired 1-1 correspondence between planar embedded graphs and link diagrams.

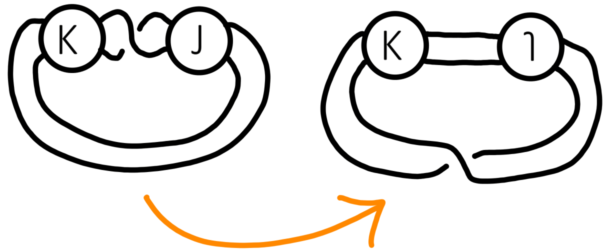

FIGURE 7. Two diagrams of the same link, related by a flype. The flype moves a crossing of two strands betwen tangles J, K to the other two strands between J, K while flipping J.

1.3. **The three-variable bracket of a link diagram.**

We give a terse definition of the *three-variable bracket* $\langle D\rangle(A, B, d)$ of a link diagram D (for more information see e.g. [Ka],[L]). Let S be the set of all states of D, and for each $s \in S$ let $a(s)$ be the number of A-smoothings in s and $b(s)$ the number of B-smoothings. Then

$$\langle D\rangle = \sum_{s\in S} A^{a(s)} B^{b(s)} d^{|s|-1}.$$

This is *not* a link invariant, as three-variable brackets of two different diagrams of a link may not agree. However, it is invariant under a flype (illustrated in Fig. 7): it is easy to see that if s, s' are states of the two diagrams in Figure 7 which agree in K, J and both have A-smoothings or both have B-smoothings at the extra crossing, then $a(s) = a(s'), b(s) = b(s'), |s| = |s'|$.

Tait's Flyping theorem [Me], whose proof is beyond the scope of this paper, says that any two reduced, alternating diagrams of the same link are related by a sequence of flypes. Therefore, the three-variable bracket is an invariant of reduced, alternating links. If L is an alternating link, we may write $\langle L\rangle$ to denote $\langle D\rangle$, where D is any reduced, alternating diagram of L.

1.4. **The Tutte polynomial of a graph.**

We give a similarly brief definition of the *Tutte polynomial* of a connected graph $G = (V, E)$. Let $\mathcal{T}$ be the set of all maximal subtrees of G. Label the edges of G $1, 2, \ldots, |E|$, and let $l(e)$ denote the label of edge e. For any $T \in \mathcal{T}$, edge e is said to be

- *internally active* with respect to T if $e \in T$ and $l(e) \leq l(f)$ for all f with endpoints in both components of $T - e$,
- *externally active* with respect to T if $e \notin T$ and $l(e) \leq l(f)$ for all f in the unique curcuit of $T + e$,
- *inactive* otherwise.

See Figure 8 to avoid confusion.

For $T \in \mathcal{T}$, define $i(T)$ to be the number of edges which are internally active with respect to T and $e(T)$ to be the number of edges which are externally active with respect to T. Then the Tutte Polynomial of G is

$$T[G](x, y) = \sum_{T\in\mathcal{T}} x^{i(T)} y^{e(T)}.$$

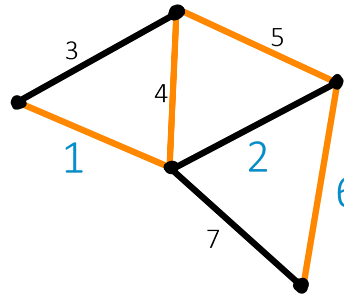

FIGURE 8. A planar graph with edges labeled $1, 2, \ldots, 7$, with maximal subtree T consisting of the orange edges. Edges $1, 6$ are internally active with respect to T while edge 2 is externally active with respect to T. Edges $3, 4, 5, 7$ are inactive with respect to T.

If G is the disjoint union of graphs G_1, G_2, we say $T[G](x,y) = T[G_1](x,y) \cdot T[G_2](x,y)$. We do not prove that the Tutte polynomial is independent of the original choice of edge labelings; see e.g. [Bo],[Ka] for a more thorough definition of this polynomial.

2. Relation between the three-variable bracket and Tutte polynomial

2.1. Expressing the coefficients of the three-variable bracket explicitly with the Tutte polynomial.

Theorem 2.1. *Let D be a reduced, alternating diagram of non-split link L, with B-graph $G = (V, E)$. Then*

$$\langle L\rangle(A, B, d) = \sum_{k=0}^{\infty} d^k \left(\sum_{i=0}^{k} \frac{A^{|V|-1-2i+k} B^{|E|-|V|+1+2i-k}}{i!(k-i)!} \frac{\partial^k T[G](1,1)}{\partial x^i \partial y^{k-i}} \right).$$

We need some new notation: given an (embedded) planar graph $H = (V_H, E_H)$, let $k(H)$ be the number of connected components of H. Let $f(H) = |E_H| - |V_H| + 1 + k(H)$ be the number of faces of H (i.e. the number of regions into which H divides the plane).

Lemma 2.2. *To each state s of D, associate the spanning (i.e. including every vertex) subgraph $G_s = (E_s, V)$ of G where E_s consists of exactly the edges corresponding to A-smoothings of s. Then $|s| = f(G_s) + k(G_s) - 1$.*

Proof of Lemma 2.2. Let s_B be the all-B-smoothing state of D. Then $f(G_{s_B}) = 1, k(G_{s_B}) = |V|$, so $f(G_{s_B}) + k(G_{s_B}) - 1 = |V| = |s_B|$ as desired. We proceed inductively. Suppose the claim holds for a state s and that s' differs from s only at crossing c, where s has a B-smoothing and s' an A-smoothing. Let $e \in E$ be the edge corresponding to c; then e is not in G_s and $G_{s'} = G_s + e$ (where of course by $G_s + e$ we mean $(E_s + e, V)$).

If two distinct loops of s meet at c, then these two loops are not distinct in s' and e has endpoints in distinct components of G_s. Then

$$f(G_{s'}) + k(G_{s'}) - 1 = f(G_s) + (k(G_s) - 1) - 1 = |s| - 1 = |s'|,$$

as desired. If in s a loop meets itself at c, then distinct loops in s' meet at c and e has endpoints in once component of G_s. Then

$$f(G_{s'}) + k(G_{s'}) - 1 = (f(G_s) + 1) + k(G_s) - 1 = |s| + 1 = |s'|,$$

as desired. □

Let $\langle D\rangle^j_k$ refer to the coefficient of $A^j B^{c(D)-j} d^k$ in $\langle D\rangle$. Referring to the definition of the three-variable bracket, we see that $\langle D\rangle^j_k$ is the number of states s of D so that $a(s) = j$ and $|s| = k + 1$. Associating a state s to a spanning subgraph G_s of G as in Lemma 2.2, we find $\langle D\rangle^j_k$ is therefore the number of spanning subgraphs of G that have j edges, i components, and $k + 2 - i$ faces for some i.

