A Look at Constant Vector Curvature on Three-Dimensional Model Spaces according to Curvature Tensor

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Abstract

This paper proves that all model spaces in dimension three with positive definite inner products have content vector curvature ϵ for some ϵ. This is done by classifying the curvature tensor associated with the model space.

1 Introduction

The study of geometric properties of manifolds using the techniques and tools of calculus is known as differential geometry. Manifolds are topological spaces that if viewed very closely behave in a similar manner to Euclidean spaces. This is known as being locally Euclidean. This study lends precision to the fields of physics, economics, and engineering.

One of the most studied geometric properties in this discipline is that of Riemann curvature tensors, and more specifically the curvature of model spaces. An algebraic curvature tensor is a function that measures the curvature of a manifold at a point.

Definition 1.1. *Let $x, y, z, w \in V$. An* algebraic curvature tensor $R : V \times V \times V \times V \to \mathbb{R}$ *such that the following properties hold:*

1. *R is multilinear in every slot,*
2. $R(x, y, z, w) = -R(y, x, z, w)$,
3. $R(x, y, z, w) = R(z, w, x, y)$,
4. $R(x, y, z, w) + R(z, x, y, w) + R(y, z, x, w) = 0$.

A manifold observed at a single point is an object known as a model space.

Definition 1.2. *Let $\langle \cdot, \cdot \rangle$ be an inner product on a vector space V and R be and algebraic curvature tensor. Then $M = (V, \langle \cdot, \cdot \rangle, R)$ is called a* model space*.*

All inner products considered in this paper are positive definite. One way to describe, classify, and study a model space is by its sectional curvature.

Definition 1.3. *Let $M = (V, \langle\cdot,\cdot\rangle, R)$ be a model space and v, $w \in V$ where ϕ is a two plane equal to span$\{v, w\}$. The* sectional curvature *of ϕ is defined to be $k(\phi) = \frac{R(v,w,w,v)}{\langle v,v\rangle\langle w,w\rangle - \langle v,w\rangle^2}$.*

The sectional curvature of a 2-plane is independent of the basis chosen for V. If all of the 2-planes of a model space have the same curvature then that model space has constant sectional curvature.

Definition 1.4. *Let $M = (V, \langle\cdot,\cdot\rangle, R)$ be a model space. M is said to have* constant sectional curvature ϵ *(csc(ϵ)) if for all 2-planes, ϕ, we have $k(\phi) = \epsilon$.*

A more specific way to study the sectional curvature of model spaces is by a measure called constant vector curvature.

Definition 1.5. *Let $M = (V, \langle\cdot,\cdot\rangle, R)$ be a model space. M is said to have* constant vector curvature ϵ *(cvc(ϵ)) if for all $v \in V$ where $v \neq 0$ there exists a $w \in V$ such that $k(span\{v, w\}) = \epsilon$.*

While constant sectional curvature implies constant vector curvature it is rather rare among model spaces. Constant vector curvature is a less stringent condition and therefore more common, thus it merits study even though it is a weaker condition.

Theorem 1.6. *Constant vector curvature is well-defined on model spaces in dimension three with positive definite inner products.[5]*

This paper will show by a combination of new and previously proven results that every model space in three-dimensions with a positive definite inner product has constant vector curvature(ϵ) for some ϵ. In some cases it can be proven that an even stronger result is true, such as constant sectional curvature or extremal constant vector curvature.

Definition 1.7. *Let $M = (V, \langle\cdot,\cdot\rangle, R)$ be a model space. M is said to have* extremal constant vector curvature ϵ (ecvc(ϵ)) *if for all 2-planes ϕ it is the case that ϵ is a bound on the sectional curvature tensor values.*

This paper will consider all of the different cases for model spaces in three-dimensions with positive definite inner products by looking at the qualities of the curvature tensor of a model space. We will look at the number of symmetric forms associated with the curvature tensor and the rank of these symmetric forms to classify all models spaces into a specific category of constant vector curvature.

The area of constant vector curvature in differential geometry has been studied only very sparingly. Most of the work in this area was done by Kelci Mumford in 2013. In her paper she proves that constant vector curvature is well-defined and many other results that we will rely upon heavily in this paper.

