Structure Groups and Linear Transformations

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1 Preliminaries

Definition 1. *Let V be a vector space and $R : V \times V \times V \times V \to \mathbb{R}$ be linear in each input. R is an* ***algebraic curvature tensor*** *if it satisfies the following properties for all x,y,z,w $\in V$:*

1. $R(x, y, z, w) = -R(y, x, z, w)$,

2. $R(x, y, z, w) = R(z, w, x, y)$

3. $R(x, y, z, w) + R(z, x, y, w) + R(y, z, x, w) = 0$

The vector space of algebraic curvature tensors on V is denoted $\mathcal{A}(V)$.

Definition 2. *Let $A \in End(V)$. The* ***precomposition of A****, denoted $A^\dagger$, with an algebraic curvature tensor $R(x, y, z, w)$ is defined $A^\dagger R(x, y, z, w) = R(Ax, Ay, Az, Aw)$. If A is invertible, we may instead define the precomposition of A with R as $A^{\dagger\dagger} R(x, y, z, w) = R(A^{-1}x, A^{-1}y, A^{-1}z, A^{-1}w)$. In some cases, the precompositions $A^\dagger$ and $A^{\dagger\dagger}$ may be used interchangably. In these cases, we denote the precomposition as A^*, and assume the form $A^{\dagger\dagger}$ in any arguments given, although the corresponding arguments for $A^\dagger$ can be easily verified.*

Definition 3. *Let R be an algebraic curvature tensor on a vector space V of dimension n. The* ***structure group*** *of R is denoted G_R and defined $G_R = \{A \in GL(n) | A^* R = R\}$.*

Definition 4. *A* ***Jordan block*** *of size k corresponding to some eigenvalue $\lambda \in \mathbb{R}$ on $\mathbb{R}^k$ is defined:*

$$J(k, \lambda) = \begin{bmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \lambda & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda \end{bmatrix}$$

The Jordan block corresponding to a pair of complex conjugate eigenvalues $a \pm b\sqrt{-1}$ is defined in the construction of the size 2k matrix:

$$J(k,a,b) = \begin{bmatrix} A & I & 0 & \dots & 0 \\ 0 & A & I & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & A \end{bmatrix}$$

where

$$A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

and

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Definition 5. *Let $\{A_i\}$ be a collection of square matrices, i=1,...,n. The **direct sum** of A_i is:*

$$\bigoplus_{i=1}^{n} A_i = \begin{bmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_n \end{bmatrix}$$

Lemma 1. *Let $A \in$ End(V). Choosing an appropriate basis for V, A will decompose as the direct sum of Jordan blocks. The unordered collection of these blocks is determined by A.*

Definition 6. *The **Jordan normal form** of A is the unordered collection of Jordan blocks from the preceding lemma.*

2 A Generalization of Previous Results

Lemma 2. *Let $A \in$ End(V). Define $T_A : \mathcal{A}(V) \to \mathcal{A}(V)$ by $T_A(R) = A^*R$. Then T_A is a linear transformation on $\mathcal{A}(V)$.*

Proof. Let R_1,$R_2 \in \mathcal{A}(V)$ and let c $\in \mathbb{R}$. Then:
$T_A((R_1 + R_2)(x,y,z,w)) = A^*(R_1 + R_2)(x,y,z,w) = (R_1 + R_2)(Ax,Ay,Az,Aw)$
$= R_1(Ax,Ay,Az,Aw) + R_2(Ax,Ay,Az,Aw) = A^*R_1(x,y,z,w) + A^*R_2(x,y,z,w)$
$= T_A(R_1(x,y,z,w)) + T_A(R_2(x,y,z,w))$.

Furthermore,
$T_A(cR_1(x,y,z,w)) = T_A(R_1(cx,y,z,w)) = A^*R_1(cx,y,z,w) = R_1(A(cx),Ay,Az,Aw)$
$= R_1(cAx,Ay,Az,Aw) = cR_1(Ax,Ay,Az,Aw) = cA^*R_1(x,y,z,w)$
$= cT_A(R_1(x,y,z,w))$. □

The following constitutes a generalization of the work done by Kaylor [1].

Definition 7. *Let $\mathscr{B} = \{\mathscr{R}_1, \mathscr{R}_2, ..., \mathscr{R}_N\}$ be an ordered basis for $\mathcal{A}(V)$ and let $A \in End(V)$. Define the* ***Kaylor matrix*** *of A with respect to $\mathscr{B}$:*

$$K_A = \begin{bmatrix} \beta_{11} & \beta_{21} & \dots & \beta_{N1} \\ \beta_{12} & \beta_{22} & \dots & \beta_{N2} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{1N} & \beta_{2N} & \dots & \beta_{NN} \end{bmatrix}$$

where β_{ij} is the coefficient corresponding to $\mathscr{R}_j$ when $A^*\mathscr{R}_i$ is expressed in terms of the basis $\mathscr{B}$.