In fact, i is determined by j and k: any such subgraph can be formed by removing $i - 1$ edges E_- from some maximal subtree T of G and then adding $k + 1 - i$ edges, each of whom have endpoints in only one component of $T - E_-$. Therefore, $j = |V| - 2i + k + 1$. In particular, when $|V| + k + 1 - j$ is odd, then $\langle D\rangle^j_k = 0$. We will show now that when $|V| + k + 1 - j$ is even, then

$$\langle D\rangle^j_k = \sum_{T\in\mathcal{T}} \binom{i(T)}{i-1}\binom{e(T)}{k+1-i},$$

by showing that given a labling of E, a spanning subgraph with j edges, i components, and $k + 2 - i$ faces can be formed *uniquely* by removing $i - 1$ internally active and adding $k + 1 - i$ externally active edges to some maximal subtree of G.

For the next two lemmas, we assume that the edges of G are labeled $1, 2, \ldots, |E|$. We denote the labeling of edge e by $l(e)$.

Lemma 2.3. *Let G' be a spanning subgraph of G that has j edges, i components, and $k + 2 - i$ faces. Then there exists a maximal subtree T of G, edges $e_1, \ldots, e_{i-1}$ which are internally active with respect to T, edges $f_1, \ldots, f_{k+1-i}$ which are externally active with respect to T, so that $T - e_1 - \cdots - e_{i-1} + f_1 + \cdots + f_{k+1-i} = G'$.*

Proof of Lemma 2.3. If $i > 1$, let e_1 be the edge between distinct components of G' with least labeling. For $m = 2, \ldots, i - 1$, let e_m be the edge between distinct components of $G' + e_1 + \cdots + e_{m-1}$ of least labeling. If $i < k + 1$, let f_1 be the edge in a cycle of G' with least labeling. For $m = 2, \ldots, k - i + 1$, let f_m be the edge in a cycle of $G' - f_1 - \cdots - f_{m-1}$ of least labeling. Then $G' + e_1 + \cdots + e_{i-1} - f_1 - \cdots - f_{k-i+1}$ is a maximal subtree of G with respect to which each e_m is internally active and each f_m is externally active. □

Lemma 2.4. *Let G' be a spanning subgraph of G that has j edges, i components, and $k + 2 - i$ faces. Then there exists at most one maximal subtree T of G so that there exist edges $e_1, \ldots, e_n$ which are internally active with respect to T, edges $f_1, \ldots, f_m$ which are externally active with respect to T, so that $T - e_1 - \cdots - e_n + f_1 + \cdots + f_m = G'$.*

Proof of Lemma 2.4. By lemma 2.3, there exists some maximal subtree T of G, internally active edges $e_1, \dots, e_{i-1}$, externally active edges $f_1, \dots, f_{k+1-i}$ so that $T - e_1 - \dots - e_{i-1} + f_1 + \dots + f_{k+1-i} - G'$. Choose e_a, f_a as in lemma treeexists.

Suppose T' is a maximal subtree of G so that for some $e'_1, \dots, e'_n$ which are internally active with respect to H and some $f'_1, \dots, f'_m$ which are externally active with respect to T', $G' = T' - e'_1, \dots - e'_n + f'_1 + \dots + f'_m$. If $n == 0$, take $\{e'_i\} = \emptyset$; similarly if $m = 0$ take $\{f'_i\} = \emptyset$. We will show that $T' = T, \{e'_i\} = \{e_i\}, \{f'_i\} = \{f_i\}$.

Note that since the e'_i, f'_i are all active, each f'_i must have endpoints in one component of $T' - e'_1 - \dots - e'_n$. Therefore, since $k(G') = i, f(G') = k + 2 - i$, we have $n = i - 1, m = k - i + 1$ as desired.

Each e'_a has endpoints in distinct components of G', so $l(e'_i) \geq l(e_1)$ for all a. But e_1 has endpoints in distinct components of $H - e'_a$ for some a; since e'_a is internally active we find $e_1 = e'_a$. Without loss of generality, take $e_1 = e'_1$. Then for $i > 1$, e'_i has endpoints in distinct components of $G' + e_1$, so $l(e'_i) \geq l(e_2)$. Similarly, since each e'_i is internally active we find $e_2 = e'_a$ for some $a \geq 2$; without loss of generality we take $a = 2$. Continue inductively to find $\{e_1, \dots, e_n\} = \{e'_1, \dots, e'_n\}$.

Each f'_a is included in a cycle of G', so $l(f'_a) \geq l(f_1)$ for all i. Every cycle of G' includes some f'_a, so in particular for some a, f'_a is included in a cycle containing f_1. Then by the definition of externally active, $l(f'_a) \leq l(f_1)$, implying $l(f'_a) = l(f_1)$, so $f'_a = f_1$. Without loss of generality take $f_1 = f'_1$. Now for $a \geq 2$, each f'_a is included in a cycle of $G' - f_1$, so $l(f'_a) \geq n(f_2)$. Again, every cycle of $G' - f_1$ includes some f'_a with $a \geq 2$, so in particular for some $a \geq 2$, f'_a is included in a cycle containing f_2. Again by definition of externally active, $l(f'_a) \leq l(f_2)$, implying $l(f'_a) = l(f_2)$, so $f'_a = f_2$. Without loss of generality take $f_2 = f'_2$. Continue inductively to find $\{f_1, \dots, f_{k-a+1}\} = \{f'_1, \dots, f'_{k-a+1}\}$. $\square$

We continue to our main proof.

Proof of Theorem 2.1. Let G' be a spanning subgraph of G which has j edges, i components, and $k - i + 2$ faces. As previously mentioned, if $|V| + k + 1 - j$ is odd then no such subgraph exists.

With lemmas 2.3 and 2.4 we have shown that G' can be formed *uniquely* by removing $i - 1$ internally active and adding $k - i + 1$ externally active edges to some maximal subtree of G. Moreover, as a scholium of Lemma 2.4, removing $i - 1$ internally active and adding $k - i + 1$ externally active edges to/from any maximal subtree of G will result in a spanning subgraph of G which has i components and $k - i + 2$ faces (and thus necessarily j edges). Therefore,

$$\begin{aligned}\langle D \rangle_k^j &= \sum_{H \in \mathcal{T}} \binom{i(H)}{i-1} \binom{e(H)}{k-i+1} \\ &= \frac{1}{(i-1)!(k-i+1)!} \frac{\partial^k T[G](1,1)}{\partial x^{i-1} y^{k-i+1}}.\end{aligned}$$

Thus,

$$\begin{aligned}\langle L\rangle(A,B,d) &= \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\langle D\rangle_k^j A^j B^{c(D)-j} d^k \\ &= \sum_{k=0}^{\infty}\sum_{i=1}^{k+1}\left(\frac{1}{(i-1)!(k-i+1)!}\frac{\partial^k T[G](1,1)}{\partial x^{i-1} y^{k-i+1}}\right) A^{|V|+1-2i+k} B^{|E|-|V|-1+2i-k} d^k \\ &= \sum_{k=0}^{\infty} d^k \left(\sum_{i=0}^{k}\frac{A^{|V|-1-2i+k} B^{|E|-|V|+1+2i-k}}{i!(k-i)!}\frac{\partial^k T[G](1,1)}{\partial x^i y^{k-i}}\right).\end{aligned}$$

□

Remark 2.5. Given a split alternating link L, call the split components $L_1, \ldots, L_n$. Then $\langle L\rangle(A,B,d) = \prod_{i=1}^n \langle L_i\rangle(A,B,d)$, so we may apply theorem 2.1 to find $\langle L\rangle(A,B,d)$ in terms of the Tutte polynomials of the B-graphs of its split components (i.e. the connected components of the B-graph of L.)

