Three-variable bracket polynomial for algebraic links

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Abstract

This paper examines the three-variable bracket polynomial for different families of links. The structure is first developed through analysis of two-bridge links and pretzel links, then these results are generalized to provide a bound for the maximum degree of d in the three-variable bracket polynomial for algebraic links.

1 Introduction

Knot theory is a field of math that has been gaining popularity and attention for quite some time. Its applications are very diverse; knots can be used both to model DNA strands and to gather information about the sun's corona.

For our purposes, we define a *knot* to be any closed loop in $\mathbb{R}^3$. We can represent knots as projections in the plane, where over-crossings are denoted by a solid line and under-crossings are denoted by a break in the line. Such projections are referred to as knot diagrams (see Figure 4). A knot without any crossings is the *unknot*. Similarly, a *link* is a collection of knots in space that may or may not be intertwined with one another. We can describe a knot as a link with one component.

We say a knot diagram is *reduced* if there are no nugatory crossings of the form in Figure 1. A knot is *alternating* if, when following any arbitrary path along the knot, one alternates over-crossings with under-crossings. This paper will be concerned solely with those links that are reduced and alternating.

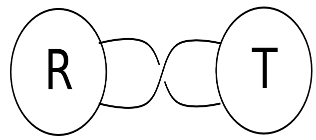


Figure 1: Nugatory crossing

A *twist* (see Figure 5) is a region of a knot diagram in which only two strands intertwine with each other. We will refer to twists with one crossing as one-crossing twists and twists with more than one crossing as multiple-crossing twists. The *twist number* is an invariant that represents the fewest number of twists of any diagram of a knot. A *tangle* of a link is any region that we can draw a circle around such that the circle intersects the link in four places. In Figure 2 (I) we see a *zero-tangle* and in Figure 2 (II) we see an ∞-*tangle*. For more on knots and links, see [2].

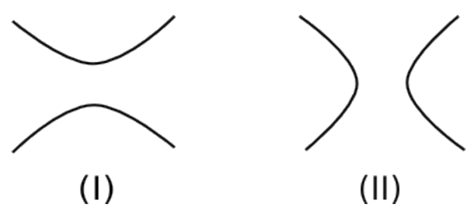


Figure 2: A zero-tangle and ∞-tangle

It will also be helpful for us to introduce flypes. A *flype* is a move on a link diagram that flips one specific part of the link. Flypes can be helpful in moving crossings from one area of a link to another. An example of a flype is shown in Figure 3.

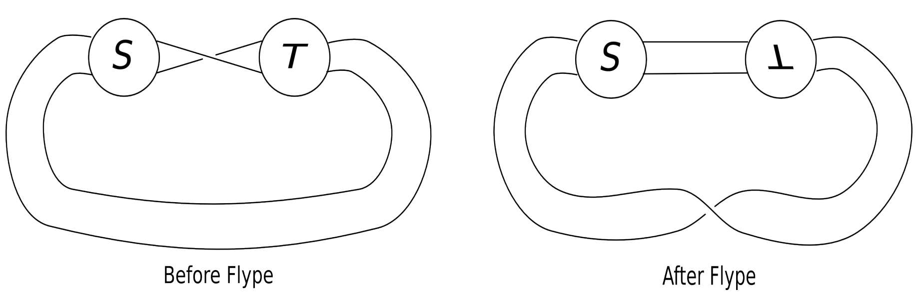


Figure 3: Example of a flype

For our analysis we must also include a brief discussion of regions of a link. We say that an *A region* is the area to the left of an under-crossing, whereas a *B region* is the area to the right of an under-crossing. In Figure 4 the regions of a 5-crossing knot are labelled for the reader. The concept of A regions and B regions also brings up the idea of smoothings. We perform a *smoothing* by cutting the knot at a crossing and glueing the ends in such a way as to connect the A regions or the B regions. If our smoothing connects two A regions, we refer to it as an *A smoothing*; otherwise we refer to it as a *B smoothing*.

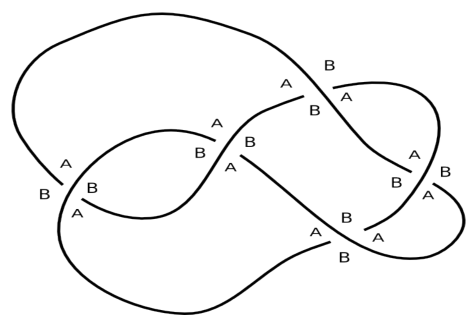


Figure 4: A and B regions of a reduced, alternating 5-crossing knot

We define *smoothing across the twist* to be a particular type of smoothing demonstrated in Figure 5. Here we see two different cases of this smoothing. In Figure 5 (I) we see a smoothing across a multiple-crossing twist, whereas in Figure 5 (II) a smoothing across a one-crossing twist is demonstrated.

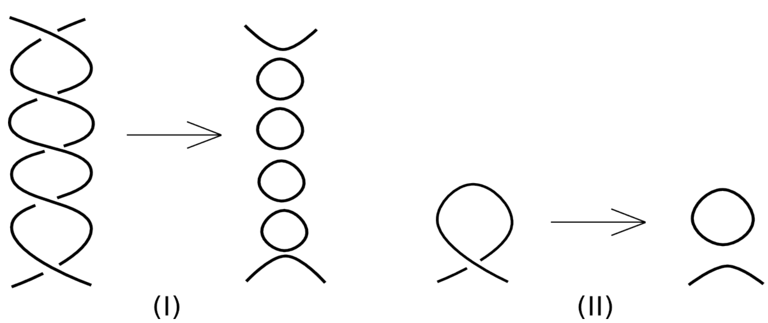


Figure 5: Smoothing across the twist

1.1 The Three-Variable Bracket Polynomial

Now we introduce an important tool that we will be using in our analysis of links. To each link we can assign a polynomial that is derived from a process of performing all possible combinations of smoothings on our link. This polynomial can tell us whether two alternating links are not isotopic to each other. For the rest of this paper, we will refer to the three-variable bracket polynomial simply as the bracket polynomial. There are two possible ways of defining this polynomial.