2 Setting the Stage

Let V be a real vector space. We need to define some of the tools that we will be using in this paper. The first of these is how to more simply calculate the curvature of a given 2-plane.

Lemma 2.1. *Let $M = (V, \langle\cdot,\cdot\rangle, R_\phi)$ be a 3-dimensional model space with e_1, e_2, e_3 as an orthonormal basis and $a, b, c, x, y, z \in \mathbb{R}$. Then*

$$[equ1]k(span\{ae_1+be_2+ce_3, xe_1+ye_2+ze_3) = \frac{R_{1221}(ay-bx)^2 + R_{1331}(az-cx)^2 + R_{2332}(bz-cy)^2}{(a^2+b^2+c^2)(x^2+y^2+z^2) - (ax-by-cz)^2} \quad (1)$$

Another of the tools that we will be using is symmetric bilinear forms.

Definition 2.2. *Let $x, y, z \in V$ and $a, b \in \mathbb{R}$. A* symmetric bilinear form *$\phi : V \times V \to \mathbb{R}$ such that the following properties hold true:*
1) $\phi(ax + by, z) = a\phi(x, z) + b\phi(y, z)$ (bilinearity),
2) $\phi(x, y) = \phi(y, x)$ (symmetry).

We can use symmetric bilinear forms to define a special type of curvature tensor.

Definition 2.3. *Let ϕ be a symmetric bilinear form and $x, y, z, w \in V$. A* canonical algebraic curvature tensor *$R_\phi = \phi(x, w)\phi(y, z) - \phi(x, z)\phi(y, w)$.*

It is important to note that R_ϕ is an algebraic curvature tensor.

Theorem 2.4. *Let R be an algebraic curvature tensor in a three-dimensional vector space. Then there exists either 1) exactly one symmetric bilinear form ϕ such that $R = \pm R_\phi$ or 2) exactly two distinct symmetric bilinear symmetric forms ϕ and ψ such that $R = R_\phi + R_\psi$.[1]*

This result allows us to classify curvature tensors according the symmetric forms associated with them. One should note that there is more than one way to measure the curvature of a model space.

Definition 2.5. *Let $M = (V, \langle\cdot,\cdot\rangle, R)$ be a model space, $v, w \in V$ and $\{e_1, ..., e_n\}$ be an orthonormal basis for V. The* Ricci tensor *is a symmetric bilinear form (ρ) defined in the following way:*

$$\rho(v, w) = \sum_{i=1}^{n} R(v, e_i, e_i, w).$$

Note that the Ricci tensor is independent of the basis chosen for V. Also, that there is a direct link between the curvature tensor and the Ricci tensor in dimension three.

Theorem 2.6. *There exists a basis that is orthonormal with respect to the inner product and diagonalizes the Ricci tensor. [2]*

The existence of this basis allows us to work with the Ricci tensor only in terms of its eigenvalues.

Lemma 2.7. *The Ricci tensor with eigenvalues $\{\lambda_i, \lambda_j, \lambda_k\}$ completely determines the curvature tensor in three-dimensions in the following way:*

$$R(e_i, e_j, e_j, e_i) = \frac{\lambda_i + \lambda_j - \lambda_k}{2}$$

[5]

This result allows us to limit the number of nonzero curvature tensor entries.

Lemma 2.8. *There are only three possible nonzero entries of the curvature tensor in three-dimensions. [5]*

The curvature tensor and Ricci tensor and connected in both their composition and number.

Theorem 2.9. *Let $M = (V, \langle\cdot,\cdot\rangle, R)$ be a model space. Let $\{e_1, e_2, e_3\}$ be a basis that orthonormal with respect to $\langle\cdot,\cdot\rangle$ and also diagonalizes the Ricci tensor. Then,*

$$||R(e_1, e_2, e_2, e_1), R(e_1, e_3, e_3, e_1), R(e_2, e_3, e_3, e_2)|| = ||spec(\rho)||.$$

[5]

This result gives us information that we can use to help us find the possible value of ϵ for a model space.

Theorem 2.10. *Let $M = (V, \langle\cdot,\cdot\rangle, R)$ be a model space. Let $\{e_1, e_2, e_3\}$ be an orthonormal basis on V with respect to $\langle\cdot,\cdot\rangle$, diagonalized the Ricci tensor and orders the curvature tensor entries in the following way: $R_{1221} \geq R_{1331} \geq R_{2332}$. If M has $cvc(\epsilon)$ then $R_{1331} = \epsilon$. [5]*

Along with knowing the possible value of ϵ we can also tell exactly when a model space has *ecvc*(ϵ).