Lemma 3. *K_A is the matrix representation of the linear transformation T_A with respect to the ordered basis $\mathscr{B}$.*

Proof. Let $R \in \mathcal{A}(V)$. Then $R = \sum_{i=1}^{N} \alpha_i \mathscr{R}_i$ where $\alpha_i \in \mathbb{R}$. Express R as a column vector with respect to $\mathscr{B}$:

$$R = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_N \end{bmatrix}$$

$T_A : \mathcal{A}(V) \to \mathcal{A}(V)$, so given $\mathscr{R}_i \in \mathscr{B}$, $T_A(\mathscr{R}_i) = A^*\mathscr{R}_i = \sum_{j=1}^{N} \beta_{ij}\mathscr{R}_j$ where $\beta_{ij} \in \mathbb{R}$. Then $T_A(R) = T_A(\sum_{i=1}^{N} \alpha_i \mathscr{R}_i) = \sum_{i=1}^{N} \alpha_i T_A(\mathscr{R}_i)$, since T_A is linear. But then $\sum_{i=1}^{N} \alpha_i T_A(\mathscr{R}_i) = \sum_{i=1}^{N} \alpha_i \sum_{j=1}^{N} \beta_{ij}\mathscr{R}_j = \sum_{j=1}^{N} \mathscr{R}_j \sum_{i=1}^{N} \alpha_i \beta_{ij}$. Expressed as a column vector with respect to $\mathscr{B}$:

$$T_A(R) = \begin{bmatrix} \sum_{i=1}^{N} \alpha_i \beta_{i1} \\ \sum_{i=1}^{N} \alpha_i \beta_{i2} \\ \vdots \\ \sum_{i=1}^{N} \alpha_i \beta_{iN} \end{bmatrix} = \begin{bmatrix} \beta_{11} & \beta_{21} & \dots & \beta_{N1} \\ \beta_{12} & \beta_{22} & \dots & \beta_{N2} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{1N} & \beta_{2N} & \dots & \beta_{NN} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_N \end{bmatrix} = K_A R$$

□

Theorem 1. *Let $\boldsymbol{x} = x_1\mathscr{R}_1 + ... + x_N\mathscr{R}_N$. If there exists a nonzero algebraic curvature tensor R such that $A \in G_R$, then $K_A\boldsymbol{x} = \boldsymbol{x}$ has a non-trivial solution. [1]*

Corollary 1. *The solution space of $K_A\boldsymbol{x} = \boldsymbol{x}$ is the set of all algebraic curvature tensors R such that $A \in G_R$. [1]*

3 Results

Fact. $K_I = I$

Proof. Let R $\in \mathcal{A}(V)$. Then:
$K_I R(x,y,z,w) = I^* R(x,y,z,w) = R(I^{-1}x, I^{-1}y, I^{-1}z, I^{-1}w) = R(x,y,z,w) = IR(x,y,z,w).$ $\square$

Fact. $K_{-A} = K_A$

Proof. Let R $\in \mathcal{A}(V)$. Then:
$K_{-A}R(x,y,z,w) = (-A)^* R(x,y,z,w) = R((-A)^{-1}x, (-A)^{-1}y, (-A)^{-1}z, (-A)^{-1}w)$
$= R(-A^{-1}x, -A^{-1}y, -A^{-1}z, -A^{-1}w) = (-1)^4 R(A^{-1}x, A^{-1}y, A^{-1}z, A^{-1}w)$
$= A^* R(x,y,z,w) = K_A R(x,y,z,w).$ $\square$

Theorem 2. *Let A,B $\in$ End(V) and let K_A and K_B be the matrix representations of $T_A(R) = A^\dagger R$ and $T_B(R) = B^\dagger R$ with respect to the basis $\mathscr{B}$. Then the matrix representation of $T_{AB}(R) = (AB)^\dagger R$ with respect to $\mathscr{B}$ is $K_{AB} = K_B K_A$.*

Proof. Let R $\in \mathcal{A}(V)$. Then $K_{AB}R(x,y,z,w) = (AB)^\dagger R(x,y,z,w)$
$= R(ABx, ABy, ABz, ABw) = A^\dagger R(Bx, By, Bz, Bw) = B^\dagger(A^\dagger R(x,y,z,w))$
$= K_B K_A R(x,y,z,w).$ $\square$

Corollary 2. *Let A,B $\in$ GL(n) and let K_A and K_B be the matrix representations of $T_A(R) = A^{\dagger\dagger} R$ and $T_B(R) = B^{\dagger\dagger} R$ with respect to the basis $\mathscr{B}$. Then the matrix representation of $T_{AB}(R) = (AB)^{\dagger\dagger} R$ with respect to $\mathscr{B}$ is $K_{AB} = K_A K_B$.*