Recursive Definition: $\langle K \rangle$ assigns a three-variable polynomial to every unoriented diagram K according to

$$I. \langle \times \rangle = A\langle)(\rangle + B\langle \asymp \rangle$$
$$II. \langle L\bigcirc \rangle = d\langle L \rangle$$
$$III. \langle \bigcirc \rangle = 1$$

This paper will be focusing on the other definition: the state model approach.

State Model Approach: A *state* is a specific choice of smoothing for each crossing. Since there are are two choices for each smoothing, a link with n crossings would have 2^n states. Each state of the link is assigned a term $A^x B^y d^m$ where x is the number of A smoothings performed to reach this state, y is the number of B smoothings performed, and m is the number of disjoint loops in this state minus one. We take the sum of all the terms corresponding to different states to get the bracket polynomial. This process is demonstrated on the Hopf link in Figure 6.

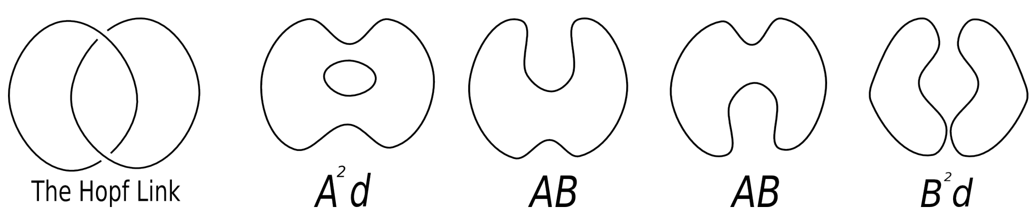


Figure 6: Bracket polynomial of the Hopf link: $A^2d + 2AB + B^2d$

The introduction of the bracket polynomial brings up another important concept: link invariants. These are values attributed to links that are identical for any two equivalent reduced, alternating links. It is important to note that the bracket polynomial is invariant under flypes. This tells us, by Tait's Flyping Theorem, that the bracket polynomial is an invariant of reduced, alternating links [4].

1.2 Families of links

Now we will introduce some different families of links that we will be performing analysis on. First we have the two-bridge links. A *two-bridge link* is a link which can be isotoped so that there are only two maxima and two minima given from the natural height function of the z-coordinate. An example of a two-bridge knot is given in Figure 7.

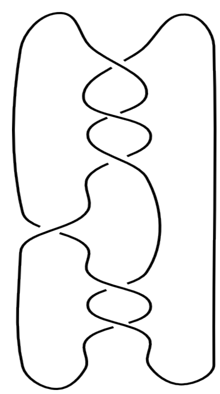


Figure 7: A 2-bridge knot

Two-bridge links are best constructed using their rational link representation. Consider the following construction for the knot represented by x_1 x_2 ... x_j, where j is even. We begin with an infinity tangle and twist the bottom two strands x_1 many times. Next we take the two strands on the right and twist them x_2 many times. Then we twist the bottom two strands x_3 many times. This process continues for our entire sequence of integers. If j is odd, we simply begin with a zero tangle and twist the two strands on the right, then continue the process as usual. To turn the ensuing rational tangle into a rational knot, we simply connect the NE and NW strands and the SE and SW strands. We will refer to this process of turning a tangle into a knot as the *numerator closure*. The process of creating the rational knot 3 1 2 is shown in Figure 8.

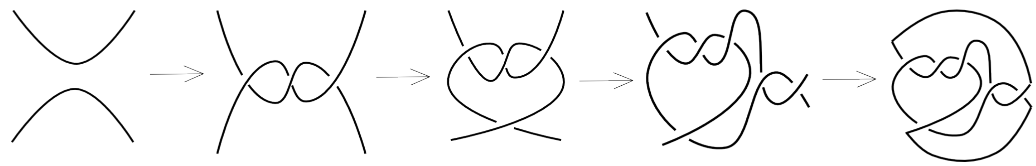


Figure 8: Construction of the 3 1 2 rational knot

We will refer to 3 1 2 as the integer sequence of the rational knot. For a rational knot to be alternating, the signs of all the integers must be the same. So for the rest of this paper we will assume, without loss of generality, that the integer sequences of all our links consist of non-negative integers. For more on the construction of rational knots, we refer the reader to [1].

Note that the rational knot we just constructed can be deformed into the knot in Figure 7 through a sequence of planar isotopies. This displays the equivalence we have between two-bridge links and rational links. We can also perform operations on rational tangles. The addition and multiplication of two rational tangles can be seen in Figure 9.

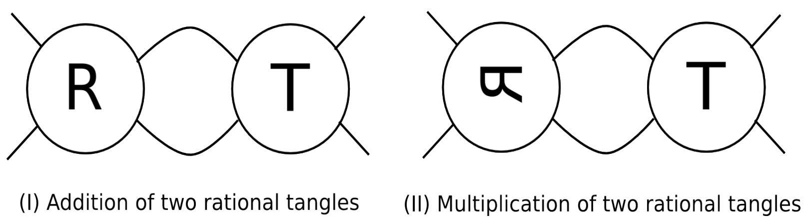


Figure 9: Addition and multiplication of rational tangles

Operations on rational tangles open up an entire new family of links known as algebraic links. *Algebraic tangles* are formed from the addition and multiplication of rational tangles. We form an *Algebraic link* by taking the numerator closure of an algebraic tangle. We can view algebraic links as rational links where each crossing has been replaced by a rational tangle. We define the *parent link*, P, of an algebraic link, L, to be the rational link obtained by replacing each rational tangle of L with a 1-tangle.

Now we introduce some additional notation. The sequence x_1,x_2,...,x_j corresponds to the algebraic link x_1 0 + x_2 0 +...+ x_j 0. We refer to any algebraic link formed from some finite sequence of integers separated by commas as a *pretzel link*. Such a pretzel link can be seen in Figure 10, where the ith box contains a vertical twist of x_i crossings.