Theorem 2.11. *Let $M = (V, \langle\cdot,\cdot\rangle, R)$ be a model space. M has $ecvc(\epsilon)$ if and only if $||spec(\rho)|| \leq 2$.[5]*

The link between the curvature tensor and Ricci tensor gives us even more information about the curvature tensor and its possible decomposition.

Theorem 2.12. *Let $M = (V, \langle\cdot,\cdot\rangle, R)$ be a model space with $\rho = \{\lambda_1, \lambda_2, \lambda_3\}$. $R = R_\phi + R_\psi$ iff $\lambda_i + \lambda_j = \lambda_k$ for some i, j, k.[1]*

3 Case 1 ($R = R_\phi$)

Lemma 3.1. *Let $M = (V, \langle\cdot,\cdot\rangle, R_\phi)$ be a three-dimensional model space. If $Rank(\phi) = 0$ or 1 then M has $csc(0)$.*

Proof. If the $Rank(\phi) = 1$ for a model space $M = (V, \langle\cdot,\cdot\rangle, R_\phi)$ then ϕ is modeled by the following matrix values:

$$\phi = \begin{vmatrix} \eta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}$$

The values of the curvature tensor are calculated below:

$R_{1221} = \phi_{11}\phi_{22} - {\phi_{12}}^2 = \eta \cdot 0 - 0 = 0$

$R_{1331} = \phi_{11}\phi_{33} - {\phi_{13}}^2 = \eta \cdot 0 - 0 = 0$

$R_{2332} = \phi_{22}\phi_{33} - {\phi_{23}}^2 = 0 \cdot -0 = 0$

If the $Rank(\phi) = 0$ then $\eta = 0$ and the result is the same. Because all of the values of the curvature tensor are zero then the curvature of any two-plane is 0. Thus, the model space has $csc(0)$.

□

Lemma 3.2. *Let $M = (R, \langle\cdot,\cdot\rangle, R_\phi)$ be a three-dimensional model space. If $Rank(\phi) = 2$ then M has ecvc(0).*

Proof. If $Rank(\phi) = 2$ then ϕ is modeled by the following matrix:

$$\phi = \begin{vmatrix} \eta_1 & 0 & 0 \\ 0 & \eta_2 & 0 \\ 0 & 0 & 0 \end{vmatrix}$$

The values of the curvature tensor are calculated below:

$R_{1221} = \phi_{11}\phi_{22} - {\phi_{12}}^2 = \eta_1\eta_2 - 0 = \eta_1\eta_2$

$R_{1331} = \phi_{11}\phi_{33} - {\phi_{13}}^2 = \eta_1 \cdot 0 - 0 = 0$

$R_{2332} = \phi_{22}\phi_{22} - {\phi_{23}}^2 = \eta_2 \cdot 0 - 0 = 0$

$||spec(\rho)|| = 2$, thus by Theorems 2.8 and 2.9 M has $ecvc(0)$.

□

It is at this point in my research that we begin to differ with Mumford's results. While her paper is meticulously written and contains many incredibly helpful and consequential results, there is an error in the proof of her Theorem 7. On Page 25 of her paper in the first paragraph of her proof she claims that one can assume without loss of generality that $\phi(v, w) = 0$. In this context this assumption cannot be made without loss of generality. Because of this, her Theorems 7, 8, and 9 and her Lemma 19 cannot be accepted as true; however, the rest of her paper stands up to scrutiny.