Proof. Let R $\in \mathcal{A}(V)$. Then:
$K_{AB}R(x,y,z,w) = (AB)^{\dagger\dagger} R(x,y,z,w) = R((AB)^{-1}x, (AB)^{-1}y, (AB)^{-1}z, (AB)^{-1}w)$
$= R(B^{-1}A^{-1}x, B^{-1}A^{-1}y, B^{-1}A^{-1}z, B^{-1}A^{-1}w) = B^{\dagger\dagger} R(A^{-1}x, A^{-1}y, A^{-1}z, A^{-1}w)$
$= A^{\dagger\dagger}(B^{\dagger\dagger} R(x,y,z,w)) = K_A K_B R(x,y,z,w).$ $\square$

Corollary 3. *If A $\in$ GL(n) then $K_{A^{-1}} = (K_A)^{-1}$.*

Proof. $K_{A^{-1}} K_A = K_{A^{-1}A} = K_I = I$ and $K_A K_{A^{-1}} = K_{AA^{-1}} = K_I = I$. $\square$

Theorem 3. *If G is a subgroup of GL(n) then $K_G = \{K_A | A \in G\}$ is also a group.*

Proof. G is a group so I $\in$ G, and so $K_I = I \in K_G$. Let $K_A \in K_G$. Then A $\in$ G and $A^{-1} \in G$ so $K_{A^{-1}} = (K_A)^{-1} \in K_G$. Let $K_A, K_B \in K_G$. Then A,B $\in G$ so AB $\in$ G and so $K_A K_B = K_{AB} \in K_G$. Let $K_A, K_B, K_C \in K_G$. Then $K_A(K_B K_C) = (K_A K_B)K_C$ since matrix multiplication is associative. $\square$

Lemma 4. *Let $\mathscr{B} = \{e_1, e_2, ..., e_n\}$ be a basis for a vector space V. Then given $e_i, e_j \in \mathscr{B}$, $e_i \neq e_j$, there exists a symmetric bilinear form ϕ such that the canonical algebraic curvature tensor $R_\phi(x,y,z,w) = \phi(x,w)\phi(y,z) - \phi(x,z)\phi(y,w)$ has $R_\phi(e_i, e_j, e_j, e_i) = 1$ and, up to symmetries of algebraic curvature tensors, all other $R_\phi(e_k, e_l, e_m, e_p) = 0$.*

Proof. Let $\phi : V \times V \to \mathbb{R}$ be defined as follows: $\phi(e_i, e_i) = \phi(e_j, e_j) = 1$ and $\phi(e_k, e_l) = 0$ for all other basis vector pairs. Then ϕ is a symmetric bilinear form, so we can form the canonical algebraic curvature tensor $R_\phi(x, y, z, w) = \phi(x, w)\phi(y, z) - \phi(x, z)\phi(y, w)$. Then $R_\phi(e_i, e_j, e_j, e_i) = 1$ and, up to symmetries of algebraic curvature tensors, all other $R_\phi(e_k, e_l, e_m, e_p) = 0$. □

Lemma 5. *Let $\mathscr{B} = \{e_1, e_2, ..., e_n\}$ be a basis for a vector space V. Then given distinct $e_i, e_j, e_k \in \mathscr{B}$, there exists an algebraic curvature tensor $\bar{R} \in \mathcal{A}(V)$ such that $\bar{R}(e_i, e_j, e_k, e_i) = 1$ and, up to symmetries of algebraic curvature tensors, all other $\bar{R}(e_l, e_m, e_p, e_q) = 0$.*

Proof. Let $\phi : V \times V \to \mathbb{R}$ be defined as follows: $\phi(e_i, e_i) = \phi(e_j, e_k) = \phi(e_k, e_j) = 1$ and $\phi(e_l, e_m) = 0$ for all other basis vector pairs. Then ϕ is a symmetric bilinear form, so we can define the canonical algebraic curvature tensor $R_\phi(x, y, z, w) = \phi(x, w)\phi(y, z) - \phi(x, z)\phi(y, w)$. Then $R_\phi(e_i, e_j, e_k, e_i) = 1$, and if E_s is any other permutation of $\{e_i, e_i, e_j, e_k\}$ then $R_\phi(E_s)$ is predetermined by the symmetries of algebraic curvature tensors. It is clear from the definition of R_ϕ that all other $R_\phi(e_l, e_m, e_p, e_q) = 0$ except $R_\phi(e_j, e_k, e_k, e_j) = -1$ and its nonzero associates. According to the previous lemma, there exists an algebraic curvature tensor $R_\psi \in \mathcal{A}(V)$ such that $R_\psi(e_j, e_k, e_k, e_j) = 1$ and all other $R_\psi(e_l, e_m, e_p, e_q) = 0$, up to symmetries of algebraic curvature tensors. Then $\bar{R} = R_\phi + R_\psi$ has:
$\bar{R}(e_i, e_j, e_k, e_i) = R_\phi(e_i, e_j, e_k, e_i) + R_\psi(e_i, e_j, e_k, e_i) = 1 + 0 = 1$,
$\bar{R}(e_j, e_k, e_k, e_j) = R_\phi(e_j, e_k, e_k, e_j) + R_\psi(e_j, e_k, e_k, e_j) = -1 + 1 = 0$, and
$\bar{R}(e_l, e_m, e_p, e_q) = R_\phi(e_l, e_m, e_p, e_q) + R_\psi(e_l, e_m, e_p, e_q) = 0 + 0 = 0$
for all other $\bar{R}(e_l, e_m, e_p, e_q)$, up to symmetries of algebraic curvature tensors. □