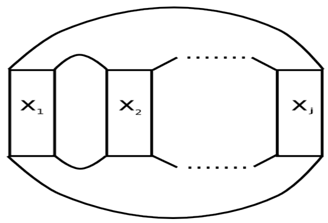


Figure 10: Pretzel link x_1 0 + x_2 0 +...+ x_j 0

2 Two-bridge links

2.1 Dual state sums

For any state, s, we denote the number of loops in s by $|s|$. We will denote the dual state of s by s_d. Then we know from [2] that

$$|s| + |s_d| \leq 2 + c.$$

Lemma 1. *Let L be a two-bridge link with at least two crossings in each twist. Let s be the maximal state of L that is obtained from progressively smoothing across the twists. Then if the twist number, n, of L is odd we get $|s| + |s_d| = c - n + 3$. If n is even, we get $|s| + |s_d| = c - n + 2$.*

Proof. Since s is the state obtained from progressively smoothing across the twists, and since every twist of L has multiple crossings, we know that s_d is the state obtained by replacing each crossing with an ∞-tangle. So s_d will be of the form in Figure 2.1(I) if n is odd, and s_d will be of the form in Figure 2.1(II) if n is even.

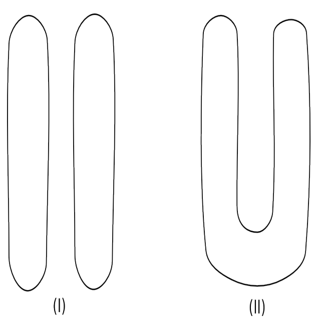


Figure 11: Possible dual states of s

We know by Lafferty that $|s| = c - n + 1$. Clearly we can see that $|s_d|$ is either 1 or 2. So if n is odd we get $|s| + |s_d| = c - n + 3$ and if n is even we get $|s| + |s_d| = c - n + 2$.

□

Lemma 2. *Let L be a two-bridge link with at least three crossings per twist. Then L has a unique maximal state.*

Proof. Let t be the maximal state obtained by progressively smoothing across the twists of L. So $|t| = c-n+1$ by Lafferty. In this state, all of our crossings have been replaced by zero-tangles. Let L_0 be the link obtained by replacing one of the crossings of L with an ∞-tangle. Then since each twist of L has at least three crossings, we can still use the formula $m = c - n$ for L_0. Let t_0 be a maximal state of L_0. Then $|t_0| = (c - 1) - n + 1$. So any state of L other than t will have fewer components than t. □

Lemma 3. *Let L be a two-bridge link with at least three crossings in each twist. Let s be a maximal state of L. Then if the twist number, n, of L is odd we get $|s| + |s_d| = c - n + 3$. If n is even, we get $|s| + |s_d| = c - n + 2$.*

Proof. Apply Lemma 1 and Lemma 2. □

2.2 Maximum degree of d

First we would like to conduct analysis on two-bridge knots that extends upon the results found in [3]. In her paper, Lafferty focused solely on two-bridge knots where each twist has multiple crossings. She established that in this particular case, if our link has c crossings, twist number n, and m is the maximal degree of d in the bracket polynomial, then $c - m = n$. We will generalize this result to encompass all reduced, alternating two-bridge links.

Part of Lafferty's analysis involved showing that smoothing across all multiple-crossing twists of a link will result in a state with the maximum degree of d. We will begin our analysis by considering the case in which smoothing across a multiple-crossing twist produces an additional multiple-crossing twist (see Figure 12).

This situation implies that at least one end of the twist must initially be adjacent to a 3-gon. Otherwise, we have a twist that is embedded within another twist as in Figure 13. Note that this case is resolved by performing a flype.

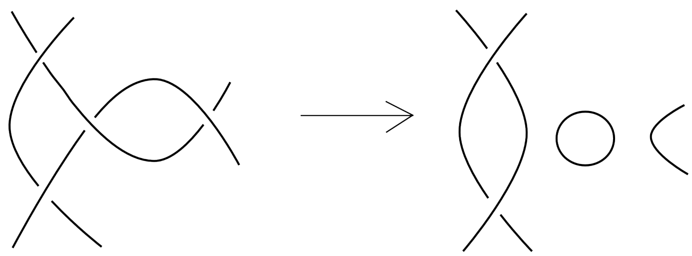


Figure 12: Smoothing across a multiple-crossing twist yields a new multiple-crossing twist

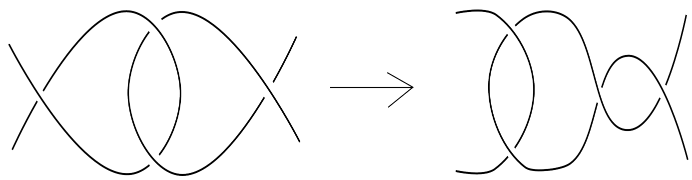


Figure 13: The case where a twist is embedded within another twist

Lemma 4. *For a knot with at least one multiple-crossing twist, suppose smoothing across a multiple-crossing twist produces an additional multiple crossing twist. Smoothing across the initial multiple-crossing twist and then smoothing across the created multiple-crossing twist leads to the maximum degree of* d*.*

Proof. We know that smoothing across the multiple-crossing twist will yield the maximum degree of d, by Theorem 5 from [3]. This smoothing can be seen in Figure 12. Now we have 4 more options to completely smooth out this diagram. These possibilities are shown in Figure 14. It is worth noting that, depending on the size of the twist in Figure 12, we may have more additional loops in these possible smoothings.

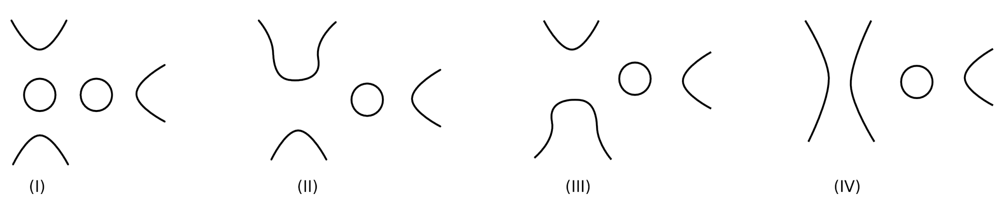


Figure 14: Possible smoothings

Clearly since I is isotopic to II and III with an additional loop, II and

III cannot yield the maximum degree of d. So the only possible options for the maximum degree of d are I and IV. Suppose we smoothed as in IV. Note that replacing an infinity tangle with a zero tangle will at worst connect two loops, decreasing the number of loops by one. However, I has an additional loop that is added, so in the worst case, replacing IV by I keeps the number of loops the same. Of course there is also the case where replacing an infinity tangle with a zero tangle will separate two loops, increasing the total number of loops. So replacing IV with I will either increase the number of loops, or keep it the same. Thus we find that smoothing across the additional twist as in I leads to the maximum degree of d.