Theorem 3.3. *Let $M = (V, \langle\cdot,\cdot\rangle, R)$ be a model space where $dim(V) = 3$ and let $\{e_1, e_2, e_3\}$ be an orthonormal basis on $\langle\cdot,\cdot\rangle$ that diagonalizes the Ricci tensor such that $R_{1221} \geq R_{1331} \geq R_{2332}$ where $R_{1331} = \epsilon$. If $R = R_\phi$ and $||spec(\rho)|| = 3$ then M has $cvc(\epsilon)$.*

Proof. Let $\phi = \begin{vmatrix} \eta_1 & 0 & 0 \\ 0 & \eta_2 & 0 \\ 0 & 0 & \eta_3 \end{vmatrix}$

$R_{1221} = \phi_{11}\phi_{22} - \phi_{12}^2 = \eta_1\eta_2 - 0 = \eta_1\eta_2 = \delta$
$R_{1331} = \phi_{11}\phi_{33} - \phi_{13}^2 = \eta_1\eta_3 - 0 = \eta_1\eta_3 = \epsilon$
$R_{2332} = \phi_{22}\phi_{33} - \phi_{23}^2 = \eta_2\eta - 3 - 0 = \eta_2\eta_3 = \tau$

If $||\{\eta_1, \eta_2, \eta_3\}|| < 3$ it follows that $||spec(\rho)|| \leq 2$. If this is the case then the model space has either $ecvc(\epsilon)$ if $||spec(\rho)|| = 2$ or $csc(\epsilon)$ if $||spec(\rho)|| = 1$. Thus, we assume that $||\{\eta_1, \eta_2, \eta_3\}|| = 3$. Let $0 \neq v \in V$ where $v = ae_1 + be_2 + ce_3$ and let $w = \frac{-\sqrt{(\delta-\epsilon)(\epsilon-\tau)}}{\epsilon-\delta}e_1 + e_3$.

Consider

$$k(span\{v, w\}) = \frac{R(v, w, w, v)}{\langle v, v\rangle\langle w, w\rangle - \langle v, w\rangle^2}$$

Note that $\delta > \epsilon > \tau$ so w is well defined. We will consider the numerator and denominator of this fraction separately.

Numerator of [**?**]:

$$\begin{aligned}
&R(v, w, w, v) = \\
&\quad R\left(ae_1 + be_2 + ce_3, \frac{-\sqrt{(\delta-\epsilon)(\epsilon-\tau)}}{\epsilon-\delta}e_1 + e_3, \frac{-\sqrt{(\delta-\epsilon)(\epsilon-\tau)}}{\epsilon-\delta}e_1 + e_3, ae_1 + be_2 + ce_3\right) \\
&\quad = \delta\left(\frac{b\sqrt{(\delta-\epsilon)(\epsilon-\tau)}}{\epsilon-\delta}\right)^2 + \epsilon\left(a + \frac{c\sqrt{(\delta-\epsilon)(\epsilon-\tau)}}{\epsilon-\delta}\right)^2 + \tau b^2 \\
&\quad = \frac{\delta b^2(\delta-\epsilon)(\epsilon-\tau)}{(\epsilon-\delta)^2} + \epsilon\left(a^2 + \frac{2ac\sqrt{(\delta-\epsilon)(\epsilon-\tau)}}{\epsilon-\delta} + \frac{c^2(\delta-\epsilon)(\epsilon-\tau)}{(\epsilon-\delta)^2}\right) + \tau b^2 \\
&\quad = \frac{-\delta b^2(\epsilon-\tau)}{\epsilon-\delta} + \epsilon\left(a^2 + \frac{2ac\sqrt{(\delta-\epsilon)(\epsilon-\tau)}}{\epsilon-\delta} - \frac{c^2(\epsilon-\tau)}{\epsilon-\delta}\right) + \tau b^2 \\
&\quad = -\delta b^2(\epsilon-\tau) + \epsilon a^2(\epsilon-\delta) + 2\epsilon ac\sqrt{(\delta-\epsilon)(\epsilon-\tau)} - \epsilon c^2(\epsilon-\tau) + \tau b^2(\epsilon-\delta) \\
&\quad = \epsilon a^2(\epsilon-\delta) + 2\epsilon ac\sqrt{(\delta-\epsilon)(\epsilon-\tau)} - \epsilon c^2(\epsilon-\tau) + \epsilon b^2(\tau-\delta) \\
&\quad = \epsilon(a^2(\epsilon-\delta) + 2ac\sqrt{(\delta-\epsilon)(\epsilon-\tau)} - c^2(\epsilon-\tau) + b^2(\tau-\delta))
\end{aligned}$$