Theorem 4. *If $A \in GL(n)$, $n \geq 3$, then $K_A = I \Leftrightarrow A = \pm I$.*

Proof. ($\Leftarrow$) This follows directly from the two previously stated facts.
($\Rightarrow$) Suppose $A \neq \pm I$. Then $A^{-1} \neq \pm I$, so the Jordan Normal form of A^{-1} has a Jordan block that isn't either $J(1, 1)$ or J(1,-1). Call this block A_1. A_1 has one of the following forms:
(i) $J(1, \lambda)$, $\lambda \neq \pm 1$
(ii) $J(2, \lambda)$
(iii) $J(m, \lambda)$, $m \in \mathbb{Z}$, $3 \leq m \leq n$
(iv) $J(1, a, b)$
(v) $J(p, a, b)$, $p \in \mathbb{Z}$, $2 \leq p \leq \frac{n}{2}$

(Note that none of the Jordan blocks of A^{-1} can have eigenvalue zero, since A^{-1} is invertible, and additionally that none of the Jordan blocks with complex eigenvalues can have imaginary part zero, since they would not be complex.)

(i) Suppose $A_1 = J(1, \lambda), \lambda \neq \pm 1$. Choose an ordered basis $\mathscr{B} = \{e_1, e_2, ..., e_n\}$ for V such that, reading down and to the right along the diagonal of the matrix representation of A^{-1}, A_1 appears first and the remaining Jordan blocks appear in the following order: (1) any other $J(1,\eta)$, (2) any $J(m,\eta), m \in \mathbb{Z}, 3 \le m \le n-1$, (3) any $J(p,a,b), p \in \mathbb{Z}, 1 \le p \le \frac{n-1}{2}$. $n \ge 3$, so A^{-1} has another Jordan block, A_2, in the second diagonal entry of A^{-1}. A_2 will have form like that of (1),(2), or (3) above.

(i)(1) Suppose $A_2 = J(1,\eta)$. Then A^{-1} has another Jordan block A_3 in the third diagonal position with form either (a) $J(m,\gamma), 1 \le m \le n-2$, or (b) $J(p,a,b), p \in \mathbb{Z}, 1 \le p \le \frac{n-2}{2}$.

(i)(1)(a) Suppose $A_3 = J(m,\gamma), 1 \le m \le n-2$. Then A^{-1} has the form:

$$A^{-1} = \begin{bmatrix} \lambda & 0 & 0 & \dots & 0 \\ 0 & \eta & 0 & \dots & 0 \\ 0 & 0 & \gamma & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \end{bmatrix}$$

So $A^{-1}e_1 = \lambda e_1$, $A^{-1}e_2 = \eta e_2$, and $A^{-1}e_3 = \gamma e_3$. Then
$A^{\dagger\dagger}R(e_1,e_2,e_2,e_1) = R(A^{-1}e_1, A^{-1}e_2, A^{-1}e_2, A^{-1}e_1) = R(\lambda e_1, \eta e_2, \eta e_2, \lambda e_1)$
$= \lambda^2\eta^2 R(e_1,e_2,e_2,e_1)$.

Similarly, $A^{\dagger\dagger}R(e_1,e_3,e_3,e_1) = R(A^{-1}e_1, A^{-1}e_3, A^{-1}e_3, A^{-1}e_1) = R(\lambda e_1, \gamma e_3, \gamma e_3, \lambda e_1)$
$= \lambda^2\gamma^2 R(e_1,e_3,e_3,e_1)$ and $A^{\dagger\dagger}R(e_2,e_3,e_3,e_2) = R(A^{-1}e_2, A^{-1}e_3, A^{-1}e_3, A^{-1}e_2)$
$= R(\eta e_2, \gamma e_3, \gamma e_3, \eta e_2) = \eta^2\gamma^2 R(e_2,e_3,e_3,e_2)$.