□

In order to properly perform analysis on two-bridge links, we must introduce an algorithm that gives us the maximum degree of d. This algorithm is necessary because we are including two-bridge links with one-crossing twists in our discussion.

Definition. *We define progressively smoothing across the twists as a process that begins by placing a line across one end of your two-bridge knot diagram, and sliding it perpendicularly across the diagram, smoothing across multiple-crossing twists as you go. Any single-crossing twists not of the form in Figure 5 are left untouched. Observe Figure 15.*

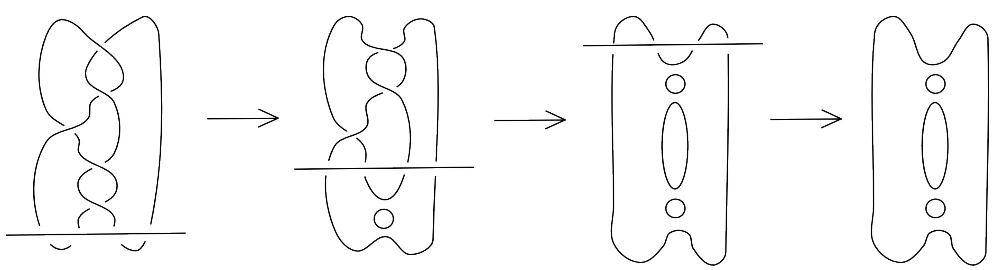


Figure 15: Progressively smoothing across the twists

Now we have the structure to begin our analysis of two-bridge links.

Lemma 5. *For a two-bridge link, starting from the bottom and progressively smoothing across the twists gives the maximum degree of d.*

Proof. First we must show that progressively smoothing across the twists actually smoothes all crossings in a two-bridge link. Without loss of generality, we can assume that the first and last twists of a two-bridge link contain multiple crossings. Starting at the bottom, we can progressively smooth across all twists of multiple crossings until we reach a twist that had only one crossing in our initial knot. If we have more than two consecutive one-crossing twists from our original knot, then we have one of the two cases in Figure 16.

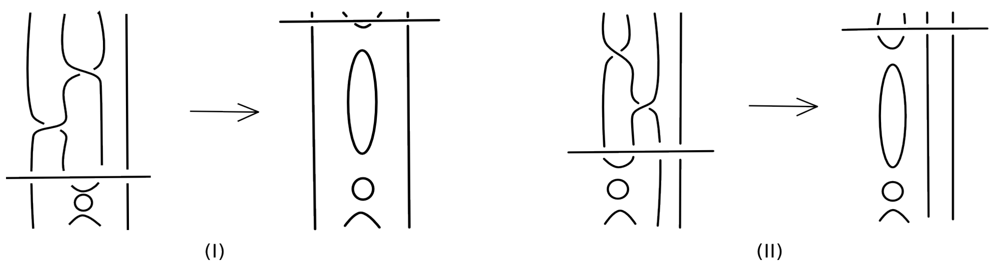


Figure 16: Possible smoothings

Notice that in each case the prior smoothing that was performed produces a new multiple-crossing twist out of the original one-crossing twists above. This process continues recursively along consecutive original one-crossing twists until we reach one of the four cases below.

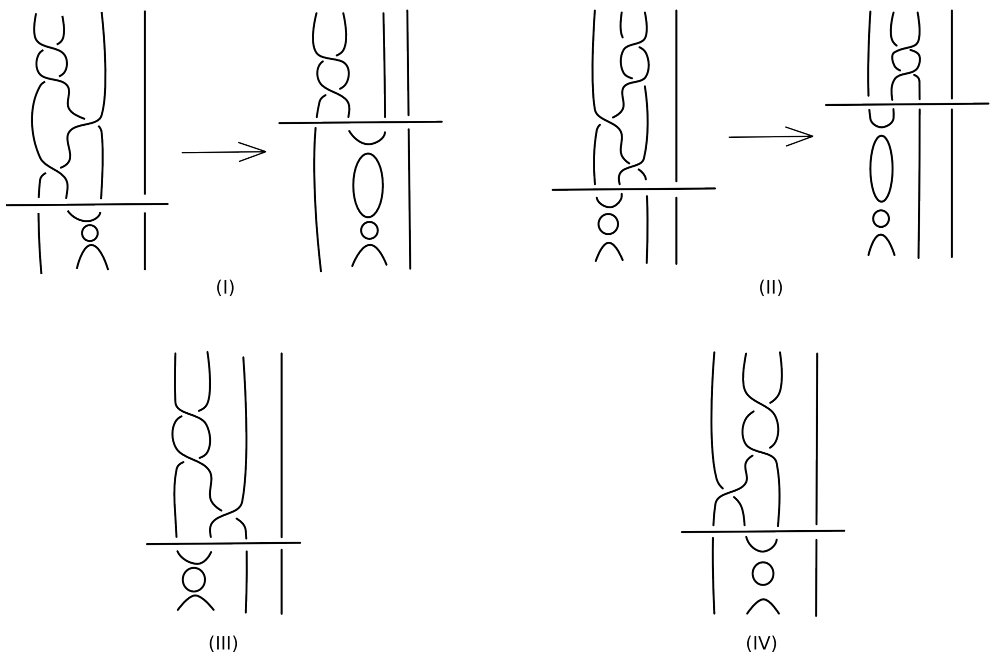


Figure 17:

Note that in case III and IV, the one-crossing twist from the original knot gets absorbed into the multiple-crossing twist above. Therefore in all cases we can proceed as usual with the next multiple-crossing twist. If there was only one or two consecutive one-crossing twists in our original knot in this region, then we would begin at case I, II,III, or IV and proceed as usual.