Denominator or [?]:

$$
\begin{aligned}
\langle v,v\rangle\langle w,w\rangle-\langle v,w\rangle^2 &= \langle ae_1+be_2+ce_3, ae_1+be_2+ce_3\rangle \\
&\left\langle \frac{-\sqrt{(\delta-\epsilon)(\epsilon-\tau)}}{\epsilon-\delta}e_1+e_3, \frac{-\sqrt{(\delta-\epsilon)(\epsilon-\tau)}}{\epsilon-\delta}e_1+e_3\right\rangle \\
&-\left\langle ae_1+be_2+ce_3, \frac{-\sqrt{(\delta-\epsilon)(\epsilon-\tau)}}{\epsilon-\delta}e_1+e_3\right\rangle^2 \\
&=(a^2+b^2+c^2)\left(\left(\frac{-\sqrt{(\delta-\epsilon)(\epsilon-\tau)}}{\epsilon-\delta}\right)^2+1\right)-\left(\frac{-a\sqrt{(\delta-\epsilon)(\epsilon-\delta)}}{\epsilon-\delta}+c\right)^2 \\
&=(a^2+b^2+c^2)\left(1-\frac{(\delta-\epsilon)(\epsilon-\tau)}{(\epsilon-\delta)^2}\right)-\left(\frac{a^2(\delta-\epsilon)(\epsilon-\tau)}{(\epsilon-\delta)^2}-\frac{2ac\sqrt{(\delta-\epsilon)(\epsilon-\tau)}}{\epsilon-\delta}+\right. \\
&=(a^2+b^2+c^2)\left(1-\frac{\epsilon-\tau}{\epsilon-\delta}\right)+\frac{a^2(\epsilon-\tau)}{\epsilon-\delta}+\frac{2ac\sqrt{(\delta-\epsilon)(\epsilon-\tau)}}{\epsilon-\delta}-c^2 \\
&=(a^2+b^2+c^2)(\tau-\delta)+a^2(\epsilon-\tau)+2ac\sqrt{(\delta-\epsilon)(\epsilon-\tau)}-c^2(\epsilon-\delta) \\
&=a^2(\tau-\delta+\epsilon-\tau)+2ac\sqrt{(\delta-\epsilon)(\epsilon-\tau)}+c^2(\tau-\delta-\epsilon+\delta)+b^2(\tau-\delta)
\end{aligned}
$$

Combining these we get:

$$
\epsilon\frac{a^2(\epsilon-\delta)+2ac\sqrt{(\delta-\epsilon)(\epsilon-\tau)}-c^2(\epsilon-\tau)+b^2(\tau-\delta)}{a^2(\epsilon-\delta)+2ac\sqrt{(\delta-\epsilon)(\epsilon-\tau)}-c^2(\epsilon-\tau)+b^2(\tau-\delta)}=\epsilon
$$

This calculation satisfies every vector where $b \neq 0$, though if $b=0$ v and w are linearly dependent. In the case where $a \neq 0$ and $b=0$ we choose w to be e_1 with the same result. In the case that $b,c=0$ we choose w to be $\frac{1}{a}e_3$. Thus, every vector in V ca be paired with another to form a 2-plane with sectional curvature ϵ.

Remark 3.4. *The situation outlined above is similar in subsequent theorems. It is treated the same way and thus we will not mention each time.*

□

4 Case 2 ($R=-R_\phi$)

Theorem 4.1. *$M=(V,\langle\cdot,\cdot\rangle,R)$ has cvc(ϵ) iff $Q=(V,\langle\cdot,\cdot\rangle,-R)$ has cvc($-\epsilon$).*

Proof. If:

Let $M = (V, \langle \cdot, \cdot \rangle, R)$ have $cvc(\epsilon)$.

It follows that for all $v \neq 0 \in V$ there exists a $w \in V$ such that $R(v, w, w, v) = \epsilon$.

Thus, $-R(v, w, w, v) = -\epsilon$

Therefore, Q has $cvc(-\epsilon)$

Only if:

Similar proof in other direction.