If $K_A = -I$ then $A^{\dagger\dagger}R(e_1,e_2,e_2,e_1) = -R(e_1,e_2,e_2,e_1)$, but $A^{\dagger\dagger}R(e_1,e_2,e_2,e_1) = \lambda^2\eta^2 R(e_1,e_2,e_2,e_1)$, so this is impossible. If $K_A = I$ then $A^{\dagger\dagger}R(e_1,e_2,e_2,e_1)$
$= R(e_1,e_2,e_2,e_1)$, $A^{\dagger\dagger}R(e_1,e_3,e_3,e_1) = R(e_1,e_3,e_3,e_1)$, and $A^{\dagger\dagger}R(e_2,e_3,e_3,e_2) = R(e_2,e_3,e_3,e_2)$, so $\lambda^2\eta^2 = \lambda^2\gamma^2 = \eta^2\gamma^2 = 1$. Thus $\lambda^2 = \gamma^2$, so $1 = \lambda^2\gamma^2 = \lambda^4$. But $\lambda \ne \pm 1$, so this is impossible.

(i)(1)(b) Suppose $A_3 = J(p,a,b), p \in \mathbb{Z}, 1 \le p \le \frac{n-2}{2}, 1 \le m \le n-2$. Then A^{-1} has the form:

$$A^{-1} = \begin{bmatrix} \lambda & 0 & 0 & 0 & \dots & 0 \\ 0 & \eta & 0 & 0 & \dots & 0 \\ 0 & 0 & a & b & \dots & 0 \\ 0 & 0 & -b & a & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \end{bmatrix}$$

So $A^{-1}e_1 = \lambda e_1$, $A^{-1}e_2 = \eta e_2$, and $A^{-1}e_3 = ae_3 - be_4$. Then $A^{\dagger\dagger}R(e_1,e_3,e_3,e_1) = R(A^{-1}e_1, A^{-1}e_3, A^{-1}e_3, A^{-1}e_1) = R(\lambda e_1, ae_3 - be_4, ae_3 - be_4, \lambda e_1) = \lambda^2a^2R(e_1,e_3,e_3,e_1) - 2\lambda^2 ab R(e_1,e_3,e_4,e_1) + \lambda^2b^2R(e_1,e_4,e_4,e_1)$.

Let R_{ijji} denote the algebraic curvature tensor constructed in lemma 3, $R_{ijji} = R_\phi$ for the appropriate ϕ, and let R_{ijki} denote the algebraic curvature tensor constructed in lemma 4, $R_{ijki} = \bar{R}$ for the appropriate $\bar{R}$. Let $R = R_{1331} + \frac{a}{2b}R_{1341}$. Then $R(e_1,e_3,e_3,e_1) = 1$, but $A^{\dagger\dagger}R(e_1,e_3,e_3,e_1) = \lambda^2a^2R(e_1,e_3,e_3,e_1) - 2\lambda^2abR(e_1,e_3,e_4,e_1) +$

$\lambda^2 b^2 R(e_1, e_4, e_4, e_1) = \lambda^2 a^2(1) - 2\lambda^2 ab(\frac{a}{2b}) + \lambda^2 b^2(0) = \lambda^2 a^2 - \lambda^2 a^2 = 0$. Thus $A^{\dagger\dagger}R \neq R$ and $A^{\dagger\dagger}R \neq -R$, so $K_A \neq \pm I$.

(i)(2) Now suppose $A_2 = J(m, \eta), 2 \leq m \leq n-1$. Then A^{-1} has the form:

$$A^{-1} = \begin{bmatrix} \lambda & 0 & 0 & \dots & 0 \\ 0 & \eta & 1 & \dots & 0 \\ 0 & 0 & \eta & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \end{bmatrix}$$

So $A^{-1}e_1 = \lambda e_1, A^{-1}e_2 = \eta e_2$, and $A^{-1}e_3 = e_2 + \eta e3$. Then $A^{\dagger\dagger}R(e_1, e_3, e_3, e_1) = R(A^{-1}e_1, A^{-1}e_3, A^{-1}e_3, A^{-1}e_1) = R(\lambda e_1, e_2 + \eta e3, e_2 + \eta e3, \lambda e_1) = \lambda^2 R(e_1, e_2, e_2, e_1) + \lambda^2\eta^2 R(e_1, e_3, e_3, e_1) + 2\lambda^2\eta R(e_1, e_2, e_3, e_1)$. Let $R = -\eta^2 R_{1221} + R_{1331}$. Then $R(e_1, e_3, e_3, e_1) = 1$, but $A^{\dagger\dagger}R(e_1, e_3, e_3, e_1) = \lambda^2 R(e_1, e_2, e_2, e_1) + \lambda^2\eta^2 R(e_1, e_3, e_3, e_1) + 2\lambda^2\eta R(e_1, e_2, e_3, e_1) = \lambda^2(-\eta^2) + \lambda^2\eta^2(1) + 2\lambda^2\eta(0) = -\lambda^2\eta^2 + \lambda^2\eta^2 = 0$. Thus $A^{\dagger\dagger}R \neq R$ and $A^{\dagger\dagger}R \neq -R$, so $K_A \neq \pm I$.