2.2. Taylor series and non-split links.

Given a reduced, alternating, non-split link L whose B-graph for some diagram is $G = (V,E)$,

$$\begin{aligned}\langle L\rangle(A,B,d) &= \sum_{k=0}^{\infty} d^k \left(\sum_{i=0}^{k}\frac{A^{|V|+k-2i-1} B^{|E|-|V|-k+2i+1}}{i!\cdot(k-i)!}\frac{\partial^k T[G](1,1)}{\partial x^i \partial y^{k-i}}\right) \\ &= A^{|V|-1}B^{|E|-|V|+1}\sum_{k=0}^{\infty}\left(\sum_{i=0}^{k}\frac{(dA^{-1}B)^i(dAB^{-1})^{k-i}}{i!\cdot(k-i)!}\frac{\partial^k T[G](1,1)}{\partial x^i \partial y^{k-i}}\right) \\ &= A^{|V|-1}B^{|E|-|V|+1}T[G](dA^{-1}B+1, dAB^{-1}+1).\end{aligned}$$

Moreover, if L is split with split components $L_1, \ldots, L_n$ whose B-graphs are $G_1 = (V_1, E_1), \ldots, G_n = (V_n, E_n)$, then

$$\begin{aligned}\langle L\rangle(A,B,d) &= \prod_{i=1}^{n}\langle L_i\rangle(A,B,d) \\ &= \prod_{i=1}^{n}\left(A^{|V_n|-1}B^{|E_n|-|V_n|+1}T[G_n](dA^{-1}B+1, dAB^{-1}+1)\right) \\ &= A^{|V|-k(G)}B^{|E|-|V|+k(G)}T[G](dA^{-1}B+1, dAB^{-1}+1).\end{aligned}$$

3. Chromatic and Flow polynomials

We have shown explicitly how to obtain $\langle L\rangle$ from $T[G](x,y)$, where L is an alternating link and G its B-graph. Much information about G is captured in $T[G](x,y)$, so is therefore captured in $\langle L\rangle$. We now investigate what information a specialization of $T[G](x,y)$ captures information about L, understanding that all such information is then contained in $\langle L\rangle$.

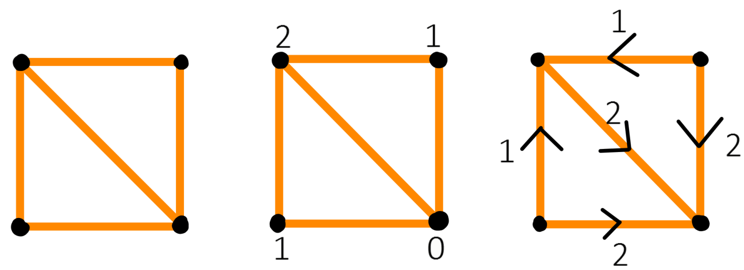

FIGURE 9. Left to right: a graph with chromatic polynomial $x(x-1)(x-2)^2$ and flow polynomial $(x-1)(x-2)$, a 3-coloring of the graph, a nowhere-zero 3-flow on the graph.

3.1. **Definitions.**

Given a graph $G = (V, E)$, an n-coloring of G is a map $f : V \to \{0, \ldots, n-1\}$ so that for any $e \in E$ with endpoints v_1, v_2, $f(v_1) \neq f(v_2)$. The *chromatic polynomial* $p_G(x)$ of G is a polynomial in $\mathbb{Z}[x]$ so that for any $n \in \mathbb{Z}^+$, $p_G(n)$ is the number of n-colorings of G (see Fig. 9 for a small example). We do not give a proof of the existence of or the general method of computing $p_G(x)$, see instead [Bo] for a deeper exposition. However, $p_G(x)$ is a specialization of $T[G](x, y)$ along the axis $y = 0$, with $p_G(x) = (-1)^{|V|-k(G)}x^{k(G)}T[G](1-x, 0)$.

An n-flow on G is a map $f : E \to \{0, \ldots, n-1\}$ so that for some choice of orientations of each edge in E and for all $v \in V$, the sum of labelings of each edge directed into v is equal to the sum of labelings of each edge directed out of v mod n (see Fig. 9). A nowhere-zero n-flow is an n-flow in which $f(e) \neq 0$ for all $e \in E$. The *flow polynomial* $q_G(y)$ of G is a polynomial in $\mathbb{Z}[y]$ so that for any $n \in \mathbb{Z}^+$, $q_G(n)$ is the number of nowhere-zero n-flows on G. Again we refer the reader to [Bo] for a deeper exposition, and simply accept that the flow polynomial is a specialization of the Tutte polynomial along the axis $x = 0$, with $q_G(y) = (-1)^{|E|-|V|+k(G)}T[G](0, 1-y)$.

Thus, when G is the B-graph of a reduced alternating link L,

$$\begin{aligned}
p_G(x) &= (-1)^{|V|-k(G)}x^{k(G)}T[G](1-x, 0) \\
&= (-1)^{|V|-k(G)}x^{k(G)}A^{-|V|+k(G)}(Ax^{1/2})^{-|E|+|V|-k(G)}\langle L\rangle(A, Ax^{1/2}, -x^{1/2}) \\
&= (-1)^{|V|-k(G)}x^{(-|E|+|V|+k(G))/2}\langle L\rangle(1, x^{1/2}, -x^{1/2}) \qquad (1)\\
q_G(y) &= (-1)^{|E|-|V|+k(G)}T[G](0, 1-y) \\
&= (-1)^{|E|-|V|+k(G)}A^{-|V|+k(G)}(Ax^{-1/2})^{-|E|+|V|-k(G)}\langle L\rangle(A, Ax^{-1/2}, -x^{1/2}) \\
&= (-1)^{|E|-|V|+k(G)}x^{(|E|-|V|+k(G)/2}\langle L\rangle(1, x^{-1/2}, -x^{1/2}).
\end{aligned}$$

We may denote $p_G(x), q_G(y)$ by $p_L(x), q_L(y)$ respectively when L is alternating.