Now we continue progressively smoothing across the multiple-crossing twists until we reach a twist that only had one crossing in our intial knot. We repeat the process described above and continue. Since, by assumption, the first twist of the knot has multiple crossings, the process will continue until all crossings have been smoothed. It is worth noting that all of our smoothings were performed on multiple-crossing twists.

Next we must show that this process gives the maximum degree of d, or that it produces a maximal state. Since all of our smoothings are either smoothing across multiple-crossing twists from the initial knot, or smoothing across multiple-crossing twists that were produced by smoothing across multiple-crossing twists, we can appeal to Lemma 4.

□

Now we must introduce another definition. We define a *one-crossing twist region*, r, of a 2-bridge link, $a_1a_2...a_n$, to be a consecutive sequence $a_i...a_{i+j}$ such that $a_i = ... = a_{i+j} = 1$, $a_{i-1} \neq 1$, and $a_{i+j+1} \neq 1$. Furthermore, we say that $|r| = j + 1$.

The proof of Lemma 5 produces another useful observation about the one-crossing twist regions.

Lemma 6. *A one crossing twist region, r, of a two-bridge link, L, adds $\left\lfloor \frac{|r|+1}{2} \right\rfloor$ more loops to a maximal state of L.*

Proof. Suppose we are progressively smoothing across the twists and we reach a one-crossing twist region, r. Suppose $|r| = 1$. Then we get the situation as in Figure 18 (I), where the crossing gets absorbed into the multiple-crossing twist above it, yielding one additional loop. If $|r| = 2$, as in Figure 18 (II), then we smooth as shown, yielding an additional two loops. If $|r| > 2$, then we smooth as in Figure 18 (II), noting that each pair of consecutive one-crossing twists contributes an additional loop. We would continue smoothing these pairs of one-crossing twists until we are left with 1 or 2, in which case we know how to finish the procedure. □

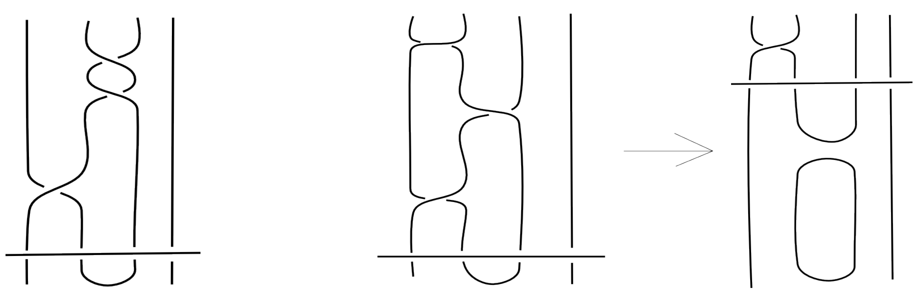


Figure 18: Possible smoothings for one-crossing twist regions

Before our next theorem we must introduce some additional notation. Let R be a rational tangle with c crossings, twist number n, and one-crossing-twist regions r_1,r_2,...,r_j. Then we define the function $\mathcal{M}(R) = c - n + \sum_{i=1}^{j} \left\lfloor \frac{|r_i|+1}{2} \right\rfloor$.

Theorem 1. *Let L be a 2-bridge link. Then if R is the rational tangle whose numerator closure is L, and if m is the maximum degree of d in the bracket polynomial, then*

$$m = \mathcal{M}(R).$$

Proof. Suppose we have a 2-bridge link L with n twists, c total crossings, j total one-crossing twist regions r_1,r_2,...r_j, and suppose m is the maximum degree of d in the bracket polynomial. We know that we can only reach a maximal state if we smooth across the multiple-crossing twists [3]. Let c_k be the number of crossings in the kth twist. Note also that a twist of c_k crossings, where $c_k > 1$, contributes a zero tangle with c_k-1 loops in the middle. From Lemma 6 we know that the ith one-crossing twist region adds $\lfloor \frac{|r_i|+1}{2} \rfloor$ more loops to a maximal state of L. So in a maximal state where l is the total number of loops, we have

$$l = (\sum_{i=1}^{n} c_i - 1) + \sum_{i=1}^{j} \left\lfloor \frac{|r_i| + 1}{2} \right\rfloor + 1.$$

The additional 1 comes from the additional unknot we obtain from smoothing across the twists. By definition, $l = m + 1$. Substitution and simplification yields

$$m = c - n + \sum_{i=1}^{j} \left\lfloor \frac{|r_i| + 1}{2} \right\rfloor .$$

So by definition, we get

$$m = \mathcal{M}(R).$$

□

Notice that if there are no one-crossing twists, then we get $m = c - n$, which is equivalent to the result found in Lafferty's paper. Thus we have extended the results in [3] to find the maximum degree of d in the bracket polynomial for general two-bridge links.

3 Pretzel links

Now we would like to conduct a similar analysis of pretzel links.

Lemma 7. *Let* $(p_1,\ldots,p_n)$ *be a pretzel link with* $p_i \geq 2$ *for all* i. *If* c *is the total number of crossings,* m *is the maximum degree of* d *in the bracket polynomial, and* n *is the twist number, then* $m = c - n + 1$.

Proof. First note that if we replace every twist of such a pretzel link with zero tangles then we will get two loops as in Figure 19.