(i)(3) Now suppose $A_2 = J(p, a, b), p \in \mathbb{Z}, 1 \leq p \leq \frac{n-1}{2}$. Then A^{-1} has the form:

$$A^{-1} = \begin{bmatrix} \lambda & 0 & 0 & \dots & 0 \\ 0 & a & b & \dots & 0 \\ 0 & -b & a & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \end{bmatrix}$$

So $A^{-1}e_1 = \lambda e_1$ and $A^{-1}e_2 = ae_2 - be_3$. Then $A^{\dagger\dagger}R(e_1, e_2, e_2, e_1)$ $= R(A^{-1}e_1, A^{-1}e_2, A^{-1}e_2, A^{-1}e_1) = R(\lambda e_1, ae_2 - be_3, ae_2 - be_3, \lambda e_1) = \lambda^2 a^2 R(e_1, e_2, e_2, e_1) + \lambda^2 b^2 R(e_1, e_3, e_3, e_1) - 2\lambda^2 ab R(e_1, e_2, e_3, e_1)$.

Let $R = R_{1221} + \frac{a}{2b} R_{1231}$. Then $R(e_1, e_2, e_2, e_1) = 1$, but $A^{\dagger\dagger}R(e_1, e_2, e_2, e_1) = \lambda^2 a^2 R(e_1, e_2, e_2, e_1) + \lambda^2 b^2 R(e_1, e_3, e_3, e_1) - 2\lambda^2 ab R(e_1, e_2, e_3, e_1) = \lambda^2 a^2(1) + \lambda^2 b^2(0) - 2\lambda^2 ab(\frac{a}{2b}) = \lambda^2 a^2 - \lambda^2 a^2 = 0$

Thus $A^{\dagger\dagger}R \neq R$ and $A^{\dagger\dagger}R \neq -R$, so $K_A \neq \pm I$.

(ii)Now, when $A_1 = J(2, \lambda)$, A_2 has one of the following forms: (1) $J(m, \eta), 1 \leq m \leq n-2$, or (2) $J(p, a, b), 1 \leq p \leq \frac{n-2}{2}$.

(ii)(1) Suppose $A_2 = J(m, \eta), 1 \leq m \leq n-2$. Then A^{-1} has the form:

$$A^{-1} = \begin{bmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \lambda & 0 & \dots & 0 \\ 0 & 0 & \eta & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \end{bmatrix}$$

So $A^{-1}e_2 = e_1 + \lambda e_2$ and $A^{-1}e_3 = \eta e_3$. Then $A^{\dagger\dagger}R(e_2, e_3, e_3, e_2)$
$= R(A^{-1}e_2, A^{-1}e_3, A^{-1}e_3, A^{-1}e_2) = R(e_1+\lambda e_2, \eta e_3, \eta e_3, e_1+\lambda e_2) = \lambda^2\eta^2 R(e_2, e_3, e_3, e_2)+ \eta^2 R(e_1, e_3, e_3, e_1) + 2\lambda\eta^2 R(e_3, e_1, e_2, e_3)$.

Let $R = R_{2332} - \lambda^2 R_{1331}$. Then $R(e_2, e_3, e_3, e_2) = 1$, but $A^{\dagger\dagger}R(e_2, e_3, e_3, e_2) = \lambda^2\eta^2 R(e_2, e_3, e_3, e_2) + \eta^2 R(e_1, e_3, e_3, e_1) + 2\lambda\eta^2 R(e_3, e_1, e_2, e_3) = \lambda^2\eta^2(1) + \eta^2(-\lambda^2) + 2\lambda\eta^2(0) = \lambda^2\eta^2 - \lambda^2\eta^2 = 0$.

Thus $A^{\dagger\dagger}R \neq R$ and $A^{\dagger\dagger}R \neq -R$, so $K_A \neq \pm I$.

(ii)(2) Now suppose $A_2 = J(p, a, b), 1 \leq p \leq \frac{n-2}{2}$. Then A^{-1} has the form:

$$A^{-1} = \begin{bmatrix} \lambda & 1 & 0 & 0 & \dots & 0 \\ 0 & \lambda & 0 & 0 & \dots & 0 \\ 0 & 0 & a & b & \dots & 0 \\ 0 & 0 & -b & a & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \end{bmatrix}$$

So $A^{-1}e_1 = \lambda e_1$ and $A^{-1}e_3 = ae_3 - be_4$. Then $A^{\dagger\dagger}R(e_1, e_3, e_3, e_1)$
$= R(A^{-1}e_1, A^{-1}e_3, A^{-1}e_3, A^{-1}e_1) = R(\lambda e_1, ae_3-be_4, ae_3-be_4, \lambda e_1) = \lambda^2 b^2 R(e_1, e_4, e_4, e_1)+ \lambda^2 a^2 R(e_1, e_3, e_3, e_1) - 2\lambda^2 ab R(e_1, e_3, e_4, e_1)$.