Remark 3.1. Given a connected planar embedded graph $G = (V, E)$, label the edges of G $1, 2, \ldots, |E|$. This naturally induces a labeling of E^*, where $G^* = (V^*, E^*)$. Maximal subtrees of G are in 1-1 correspondence with maximal subtrees of G^*, where $T \in G$ corresponds to the maximal subtree $T^*$of G^* including exactly the edges not corresponding to those in T. For $e \in T$, the edges in G between the components of $T - e$ correspond exactly the edges in G^* forming the circuit in $T^* + e^*$, where e^* is the edge corresponding to e. That is, an edge of G is internally

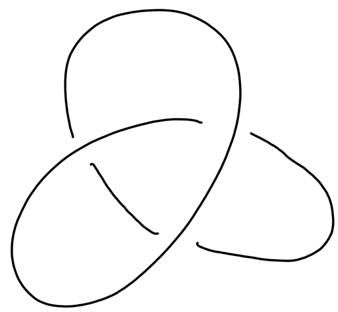

FIGURE 10. A diagram of the unknot whose three-variable polynomial does not agree with any chromatic polynomial as prescribed by Equation (1).

active with respect to T if and only if its corresponding edge in G^* is externally active with respect to T^*. Therefore,

$$T[G](x,y) = T[G^*](y,x).$$

Thus, we find a relationship between the chromatic and flow polynomials of G:

$$\begin{aligned} p_G(x) &= (-1)^{|V|-1} x^{k(G)} T[G](1-x, 0) \\ &= (-1)^{|V|-k(G)} x^{k(G)} T[G^*](0, 1-x) \\ &= (-1)^{|V|-k(G)-|E^*|+|V^*|-k(G^*)} x q_{G^*}(x) \\ &= x q_{G^*}(x). \end{aligned}$$

Rather than studying the chromatic and flow polynomial of a graph G, we will usually study the chromatic polynomials of G and G^*.

Remark 3.2. Note that the relation between chromatic polynomial and three-variable bracket in Equation (1) cannot be easily adapted to apply to non-alternating links; we provide an example (Fig. 10) for which the right-hand expression does not yield a valid chromatic polynomial for any graph. The three variable polynomial for the diagram of the unknot in Figure 10 is $\langle D\rangle^3(A,B,d) = A^3 + 3a^2Bd + AB^2d^2 + 2AB^2 + B^3d$, so $\langle D\rangle^3(1, x^{1/2}, -x^{1/2}) = 1 + x$, so the above expression would yield $p_G(x) = \pm(x^n + x^{n-1})$ for some x. Since $p_G(x) \neq x^m$ for any m, G must include some edge; but this implies $(x-1)|(x^n + x^{n-1})$, which does not hold. Therefore, no such graph G exists.

Remark 3.3. Let $G = (V,E)$ be a multigraph with no loops and $\tilde{G} = (V, \tilde{E})$ the underlying simple graph (i.e. parallel edges replaced with one edge). Then two vertices in G are connected if and only if they are connected in $\tilde{G}$. Therefore, $f : V \to \{0, 1, \ldots, n-1\}$ is an n-coloring of G if and only if it is an n-coloring of $\tilde{G}$, so $p_G(x) = p_{\tilde{G}}(x)$.

3.2. **Skein relations for the chromatic and flow polynomials.**

Let L be a non-split link L with B-graph G. Recall (or see [Bo]) $p_G(x)$ satisfies the contraction-deletion relation $p_{G-e}(x) - p_{G/e}(x)$ for any edge e in G. Note that if e does not correspond to a nugatory crossing in L, then $G - e$ is the B-graph for the link obtained by B-smoothing the crossing of L corresponding to e and G/e is the B-graph of the link obtained by

A-smoothing at that crossing. Moreover, $p_\circ(x) = x$, and to avoid ambiguity we say $p_{L\circ}(x) = xp_L(x)$. Therefore, the chromatic polynomial satisfies a nontrivial skein relation:

$$\begin{aligned} p_{\times}(x) &= p_{\asymp}(x) - p_{)(}(x) \text{ if } \times \text{ is non-nugatory} \\ p_{\circlearrowleft}(x) &= (x-1)p_{\smile}(x) \\ p_{\circlearrowright}(x) &= 0 \\ p_{L\circ}(x) &= xp_L(x) \\ p_\circ(x) &= x. \end{aligned}$$

Since $p_L(x)$ is defined only for alternating links L, $p_L(x)$ does not satisfy the HOMFLY or Kauffman skein relations. Moreover, the input of $\langle L\rangle(1, x^{1/2}, -x^{1/2})$ does not coincide with that of the Jones polynomial specialization $\langle L\rangle(A, A^{-1}, -A^2 - A^{-2})$ of the three-variable bracket for any x.

The flow polynomial satisfies a similar relation, coming from the contraction-deletion relation $q_G(y) = q_{G/e}(y) - q_{G-e}(y)$:

$$\begin{aligned} q_{\times}(y) &= -q_{\asymp}(y) + q_{)(}(y) \text{ if } \times \text{ is non-nugatory} \\ q_{\circlearrowleft}(y) &= 0 \\ q_{\circlearrowright}(y) &= (y-1)q_{\smile}(y) \\ q_{L\circ}(y) &= yq_L(x) \\ q_\circ(y) &= 1. \end{aligned}$$

These skein relations illustrate how one may compute $p_L(x), q_L(y)$ directly, rather than first computing $\langle L\rangle$ or $T[G](x, y)$. Because any nugatory crossing in a diagram D implies at least one of $p_D(x), q_D(y)$ is zero, computing $p_L(x)$ or $q_L(y)$ takes at most as many steps at computing $\langle L\rangle$, and may take many fewer.

3.3. **Chromatic polynomial yields twist number of alternating 3-braid link diagrams.**

A 3-braid consists of three twisted strands (i.e. smooth embeddings of $I \sqcup I \sqcup I$) in S^3 so that in the projection onto the equatorial two-sphere, and for some choice of x-y coordinates locally around the diagram, the only minima and maxima in y occur at the ends of strands (see Fig. 11). A 3-braid link is the link formed by pairwise connecting the ends which are maxima in y to the ends which are minima in y (again, see Fig. 11).

Theorem 3.4. *Let D be a reduced, alternating 3-braid link with $2k > 1$ twists, as in Figure 12. Let G_A, G_B be the A- and B-graphs of D. Then $2k = -a_m - b_n - \log_2(\langle D\rangle(1,1,1))$, where $p_{G_A}(x) = a_{m+1}x^{m+1} + a_m x^m + \cdots + a_0, p_{G_B} = b_{n+1}x^{n+1} + b_n x^n + \cdots + b_0$.*

Of course, the A-graph of a link diagram is formed in the same way as the B-graph, but considering the all-A-smoothing state instead of the all-B-smoothing state.

Proof of Theorem 3.4. Figure 12 illustrates the B-graph of an alternating 3-braid link diagram D with $2k > 1$ twists. Taking D to have no nugatory crossings is equivalent to requiring $k \geq 2$ or $n_1, n_2 > 1$.

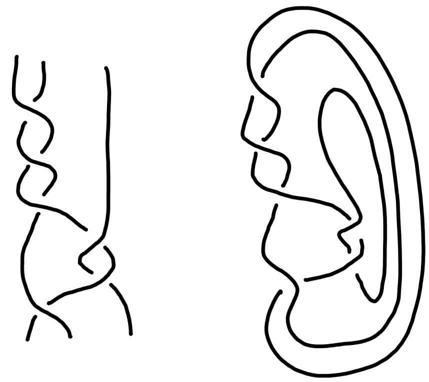

FIGURE 11. Left: A 3-braid. Right: A 3-braid link.