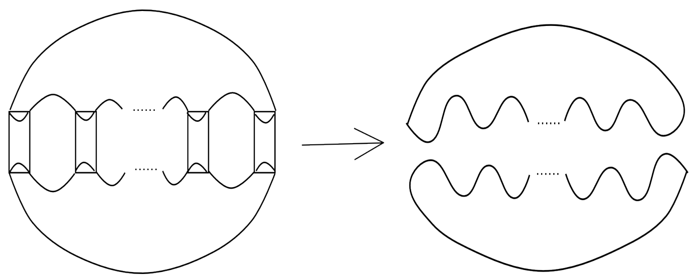


Figure 19: Replacing each twist of a pretzel link with a zero tangle

Now we must recognize that since every twist has at least two crossings, in order to get the maximal degree of d, we must smooth across the twists. Note that the ith twist turns into a zero tangle with $p_i - 1$ loops in between. So we can see that where l is the number of loops resulting from smoothing across the twists, we have

$$l = \sum_{i=1}^{n}(p_i - 1) + 2.$$

The additional 2 comes from the two components created in Figure 19. So by definition we have

$$m = \sum_{i=1}^{n} p_i - n + 1.$$

Now observe that $c = \sum_{i=1}^{n} p_i$. So after substitution we get

$$m = c - n + 1.$$

□

Lemma 8. *Let* $(p_1,\ldots,p_n)$ *be a pretzel link where there exists a unique* i *such that* $p_i = 1$*. If* c *is the total number of crossings,* m *is the maximum degree of* d *in the bracket polynomial, and* n *is the twist number, then* $m = c - n + 1$*.*

Proof. We begin by smoothing across all multiple-crossing twists. Thus we would end up with the knot in Figure 20, with $\sum_{k \neq i}(p_k - 1)$ additional loops.

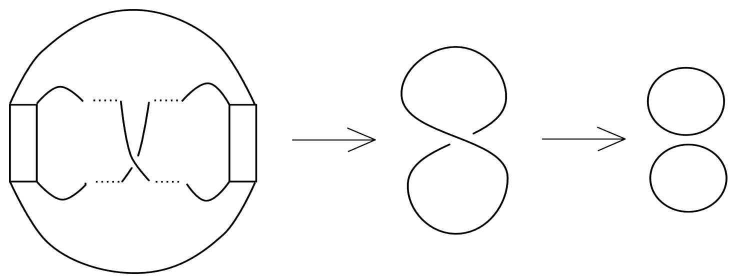


Figure 20: Smoothings in a pretzel link with a single one-crossing twist

In order to get the maximum degree of d, we would smooth the last crossing as in Figure 20. So where l is the number of loops, we get

$$l = \sum_{k \neq i}(p_k - 1) + 2 = \sum_{k \neq i}(p_k) - (n-1) + 2 = \sum_{k \neq i}(p_k) - n + 3.$$

Substituting in $l = m + 1$ yields

$$m = \sum_{k \neq i}(p_k) - n + 2.$$

Since $\sum_{k \neq i}(p_k) = c - 1$, we get

$$m = c - n + 1.$$

□

Next we must address the case where our pretzel link $(p_1,\ldots,p_n)$ contains multiple i's such that $p_i = 1$. In this case we can flype all of the one-crossings to the end of the sequence. Thus we can assume without loss of generality that in a pretzel link with multiple ones in its representation, they all appear at the end.

Theorem 2. *Let $(p_1,...,p_q)$ be a pretzel link such that there exists a $j < q$ with $p_j = p_{j+1} = ... = p_q = 1$ and for all $i < j$, $p_i \neq 1$. If there are c crossings, m is the maximal degree of d in the bracket polynomial, and n is the twist number of the link, then $m = c - n$.*

Proof. We begin by smoothing across the first $j-1$ twists. So we are left with a diagram of the form in Figure 21, with $\sum_{k<j}(p_k - 1)$ additional loops.

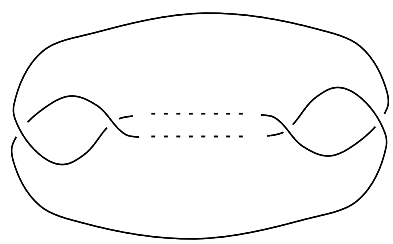


Figure 21: Smoothing of a pretzel knot with multiple ones in its integer representation

Smoothing across the one remaining twist (Lemma 4) yields $q - j + 1$ additional loops. So if l is the number of loops in a maximal state, then we have

$$l = \sum_{k<j}(p_k - 1) + q - j + 1 = \sum_{k<j}(p_k) - j + 1 + q - j + 1$$
$$= c - (q - j + 1) - j + 1 + q - j + 1 = c - j + 1.$$

Substituting in $l = m + 1$, we get

$$m = c - j.$$

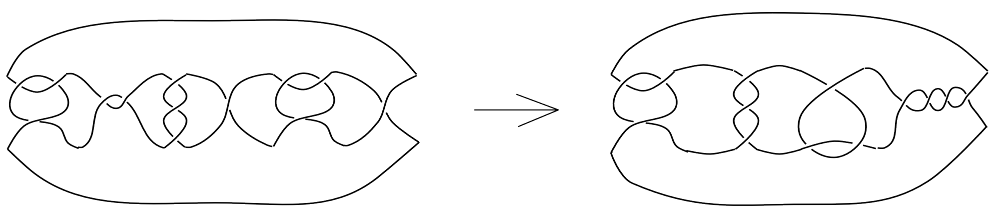
Since the sequence of ones at the end of the link forms one twist, we know that $n = j$. So we have

$$m = c - n.$$

□

4 Algebraic links

Next we will look at yet another class of links: algebraic links. For our discussion, we will assume that if an integer tangle is added to a rational tangle, we have combined them to form a new rational tangle. We must also make a note concerning the twist number of an algebraic link. Consider a sum of rational tangles $R_1+R_2+...+R_i$. Let $f_1,f_2,...,f_i$ be the sequence of the final integers from each of the rational tangles' integer sequence. Through a sequence of flypes, we can combine all of these crossings into a single twist located at the end of this sum. So the final integers of the tangles, $R_1,R_2,...,R_{i-1}$ will be zero, while the final integer of R_i will be equal to $f_1 + f_2 + ... + f_i$. An example of this process is shown in Figure 22. We will refer to diagrams for which this process has already been completed throughout the entire link as *twist minimal diagrams.* It should be noted that in a twist minimal algebraic link diagram, there may be some reflections of rational tangles present, as in Figure 22. A rational tangle, R, that has been flyped will be referred to as $F(R)$. However, this does not affect our algorithm of progressively smoothing across the twists, so we do not worry about this situation.



(I) 2 1 2 + 3 1 + 2 1 1 (II) 2 1 0 + 3 0 + F(2 1 4)

Figure 22: Obtaining a twist minimal algebraic link

Lemma 9. *Let R be a rational tangle. Then progressively smoothing across the twists of R yields an ∞-tangle, a zero-tangle, or a 1-tangle, with additional loops.*

Proof. We will take a case-by-case approach to this proof. The possible tangles are shown in Figure 23.