Let $R = R_{1331} + \frac{a}{2b} R_{1341}$. Then $R(e_1, e_3, e_3, e_1) = 1$, but $A^{\dagger\dagger}R(e_1, e_3, e_3, e_1) = \lambda^2 b^2 R(e_1, e_4, e_4, e_1) + \lambda^2 a^2 R(e_1, e_3, e_3, e_1) - 2\lambda^2 ab R(e_1, e_3, e_4, e_1) = \lambda^2 b^2(0) + \lambda^2 a^2(1) - 2\lambda^2 ab(\frac{a}{2b}) = \lambda^2 a^2 - \lambda^2 a^2 = 0$. Thus $A^{\dagger\dagger}R \neq R$ and $A^{\dagger\dagger}R \neq -R$, so $K_A \neq \pm I$.

(iii) Let $A_1 = J(m, \lambda), 3 \leq m \leq n$. Then A^{-1} has the form:

$$A^{-1} = \begin{bmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \lambda & 1 & \dots & 0 \\ 0 & 0 & \lambda & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \end{bmatrix}$$

So $A^{-1}e_1 = \lambda e_1$ and $A^{-1}e_3 = e_2 + \lambda e_3$. Then $A^{\dagger\dagger}R(e_1, e_3, e_3, e_1)$
$= R(A^{-1}e_1, A^{-1}e_3, A^{-1}e_3, A^{-1}e_1) = R(\lambda e_1, e_2+\lambda e_3, e_2+\lambda e_3, \lambda e_1) = \lambda^2 R(e_1, e_2, e_2, e_1)+ 2\lambda^3 R(e_1, e_2, e_3, e_1) + \lambda^4 R(e_1, e_3, e_3, e_1)$.

Let $R = -\lambda^2 R_{1221} + R_{1331}$. Then $R(e_1, e_3, e_3, e_1) = 1$, but $A^{\dagger\dagger}R(e_1, e_3, e_3, e_1) = \lambda^2 R(e_1, e_2, e_2, e_1) + 2\lambda^3 R(e_1, e_2, e_3, e_1) + \lambda^4 R(e_1, e_3, e_3, e_1) = \lambda^2(-\lambda^2) + 2\lambda^3(0) + \lambda^4(1) = -\lambda^4 + \lambda^4 = 0$. Thus $A^{\dagger\dagger}R \neq R$ and $A^{\dagger\dagger}R \neq -R$, so $K_A \neq \pm I$.

(iv) Let $A_1 = J(1, a, b)$. Then A_2 has one of the following forms:
(1) $J(m, \lambda), 1 \leq m \leq n-2$, or (2) $J(q, c, d), 1 \leq q \leq \frac{n-2}{2}$.

(iv)(1) Let $A_2 = J(m, \lambda), 1 \leq m \leq n-2$. Then A^{-1} has the form:

$$A^{-1} = \begin{bmatrix} a & b & 0 & \dots & 0 \\ -b & a & 0 & \dots & 0 \\ 0 & 0 & \lambda & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \end{bmatrix}$$

So $A^{-1}e_1 = ae_1 - be_2$ and $A^{-1}e_3 = \lambda e_3$. Then $A^{\dagger\dagger}R(e_1, e_3, e_3, e_1)$ $= R(A^{-1}e_3, A^{-1}e_1, A^{-1}e_2, A^{-1}e_3) = R(ae_1 - be_2, \lambda e_3, \lambda e_3, ae_1 - be_2) = \lambda^2 a^2 R(e_1, e_3, e_3, e_1) + \lambda^2 b^2 R(e_2, e_3, e_3, e_2) - 2ab\lambda^2 R(e_3, e_1, e_2, e_3)$.

Let $R = \frac{a}{2b} R_{3123} + R_{1331}$. Then $R(e_1, e_3, e_3, e_1) = 1$, but $A^{\dagger\dagger}R(e_1, e_3, e_3, e_1) = \lambda^2 a^2 R(e_1, e_3, e_3, e_1) + \lambda^2 b^2 R(e_2, e_3, e_3, e_2) - 2ab\lambda^2 R(e_3, e_1, e_2, e_3) = \lambda^2 a^2(1) + \lambda^2 b^2(0) - 2ab\lambda^2(\frac{a}{2b}) = \lambda^2 a^2 - \lambda^2 a^2 = 0$. Thus $A^{\dagger\dagger}R \neq R$ and $A^{\dagger\dagger}R \neq -R$, so $K_A \neq \pm I$.