As stated in remark 3.3, a multigraph with no loops has the same chromatic polynomial as its underlying simple graph. Therefore, the B-graph of L has chromatic polynomial equal to that of the right-most graph in Figure 12. Let $n = \sum_i n_{2i}$; note $n > 2$ We calculate the coefficient b_n of x^n in $p_{G_B}(x)$ using the contraction/deletion relation, noting that any graph with n vertices yields a coefficient of 1 and a graph with fewer than n vertices yields a coefficient of 0, and that $\tilde{G}_B$ has $n+1$ vertices. We contract/delete along each edge meeting the central vertex in $\tilde{G}$ (see Fig. 12), obtaining exactly k graphs with n vertices (one corresponding to each choice of exactly one contraction) and one graph with $n+1$ vertices (corresponding to all deletions) that is a cycle of n vertices and an isolated vertex, whose chromatic polynomial has coefficient $-\binom{n}{1} = -n$ of x^n. Thus, the coefficient of x^n is $b_n = -n - k$. Let $m = \sum_i n_{2i-1}$; note $m > 2$. By a similar argument, the coefficient a_m of x^m in $p_{G_A}(x)$ is $a_m = -m - k$. Therefore, $-a_m - b_n = 2k + c(L)$. That is,

$$2k = -a_m - b_n - \log_2(\langle L \rangle(1,1,1)).$$

□

3.4. **Chromatic polynomial yields twist number of alternating 2-bridge link diagrams.** A 2-bridge link is a link with a diagram so that for some choice of x-y coordinates around the diagram, the diagram has exactly two minima and two maxima in the y-direction (see Fig. 13). Every 2-bridge link has a 2-bridge diagram that is alternating and whose twists alternate between the center and left-hand side, as illustrated. We will refer to these diagrams as *regular* 2-bridge diagrams.

Let D be a regular 2-bridge diagram with $k \geq 2$ twists as in Figure **??**. To avoid ambiguity in the case of one-crossing twists, we say that if the B-graph is in the left format (orange) of Figure **??** then the center twists are A-twists; otherwise the left twists are A-twists. Let $\tilde{G}$ be the underlying simple graph of G, the B-graph of D. 14 illustrates all possible forms of $\tilde{G}$. We note that $\tilde{G}$ is a chain of cycles, with each cycle sharing an edge and two vertices with the one below it. Moreover, if the A-twists (top to bottom) have length $m_1, \ldots, m_l$ then the cycles (top to bottom) have length $c_1, \ldots, c_l$, where $c_i = m_i + 2$ except possibly $c_1 = m_1 + 1$ and/or $c_l = m_l + 1$. Let C_r denote a cycle of length r, so

$$p_G(x) = \frac{\prod_{i=1}^{l} p_{C_{c_i}}(x)}{(x(x-1))^{l-1}}. \qquad (2)$$

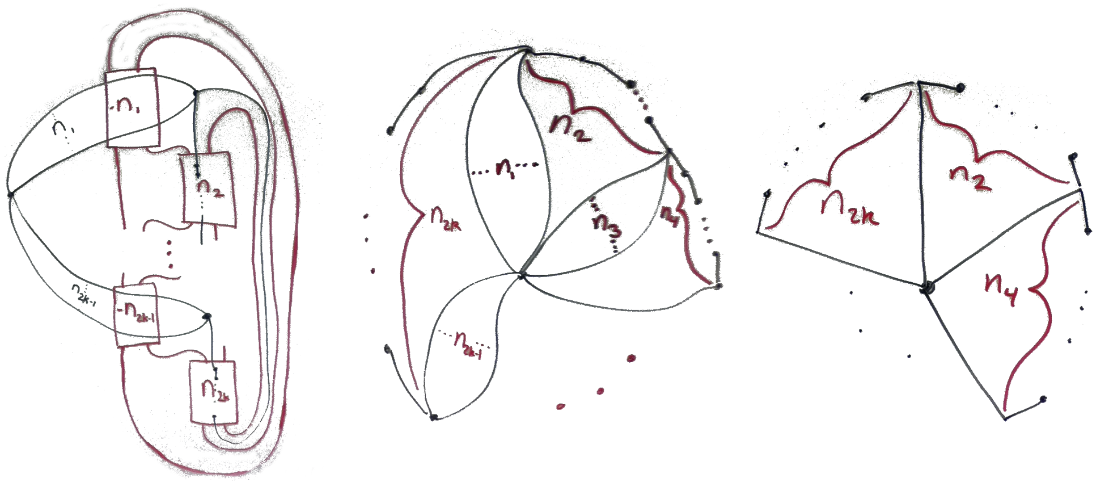

FIGURE 12. Left to right: an alternating 3-braid diagram with $2k > 1$ twists and its B-graph G_B, G_B, the underlying simple graph (multi-edges replaced by a single edge) $\tilde{G}_B$ of G_B.

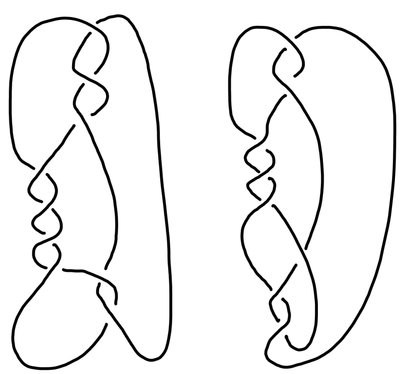

FIGURE 13. Two 2-bridge link diagrams.

It is well known that $p_{C_r} = (x-1)^r + (-1)^r(x-1) = (-1)^r(x-1)(1-(1-x)^{r-1})$; computing this inductively with the contraction/deletion relation is an easy exercise and is omitted. Therefore, from Equation 2,

$$\begin{aligned} p_G(x) &= \frac{\prod_{i=1}^{l} p_{C_{c_i}}(x)}{(x(x-1))^{l-1}} \\ &= \frac{\prod_{i=1}^{l}\left((-1)^{c_i}(x-1)(1-(1-x)^{c_i-1})\right)}{(x(x-1))^{l-1}} \\ &= (-1)^{\sum c_i} x(x-1)\prod_{i=1}^{l}\frac{1-(1-x)^{c_i-1}}{x}. \end{aligned} \tag{3}$$

Remark 3.5. Suppose D is a regular 2-bridge link diagram with more than 1 twists and B-graph G. Then from Equation 3, if $p_G(x)$ *uniquely* factors in $\mathbb{Z}[x]$ as $p_G(x) = \pm x(x-1)\prod_{i=1}^{l}\frac{1-(1-x)^{c_i-1}}{x}$,