Consider first the case where the sequence of integers defining a rational tangle is of the form x_1 ... x_n where $x_n \geq 2$. Then we must smooth across the twist corresponding to x_n, which yields an ∞-tangle.

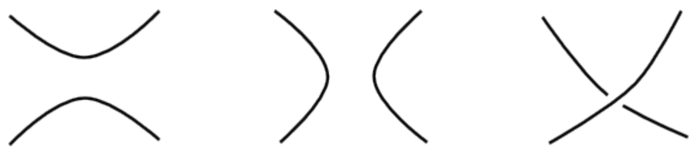


Figure 23: Possible tangles after progressively smoothing across twists

Next, consider the case in which the sequence of integers defining a rational tangle is of the form x_1 ... x_m 0 where $x_m \geq 2$. Then we must smooth across the twist corresponding to x_m, which yields a zero-tangle.

The final case to consider is when the sequence of integers defining a rational tangle is of the form x_1 ... 1 0 or x_1 ... 1. Note that if this final 1 is part of a one-crossing twist region of even magnitude, then it will get smoothed across during our algorithm. If our tangle is x_1 ... 1 0, as in Figure 24 (II), then it will get smoothed into an ∞-tangle. If our tangle is x_1 ... 1, as in Figure 24 (I), then it will get smoothed into a zero-tangle. However, if this final 1 is part of a one-crossing twist region of odd magnitude, then it will remain as part of a one-crossing twist and remain unsmoothed as in Figure 24 (III).

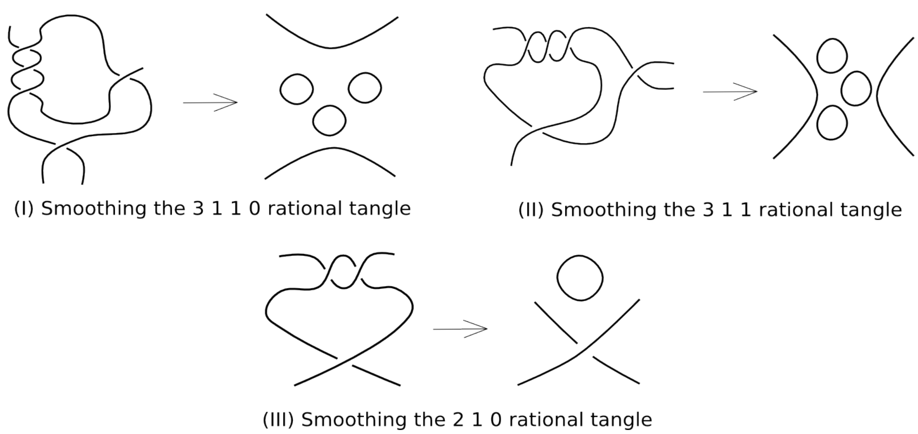


Figure 24: Progressively smoothing across the twists of some rational tangles

□

We must include the additional possibility of getting a 1-tangle from our algorithm due to the fact that we are working with rational tangles as opposed to rational links. It is also worth remarking that using our algorithm, when we end up with a 1-tangle, it will not be of the form in Figure 5 (II).

Lemma 10. *Let L be an alternating algebraic link. Then a maximal state of L can be reached by first progressively smoothing across the twists of the rational tangles.*

Proof. Suppose that we smooth across the twists of all our rational tangles. Then by Lemma 9 it follows that each rational tangle has been replaced by an ∞-tangle, a zero-tangle, or a 1-tangle, with additional loops. Note that the only way we would get a state that is not maximal is if our choice of smoothings connected two loops outside the tangle, whereas a different choice of smoothings gave the same number of loops in the tangle while separating one loop outside. Since progressively smoothing across the twists to a 1-tangle has no global effect on the knot, we know that progressively smoothing across the twists to a 1-tangle is optimal.

We will show that if smoothing progressively across the twists yields a zero-tangle, then performing alternate smoothings to arrive at an ∞-tangle will not be optimal, and vice versa. Suppose now that a rational tangle gets smoothed into a zero-tangle. This implies that the sequence of integers defining the tangle is of the form $x_1, ..., x_m, 0$ where $x_m \geq 1$. If $x_m \geq 2$ then our tangle is of the form in Figure 25, where the rest of the tangle is within T.

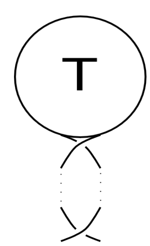


Figure 25: Rational tangle of the form $x_1, ..., x_m, 0$

If we are to smooth this tangle into an ∞-tangle, then we must perform vertical smoothings on the twist outside of T. This sequence of smoothings would simply give us T as in Figure 26. Note that if we had smoothed across the twists, we would have–depending on the size of the twist–created at least one additional loop as in Figure 26.

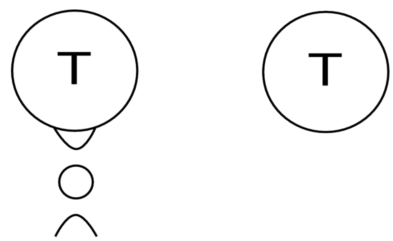


Figure 26: Possible smoothings

Therefore, as a whole, the rational tangle would have at least one additional loop inside of it if we smooth progressively across the twists. Thus, changing our zero-tangle to an ∞-tangle loses at least one loop locally, while gaining at most one loop globally. Therefore it is optimal to progressively smooth across the twists.

If $x_m = 1$ then we know that during our algorithm this final crossing will eventually become part of a 2-crossing twist. Then we have the same situation as before.

The proof follows identically if smoothing progressively across the twists yields an ∞-tangle. □

Theorem 3. *Let L be an algebraic link. In order to reach a maximal state of L, we first progressively smooth across the twists of each rational tangle composing L. Then we progressively smooth across the twists of the remaining link.*

Proof. Let L be an algebraic link. We know by Lemma 10 that in order to reach a maximal state of L we can first progressively smooth across the twists of each rational tangle composing L. It follows that the remaining link, call it K, will be the parent link of L where each crossing may have been replaced by a zero-tangle or an ∞-tangle. Suppose we progressively smooth across the twists of K. If there are no one-crossing twists, then this algorithm will smooth all of the crossings and we will have a maximal state of L by Theorem 5.