□

5 Case 3 ($R = R_\phi + R_\psi$)

If the $||spec(\rho)|| = 1$ then $R = R_\phi$ for some ϕ. Thus, be begin our investigation in this section with $||spec(\rho)|| = 2$. M has $ecvc(\epsilon)$ if and only if $||spec(\rho)|| \leq 2$.[5]

Theorem 5.1. *Let $M = (V, \langle \cdot, \cdot \rangle, R)$ where $R = R_\phi + R_\psi$, $spec(\rho) = \{\lambda_1, \lambda_2 \lambda_3\}$, and $||spec(\rho)|| = 3$. If λ_1 and λ_2 have different signs then M has $cvc(0)$.*

Proof. By Theorem 2.12 $R = R_\phi + R_\psi$ only when $\lambda_1 + \lambda_2 = \lambda_3$ where $\lambda_1, \lambda_2 \neq 0$. Thus, $R_{1221} = 0$, $R_{1331} = -\lambda_1$, $R_{2332} = -\lambda_2$. By assumption λ_1, λ_2 have different signs. Let $0 \neq v \in V$ where $v = ae_1 + be_2 + ce_3$. Let $w = \frac{\sqrt{-\lambda_1 \lambda_2}}{-\lambda_1} e_1 + e_2$. Note that by assumption $\lambda_1 \neq 0$ and λ_1 and λ_2 have different signs thus w is well-defined.

Consider

$$\frac{R(v, w, w, v)}{\langle v, v \rangle \langle w, w \rangle - \langle v, w \rangle^2} =$$

$$\frac{R\left(ae_1 + be_2 + ce_3, \frac{\sqrt{-\lambda_1 \lambda_2}}{-\lambda_1} e_1 + e_2, \frac{\sqrt{-\lambda_1 \lambda_2}}{-\lambda_1} e_1 + e_2, ae_1 + be_2 + ce_3\right)}{\langle ae_1 + be_2 + ce_3, ae_1 + be_2 + ce_3 \rangle \left\langle \frac{\sqrt{-\lambda_1 \lambda_2}}{-\lambda_1} e_1 + e_2, \frac{\sqrt{-\lambda_1 \lambda_2}}{-\lambda_1} e_1 + e_2 \right\rangle - \left\langle ae_1 + be_2 + ce_3, \frac{\sqrt{-\lambda_1 \lambda_2}}{-\lambda_1} e_1 + e_2 \right\rangle^2}$$

Since our goal is to show that this fraction is equal to 0 we shall only examine the numerator of this fraction.

$$
\begin{aligned}
R(v,w,w,v) &= R(ae_1+be_2+ce_3, \frac{\sqrt{-\lambda_1\lambda_2}}{-\lambda_1}e_1+e_2, \frac{\sqrt{-\lambda_1\lambda_2}}{-\lambda_1}e_1+e_2, ae_1+be_2+ce_3) \\
&= -\lambda_1\left(\frac{-c\sqrt{-\lambda_1\lambda_2}}{\lambda_1}\right)^2 - \lambda - 2c^2 \\
&= -\lambda_1\left(\frac{-c^2\lambda_1\lambda_2}{\lambda_1^2}\right) - \lambda_2 c^2 \\
&= \lambda_1 c^2 - \lambda_1 c^2 \\
&= 0. \qquad \square
\end{aligned}
$$

Thus, every vector in V can be paired with another to form a 2-plane with sectional curvature 0.

$\square$

Theorem 5.2. *Let $M = (V, \langle\cdot,\cdot\rangle, R)$ where $R_\phi + R_\psi$ and $\rho = \{\lambda_1, \lambda_2, \lambda_3\}$. If $\lambda_1,\ \lambda_2$ have the same sign M has $cvc(-\lambda_1)$.*

Proof. By theorem 2.12 $R = R_\phi + R_\psi$ only when $\lambda_1 + \lambda_2 = \lambda_3$ where $\lambda_1, \lambda_2 \neq 0$. Thus, $R_{1221} = 0,\ R_{1331} = -\lambda_1,\ R_{2332} = -\lambda_2$. Let $\lambda_1,\ \lambda_2$ have the same sign and $\{e_1, e_2, e_3\}$ be a basis for V that is orthonormal with respect to $\langle\cdot,\cdot\rangle$. Let $0 \neq v \in V$ where $v = ae_1+be_2+ce_3$. Let $w = \frac{-\sqrt{\lambda_1(\lambda_2-\lambda_1)}}{\lambda_1}e_1 + e_3$.