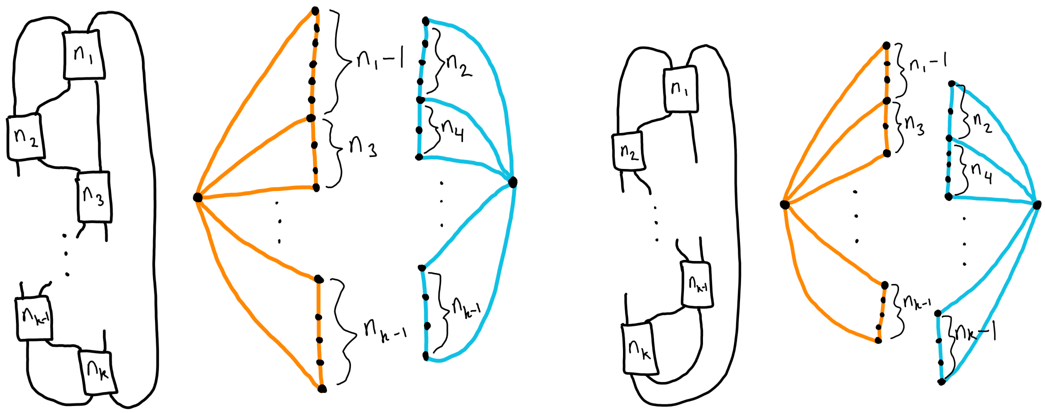

FIGURE 14. A regular two-bridge diagram and underlying simple graphs $\tilde{G}_A, \tilde{G}_B$ of its A- and B-graphs. The diagram has $k \geq 2$ twists; on the left k is odd and on the right k is even.

then we recover l (the number of A-twists in D) and the multiset $\{c_1, \ldots, c_l\}$, where $c_i = n_i + 2$ except possibly $c_1 = n_1 + 1$ and/or $c_l = n_l + 1$. However, it is not obvious that $p_G(x)$ will factor in the form of the right hand side of Equation 3 uniquely. Of course, if $c_i - 1$ is prime then $(1-(1-x)^{c_i-1})/x$ is irreducible in $\mathbb{Z}[x]$ by Eisenstein's criterion at $c_i - 1$.

Theorem 3.6. *Let D be a regular 2-bridge link diagram with more than 1 twist and B-graph G. Suppose $a_1, \ldots, a_m \in \mathbb{Z}^+$ with $p_G(x) = \pm x(x-1)\prod_{i=1}^m \frac{1-(1-x)^{a_i}}{x}$. Then D has m A-twists.*

Proof of Theorem 3.6. Suppose D has l A-twists. By Equation 3, there exist some $c_1, \ldots, c_l$ so that $f(x) = (-1)^{(\sum c_i)+l}\prod_{i=1}^l \frac{1-(1-x)^{c_i}}{x} = (-1)^{(\sum a_i)+m}\prod_{i=1}^l \frac{1-(1-x)^{a_i}}{x}$. The degree d of $f(x)$ is therefore

$$\sum_i^l (c_i - 1) = \sum_i^m (a_i - 1)$$

$$\left(\sum_i^l c_i\right) - l = \left(\sum_i^m a_i\right) - m. \tag{4}$$

Moreover, the coefficient of $d-1$ in $f(x)$ is

$$(-1)^{(\sum c_i)+l}\sum_{i=1}^l \left((-1)^{c_i-1}c_i(-1)^{(\sum_j c_j)-c_i}\right) = (-1)^{(\sum a_i)+m}\sum_{i=1}^m \left((-1)^{a_i-1}a_i(-1)^{(\sum_j a_j)-a_i}\right)$$

$$(-1)^l\sum_{i=1}^l c_i = (-1)^m\sum_{i=1}^m a_i \tag{5}$$

Suppose $l \not\equiv m \pmod 2$. Then $\sum_i c_i = -\sum_i a_i$, so from Equation 4 we find $2\sum_i c_i = l - m$, contradicting $l \not\equiv m \pmod 2$. Therefore, $l \equiv m \pmod 2$. Equation 5 yields $\sum_i c_i = \sum_i a_i$, so from Equation 4 we find $l = m$. Thus, m is the number of A-twists in D. $\square$

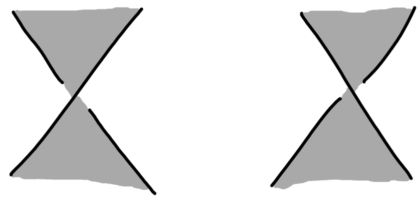

FIGURE 15. Checkerboard shade S^2 with respect to a knot diagram D. All crossings of D are labeled -1 or $+1$ as prescribed here.

Remark 3.7. If D is a regular two-bridge diagram with only one twist, then

$$\langle D \rangle = \begin{cases} A^{c(D)}\left(d - \frac{1}{d}\right) + \frac{(A+Bd)^{c(K)}}{d} & \text{if } D \text{ has an } A\text{-twist} \\ B^{c(D)}\left(d - \frac{1}{d}\right) + \frac{(Ad+B)^{c(K)}}{d} & \text{if } D \text{ has a } B\text{-twist.} \end{cases} \tag{6}$$

To avoid ambiguity in the case that D has one crossing, we say that such a diagram has an A-twist if the B-graph is a cycle and a B-twist if the A-graph is a cycle (if both are cycles then $c(D) = 2$ and the two expressions in Equation 6 agree).

4. SIGNATURE

4.1. **Definition.**

Given a knot K, the knot signature $\sigma(K)$ can be calculated from any diagram of K. We aim to recover $\sigma(K)$ from $\langle K \rangle$. We first define the knot signature by showing how to calculate it from a diagram of K, although this is not the usual definition (see [G] for further discussion and the origin of this calculation). Let D be a diagram of K, and checkerboard shade S^2 with respect to D. Label each crossing ± 1 as in Figure 15. Given a crossing c, call its label $\mu(c)$. Let G be the checkerboard graph for D corresponding to the white (unshaded) regions (i.e. G has a vertex for each white region and an edge for eachc rossing between adjacent white regions). Edges in G are weighted ± 1, according to the label of the corresponding crossing. Let L be the Laplacian matrix for G. Then the *Goeritz matrix* $\hat{G}$ for D is the upper-left $(n-1) \times (n-1)$ minor of L, where L is $n \times n$. Orient D, and label each crossing of D as either Type I or Type II as in Figure 16. Let $\mu = \sum_{c \text{ Type II}} \mu(c)$. Then

$$\sigma(K) = \sigma(\hat{G}) - \mu,$$

where $\sigma(G)$ is the usual matrix signature. The fact that this sum is an invariant of the knot is remarkable, and is proved in [G].

4.2. $\sigma(\hat{G})$ **for Alternating Knots.**

Remark 4.1. Let D be an alternating knot diagram with some checkerboard shading. If the white regions are enclosed by the loops of the all-B-smoothing state, then $\mu(c) = +1$ for all crossings c of D. If the white regions are enclosed by the loops of the all-A-smoothing state, then $\mu(c) = -1$ for all crossings c of D.