Now we must show that even if K contains one-crossing twists, we can still smooth progressively across the twists. Suppose K has a one-crossing twist. We begin by progressively smoothing across multiple-crossing twists until we reach a one-crossing twist. Then we must have one of the images in Figure 27. First note that if all of the tangles above any of these links are

∞-tangles, then we would be left with an unknot that had a Reidemeister-1 move performed on it, which we can smooth across. So we can assume that there are either more zero-tangles or more 1-tangles above this point. Note also that in Figure 27 (II) and (IV) we have formed a Reidemeister-1 move that we can smooth across.

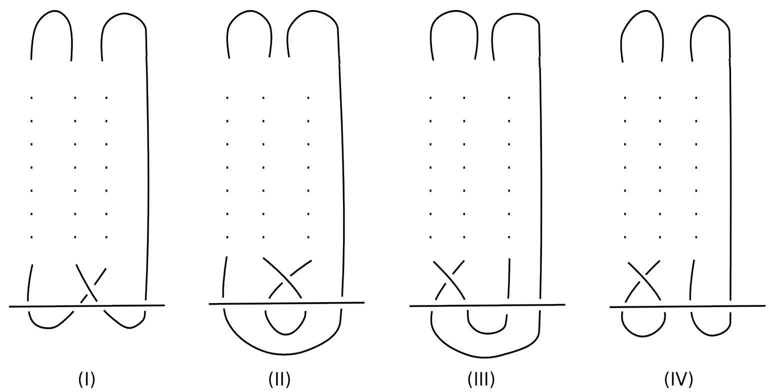


Figure 27: Possibilities upon reaching the first one-crossing twist

Suppose our link is of the form in Figure 27 (I) or (III). If the next non-∞-tangle above this point is a zero-tangle, then both of these 1-tangles will form Reidemeister-1 moves, which we can smooth across. If the next non-∞-tangle above this point is a 1-tangle, then our original 1-tangle has become part of a multiple-crossing twist, which we can smooth across.

Continuing this process we will be able to smooth every crossing in our link using this algorithm, giving us a maximal state of L.

□

Due to the fact that we are first smoothing each rational tangle locally, we sometimes end up with tangles that don't properly align with our algorithm as it is performed on the parent link. For instance, we might be forced to smooth part of a multiple-crossing twist in our parent link to an ∞-tangle, which is never optimal. This makes it difficult to find an exact formula for the maximum degree of d. However, we have established an upper bound and lower bound for the maximum degree of d, which takes these issues into consideration.

Theorem 4. *Let L be an alternating algebraic link that is composed of the rational tangles R_1,R_2,...,R_j, and suppose x of them get progressively smoothed* *to a 1-tangle. Then if m is the maximum degree of d in the bracket polynomial, and P is the parent link of L, we get*

$$\sum_{i=1}^{j} \mathcal{M}(R_i) - x \leq m \leq \sum_{i=1}^{j} \mathcal{M}(R_i) + \mathcal{M}(P) - x.$$

Proof. We can assume that we are looking at a twist minimal diagram of L. We know by Theorem 3 that in order to reach a maximal state of L we begin by progressively smoothing across the twists of the rational tangles. We will call the remaining link K, which is the parent link of L where each crossing may have been replaced by a zero-tangle or an ∞-tangle. Now we must count up all of the loops, l_0, that were created in this process. Let c_i be the number of crossings in the ith twist. Note also that a twist of c_i crossings, where $c_i > 1$, contributes a zero-tangle with $c_i - 1$ loops in the middle. Likewise, the one-crossing twist region r_i produces $\left\lfloor \frac{|r_i|+1}{2} \right\rfloor$ loops. Notice also that we must subtract one loop for each rational tangle that gets smoothed to a 1-tangle. Let n be the twist number of L, and let r_1, r_2,..., r_y be the one-crossing twist regions of L. Then we have

$$\begin{aligned} l_0 &= \sum_{i=1}^{n}(c_i - 1) + \sum_{i=1}^{y} \left\lfloor \frac{|r_i|+1}{2} \right\rfloor - x = c - n + \sum_{i=1}^{y} \left\lfloor \frac{|r_i|+1}{2} \right\rfloor - x \\ &= \sum_{i=1}^{j} \mathcal{M}(R_i) - x \end{aligned}$$

in addition to our link K. By Theorem 3 we progressively smooth across K to get a maximal state of L. By Theorem 1, we know that the maximum number of additional loops we could get, l_1, is given by

$$l_1 = \mathcal{M}(P) + 1$$

where the one comes from the additional unknot that we get at the end of the process. Thus $l_u = l_0 + l_1$ provides an upper bound for the maximum number of loops we could get in a state of L. So we have

$$l_u = \sum_{i=1}^{j} \mathcal{M}(R_i) + \mathcal{M}(P) - x + 1.$$

Thus, by definition we get

$$m \leq \sum_{i=1}^{j} \mathcal{M}(R_i) + \mathcal{M}(P) - x.$$

Note also that the minimum number of additional loops we could get, l_2, is 1. So $l_m = l_0 + l_2$ provides a lower bound for the maximum number of loops we could get in a state of L. Therefore, after substitution, we get

$$\sum_{i=1}^{j} \mathcal{M}(R_i) - x \leq m \leq \sum_{i=1}^{j} \mathcal{M}(R_i) + \mathcal{M}(P) - x. \tag{1}$$

□

Through recognition of what gives us the maximum number of loops for l_1 in the proof of Theorem 4, we arrive at the following corollary.

Corollary 1. *Let L be an alternating algebraic link that is composed of the rational tangles R_1,R_2,...,R_j, and suppose that all of them get progressively smoothed to a* 1*-tangle. Then if m is the maximum degree of d in the bracket polynomial, and P is the parent link of L, we get*

$$m = \sum_{i=1}^{j} \mathcal{M}(R_i) + \mathcal{M}(P) - x.$$

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