Consider
$$k(span\{v,w\}) = \frac{R(v,w,w,v)}{\langle v,v\rangle\langle w,w\rangle - \langle v,w\rangle^2} =$$

Numerator of [?]:

$$
\begin{aligned}
R(v,w,w,v) &= R\left(ae_1+be_2+ce_3, \frac{-\sqrt{\lambda_1(\lambda_2-\lambda_1)}}{\lambda_1}e_1+e_3, \frac{-\sqrt{\lambda_1(\lambda_2-\lambda_1)}}{\lambda_1}e_1+e_3, ae_1+be_2+ce_3\right) \\
&= a^2R_{1331} + 2ac\left(\frac{\sqrt{\lambda_1(\lambda_2-\lambda_1)}}{\lambda_1}\right)R_{1331} + b^2R_{2332} + c^2\frac{\sqrt{\lambda_1(\lambda_2-\lambda_1)}^2}{\lambda_1}R_{1331} \\
&= -\lambda_1(a + c\left(\frac{\sqrt{\lambda_1(\lambda_2-\lambda_1)}}{\lambda_1}\right)^2 - \lambda_2 b^2 \\
&= -\left(a^2\lambda_1 + 2ac\sqrt{\lambda - 1(\lambda_2-\lambda_1)} + c^2(\lambda_2-\lambda_1) + b^2\lambda_2\right) \\
&= -\lambda_1\left(a^2\lambda_1 + 2ac\sqrt{\lambda_1(\lambda_2-\lambda_1)} + c^2(\lambda_2-\lambda_1) + b^2\lambda_2\right)
\end{aligned}
$$

Denominator of [?]:

$$\begin{aligned}
&\langle v,v\rangle\langle w,w\rangle-\langle v,w\rangle^2\\
&\quad=\langle ae_1+be_2+ce_3, ae_1+be_2+ce_3\rangle\\
&\quad\left\langle \frac{-\sqrt{\lambda_1(\lambda_2-\lambda_1)}}{\lambda_1}e_1+e_3, \frac{-\sqrt{\lambda_1(\lambda_2-\lambda_1)}}{\lambda_1}e_1+e_3\right\rangle\\
&\quad-\left\langle ae_1+be_2+ce_3, \frac{-\sqrt{\lambda_1(\lambda_2-\lambda_1)}}{\lambda_1}e_1+e_3\right\rangle^2\\
&\quad=(a^2+b^2+c^2)\left(\left(\frac{-\sqrt{\lambda_1(\lambda_2-\lambda_1)}}{\lambda_1}\right)^2-1\right)-\left(a\left(\frac{-\sqrt{\lambda_1(\lambda_2-\lambda_1)}}{\lambda_1}\right)+c\right)^2\\
&\quad=(a^2+b^2+c^2)\left(\frac{\lambda_2}{\lambda_1}\right)-\frac{a^2\lambda_2}{\lambda_1}+a^2+\frac{2ac\sqrt{\lambda_1(\lambda_2-\lambda_1)}}{\lambda_1}\\
&\quad=a^2\lambda_1+2ac\sqrt{\lambda_1(\lambda_2-\lambda_1)}+c^2(\lambda_2-\lambda_1)+b^2\lambda_2 \qquad \square
\end{aligned}$$

Combining we have:

$$\frac{-\lambda_1\left(a^2\lambda_1+2ac\sqrt{\lambda_1(\lambda_2-\lambda_1)}+c^2(\lambda_2-\lambda_1)+b^2\lambda_2\right)}{a^2\lambda_1+2ac\sqrt{\lambda_1(\lambda_2-\lambda_1)}+c^2(\lambda_2-\lambda_1)+b^2\lambda_2}=-\lambda_1$$

Therefore, M has $cvc(-\lambda_1)$.

$\square$

6 Conclusion

Every model space with a positive definite inner product in three-dimensions has $cvc(\epsilon)$ for some ϵ. If the values of the curvature tensor are known then it is possible to find the value of ϵ and in some cases the stronger results of extremal vector curvature or constant sectional curvature. For every ϵ there exists a model space in 3-dimensions with $cvc(\epsilon)$ and $ecvc(\epsilon)$ where ϵ can be either an upper or lower bound.

7 Open Questions

- Is constant vector curvature well defined in dimensions greater than 3 or when the inner product of the model space is nondegenerate?
- To what extent to the eigenvalues of the Ricci tensor effect the *cvc* condition in higher dimensions or other situations?

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