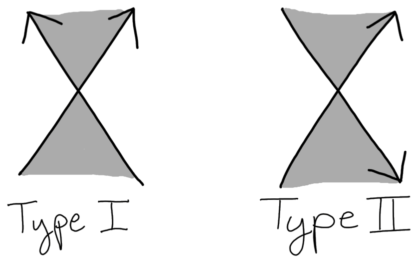

FIGURE 16. Checkerboard shade S^2 with respect to a knot diagram D and orient D. All crossings of D are labeled Type I and Type II as prescribed here. Note that this label does not depend on the choice of orientation pf D.

Theorem 4.2. *Let D be an alternating knot diagram and $\hat{G}$ the Goeritz matrix for D corresponding to checkerboard graph $G = (V, E)$. Then $\sigma(\hat{G}) = |V| - 1$ if $\mu(c) = +1$ for all crossings c of D, and $\sigma(\hat{G}) = -|V| + 1$ if $\mu(c) = -1$ for all crossings c of D.*

Proof of Theorem 4.2. If $\mu(c) = +1$ for all c, then the white regions are enclosed by the loops of the all-B-smoothing state, so G is the B-graph of D with all edges weighted $+1$. Then $\hat{G}$ is the upper-left $(|V| - 1) \times (|V| - 1)$ minor of L, where L is the Laplacian of the B-graph.

If $\mu(c) = -1$, then G is the A-graph of D with edges weighted -1. Then $\hat{G}$ is the upper-left $(|V| - 1) \times (|V| - 1)$ minor of $-L$, where L is the Laplacian of the A-graph.

Diagonalize $\hat{G}$ to a matrix with n_0 0s on the diagonal, n_+ positive entries, and n_- negative entries. Since $\hat{G}$ is the upper left $(|V| - 1) \times (|V| - 1)$ minor of a Laplacian of G, an unweighted graph, $|\det(\hat{G})|$ is the number of maximal subtrees of G (see e.g. [T]). Since D is a knot diagram, G is connected, so $|\det(\hat{G})| > 0$. That is, $n_0 = 0$. for $i = 0, 1, \ldots, |V| - 1$ let Δ_i be the determinant of the upper-left $i \times i$ minor of $\hat{G}$. Since $\hat{G}$ is symmetric, n_- is the number of sign changes in the finite sequence $1 = \Delta_0, \Delta_1, \ldots, \Delta_{n-1}, \Delta_n = \det(\hat{G})$. Any upper-left minor of a Laplacian will have positive determinant, by a generalization of the matrix-tree theorem [Mo, Theorem 3.1]. Therefore,if $\mu(c) = +1$ then $\hat{G}$ is positive-definite, so $\sigma(G) = |V| - 1$. If $\mu(c) = -1$ then $\hat{G}$ is negative-definite, so $\sigma(G) = -|V| + 1$ □

Remark 4.3. From theorem 4.2, we can obtain $\sigma(\hat{G})$ from $\langle D\rangle$, when D is an alternating knot diagram and $\hat{G}$ is the Goeritz matrix for D for some checkerboard shading. If $\mu(c) = +1$ for all crossings c then the one term in $\langle D\rangle$ of the form $B^{c(D)}d^j$ satisfies $j = \sigma(\hat{G})$. If $\mu = -1$ for all crossings c then the one term in $\langle D\rangle$ of the form $A^{c(D)}d^j$ satisfies $j = -\sigma(\hat{G})$.

4.3. Can we obtain μ from $\langle D\rangle$ when D is alternating?

If D is a knot diagram and we have already indicated how S^2 is checkerboard shaded around D, we write μ_D to designate μ for diagram D with that choice of shading.

Theorem 4.4. *Let $D, \hat{D}$ be reduced alternating diagrams of some knot K. Checkerboard shade both diagrams so that the sign of all crossings in $D, \hat{D}$ agree. Then $\mu_D = \mu_{\hat{D}}$.*

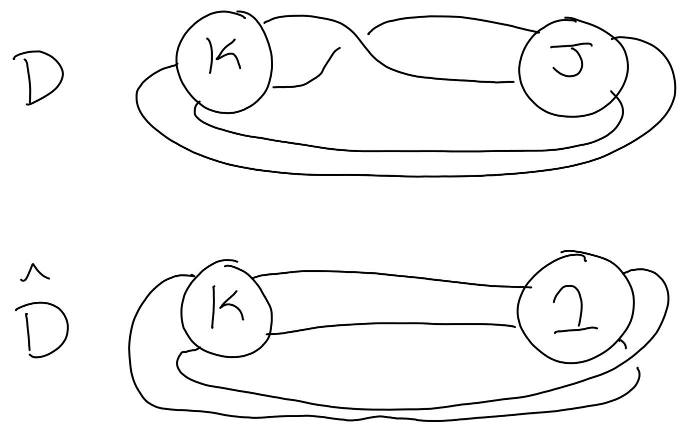

FIGURE 17. Diagrams D and $\hat{D}$ of a knot K, related by a flype.

Proof of Theorem 4.4. By the Tait Flyping Theorem [Me] it is sufficient to consider the case when $D, \hat{D}$ differ by a flype, as in Figure 17. Since the signs of crossings of $D, \hat{D}$ agree, the "unbounded" faces in Figure 17 are both shaded or both unshaded. Orient $D, \hat{D}$ so that the orientations of K, J agree between diagrams. Then the shadings and orientations in each copy of K and J agree, so all crossings in K or J will have the same type in D and $\hat{D}$. Moreover, the two strands of the other crossing in D are both oriented in or both oriented out of J if and only if the two strands of the other crossing in $\hat{D}$ are both oriented in or both oriented out of J. Therefore, the two crossings in $D, \hat{D}$ which are not in J or K have the same type. Thus, μ is the same for D as for $\hat{D}$, so μ is invariant under flypes. Therefore, μ along with the choice of positive- or negative-sign checkerboard shading is a reduced alternating knot invariant. □

Corollary 4.5. *Let $D, \hat{D}$ be reduced alternating diagrams of some knot K. Checkerboard shade both diagrams so that the sign of all crossings in $D, \hat{D}$ agree. Then $\sigma(\hat{G}_D) = \sigma(\hat{G}_{\hat{D}})$, where $\hat{G}_D, \hat{G}_{\hat{D}}$ are the Goeritz matrices for $D, \hat{D}$ correspondingly.*

Proof of Corollary 4.5. We have $\sigma(\hat{G}_D) - \mu_D = \sigma(K) = \sigma(\hat{G}_{\hat{D}}) - \mu_{\hat{D}}$. Theorem 4.4 implies the corollary. □

For the next remark, we must define the writhe of a knot diagram.

Given a knot diagram D, orient the diagram. We refer to a crossing c of D as "positive" or "negative" according to Figure 18. This does not coincide with the definition of $\mu(c)$, so we admit this to be poor notation. We write $\mathrm{sgn}(c) = 1$ if c is positive and $\mathrm{sgn}(c) = -1$ if c is negative. Note $\mathrm{sgn}(c)$ is independent of the choice of orientation of D. The *writhe* of D is given by $w(D) = \sum_c \mathrm{sgn}(c)$.

Remark 4.6. Let D be an alternating diagram with some checkerboard shading. If the white regions correspond to the B-graph of G, then a crossing is of Type II if and only if it is positive; otherwise a crossing is of Type II if and only if it is negative. In the first case $\mu_D = (w(D) + c(D))/2$ and in the second $\mu_D = (w(D) - c(D))/2$.

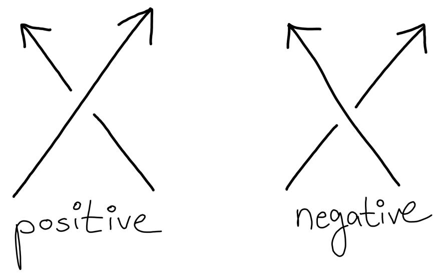

FIGURE 18. A positive crossing (left) and a negative crossing (right).

Thus, from Remark 4.6 we see that to obtain μ for an alternating link diagram, it is sufficient to find the writhe of the diagram. This leads to the following vague conjecture.

Conjecture 4.7. *Given an alternating knot diagram D with B-graph G, $w(D)$ can be obtained from G.*

The B-graph and its planar embedding certainly determine $w(D)$, since they together determine D, but many planar graphs have distinct planar embeddings. Finding two alternating knot diagrams with different writhes but whose B-graphs are isomorphic would provide a counterexample, but so far no such pair has been found. Whitney's Uniqueness Theorem [W] states that any 3-connected planar graph has a unique embedding into S^2, implying conjecture 4.7 holds true if G is 3-connected.

5. IS IT POSSIBLE TO RECOVER THE HOMOLOGY OF THE DOUBLE CYCLIC COVER OF S^3 BRANCHED OVER L FROM $\langle L\rangle$ FOR ALTERNATING L?

We assume knowledge of undergraduate-level algebraic structures for this section.

Given a link L, let X_L^2 refer to the double cover of S^3 branched over L.

Let L be an alternating link with diagram D. Checkerboard shade D so that the white regions correspond to the B-graph $G = (V, E)$ of D to obtain Laplacian L and Goeritz matrix $\hat{G}$. Label the vertices of G by $v_1, \ldots, v_n$ (where $n = |V|$) so that v_n corresponds to the row and column eliminated from L to form $\hat{G}$. Then $\hat{G}$ is a presentation matrix for $H_1(X_L^2)$ as a $\mathbb{Z}$-module (see [L]).

Let $m_1, \ldots, m_k$ be the invariant factors of $H_1(X_2)$, so that $H_1(X_2) \cong \mathbb{Z}/m_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/m_k\mathbb{Z}$ and $1 \neq m_1|\cdots|m_k$. The first elementary ideal of $H_1(X_2)$ is $|\det(G)| = (m_1 \cdots m_k)$, while the second is $(m_1 \cdots m_{k-1})$. Moreover, the second elementary ideal of $H_1(X_2)$ is generated by the determinants of all $(n-2) \times (n-2)$ minors of $\hat{G}$ (see [L, Def. 2]. Therefore, to recover m_k it is sufficient to recover the greatest common factor of these determinants.

Let $\hat{G}^i_j$ be the minor of $\hat{G}$ obtained by removing the row corresponding to v_i and column corresponding to v_j. Assume $i \neq j$. By the all minors matrix tree theorem, $|\det(\hat{G}^i_j)|$ is the number of spanning 2-tree forests in G which include v_i, v_j in one tree and v_n in the other [Ch]. Therefore, $2|\det(\hat{G}^i_j)| = |\det(\hat{G}^i_i)| + |\det(\hat{G}^j_j)| - |\det L^{i,j}_{i,j}|$, since $|\det L^{i,j}_{i,j}|$ is the number of spanning two-tree forests in G in which v_i, v_j are in different trees. Note $\hat{G}^i_i = L^{i,n}_{i,n}, \hat{G}^j_j = L^{j,n}_{j,n}$. Of course, if $i = j$, then $\hat{G}^i_j = L^{i,n}_{i,n}$. Therefore,

$$\left(2\det(\hat{G}^i_j) \mid i, j \in [n-1]\right) \subset \left(\det(L^{i,j}_{i,j}) \mid i \neq j \in [n]\right) \subset \mathbb{Z}.$$

Given $a \neq b \in [n]$, let G' be the Goeritz matrix obtained by removing the row and column corresponding to v_a from L. Then the 2nd elementary ideal of $H_1(X_2)$ is generated by the determinants of $(n-2) \times (n-2)$ minors of G', including $G'^a_a = L^{a,b}_{a,b}$. Therefore,

$$\left((\det(L^{i,j}_{i,j}) \mid i \neq j \in [n]\right) \subset \left(\det(G'^i_j) \mid i, j \in \{1, 2, \ldots, \hat{a}, \ldots, n\}\right) = \left(\det(\hat{G}^i_j) \mid i, j \in [n-1]\right).$$

Moreover, we note $G^i_i = \hat{G}^{i,n}_{i,n}$. Therefore,

$$\left(2\det(G^i_j) \mid i, j \in [n-1]\right) \subset \left(\det(\hat{G}^{i,j}_{i,j}) \mid i \neq j \in [n]\right) \subset \left(\det(G^i_j) \mid i, j \in [n-1]\right) \subset \mathbb{Z}.$$

Thus,

$$2\left(\det(\hat{G}^i_j) \mid i, j \in [n-1]\right) \subset \left(\det(L^{i,j}_{i,j}) \mid i \neq j \in [n]\right) \subset \left(\det(\hat{G}^i_j) \mid i, j \in [n-1]\right).$$

Thus, if $\left(\det(L^{i,j}_{i,j}) \mid i \neq j \in [n]\right) = m\mathbb{Z}$, then either $m_k = |\det(\hat{G})/m|$ or $m_k = 2|\det(\hat{G})/m|$. Of course, $m_k | \det(\hat{G})$, so if $\det(\hat{G})$ is odd then $m_k = |\det(\hat{G})/m|$. Similarly, since m_k is the largest invariant factor, if $\hat{G}$ is even and $\hat{G}/m$ is odd then $m_k = 2|\hat{G}/m|$. More generally, if $2^i \mid \hat{G}$ where $i > kn$ for some n but $2^n \nmid \hat{G}/m$, then $m_k = 2|\hat{G}/m|$.

The graphic interpretation of $|\det(L^{i,j}_{i,j})|$ is seemingly nicer than that of $|\det(\hat{G}^i_j)|$ for $i \neq j$: recall that $|\det(L^{i,j}_{i,j})|$ is the number of spanning 2-tree forests of G in which v_i, v_j are in different trees, while $|\det(\hat{G}^i_j)|$ is the number of spanning 2-tree forests in G which include v_i, v_j in one tree and v_n in the other. However, we have been unable to find $\det(L^{i,j}_{i,j})$ for some $i \neq j$ with $T[G](x, y)$, nor any other graph polynomial.

Conjecture 5.1. *For $i \neq j$, $\det(L^{i,j}_{i,j})$ can be found from $T[G](x, y)$ or a related graph polynomial.*

If Conjecture 5.1 holds true, then we can recover m_k up to a factor of 2 from $T[G](x, y)$.

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