(iv)(2) Let $A_2 = J(q, c, d), 1 \leq q \leq \frac{n-2}{2}$. Then A^{-1} has the form:

$$A^{-1} = \begin{bmatrix} a & b & 0 & 0 & \dots & 0 \\ -b & a & 0 & 0 & \dots & 0 \\ 0 & 0 & c & d & \dots & 0 \\ 0 & 0 & -d & c & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \end{bmatrix}$$

So $A^{-1}e_1 = ae_1 - be_2$, $A^{-1}e_2 = be_1 + ae_2$ and $A^{-1}e_3 = ce_3 - de_4$. Then $A^{\dagger\dagger}R(e_1, e_2, e_3, e_1) = R(A^{-1}e_1, A^{-1}e_2, A^{-1}e_3, A^{-1}e_1) = R(ae_1 - be_2, be_1 + ae_2, ce_3 - de_4, ae_1 - be_2) = ac(a^2 + b^2)R(e_1, e_2, e_3, e_1) + bc(a^2 + b^2)R(e_2, e_1, e_3, e_2) - ad(a^2 + b^2)R(e_1, e_2, e_4, e_1) - bd(a^2 + b^2)R(e_2, e_1, e_4, e_2)$.

Let $R = \frac{ac}{bd} R_{2142} + R_{1231}$. Then $R(e_1, e_2, e_3, e_1) = 1$, but $A^{\dagger\dagger}R(e_1, e_2, e_3, e_1) = ac(a^2+b^2)R(e_1, e_2, e_3, e_1) + bc(a^2+b^2)R(e_2, e_1, e_3, e_2) - ad(a^2+b^2)R(e_1, e_2, e_4, e_1) - bd(a^2+b^2)R(e_2, e_1, e_4, e_2) = ac(a^2 + b^2)(1) + bc(a^2 + b^2)(0) - ad(a^2 + b^2)(0) - bd(a^2 + b^2)(\frac{ac}{bd}) = ac(a^2 + b^2) - ac(a^2 + b^2) = 0$. Thus $A^{\dagger\dagger}R \neq R$ and $A^{\dagger\dagger}R \neq -R$, so $K_A \neq \pm I$.

(v) Let $J(p, a, b)$, $2 \leq p \leq \frac{n}{2}$. Then A^{-1} has the form:

$$A^{-1} = \begin{bmatrix} a & b & 1 & 0 & \dots & 0 \\ -b & a & 0 & 1 & \dots & 0 \\ 0 & 0 & a & b & \dots & 0 \\ 0 & 0 & -b & a & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \end{bmatrix}$$

So $A^{-1}e_1 = ae_1 - be_2$, $A^{-1}e_2 = be_1 + ae_2$ and $A^{-1}e_3 = e_1 + ae_3 - be_4$. Then $A^{\dagger\dagger}R(e_1, e_2, e_3, e_1) = R(A^{-1}e_1, A^{-1}e_2, A^{-1}e_3, A^{-1}e_1) = R(ae_1 - be_2, be_1 + ae_2, e_1 +$

$ae_3 - be_4, ae_1 - be_2) = b(a^2+b^2)R(e_1,e_2,e_2,e_1) + a^2(a^2+b^2)R(e_1,e_2,e_3,e_1) + ab(a^2+b^2)R(e_2,e_1,e_3,e_2) - ab(a^2+b^2)R(e_1,e_2,e_4,e_1) - b^2(a^2+b^2)R(e_2,e_1,e_4,e_2)$.

Let $R = \frac{a^2}{b^2}R_{2142} + R_{1231}$. Then $R(e_1,e_2,e_3,e_1) = 1$, but $A^{\dagger\dagger}R(e_1,e_2,e_3,e_1) = b(a^2+b^2)R(e_1,e_2,e_2,e_1)+a^2(a^2+b^2)R(e_1,e_2,e_3,e_1)+ab(a^2+b^2)R(e_2,e_1,e_3,e_2)-ab(a^2+b^2)R(e_1,e_2,e_4,e_1) - b^2(a^2+b^2)R(e_2,e_1,e_4,e_2) = b(a^2+b^2)(0) + a^2(a^2+b^2)(1) + ab(a^2+b^2)(0) - ab(a^2+b^2)(0) - b^2(a^2+b^2)(\frac{a^2}{b^2}) = a^2(a^2+b^2) - a^2(a^2+b^2)$=0.

Thus $A^{\dagger\dagger}R \neq R$ and $A^{\dagger\dagger}R \neq -R$, so $K_A \neq \pm I$.

□

Corollary 4. *Define $U : G \to K_G$ by $U(A) = K_A$. Then U is an onto homomorphism of groups, $Ker(U) = \{\pm I\}$, and so $G/\{\pm I\} \cong K_G$.*

Proof. By construction, $K_A \in K_G \Rightarrow A \in G$ and $U(A) = K_A$ so U is onto. Let $A, B \in G$. Then $U(AB) = K_{AB} = K_A K_B = U(A)U(B)$. So U is a group homomorphism.
$Ker(U) = \{A \in G | U(A) = K_A = I\} = \{\pm I\}$ by the previous theorem, so $G/\{\pm I\} \cong K_G$ by the First Isomorphism Theorem. □

4 Open Questions

1. Given $A \in End(V)$, A not necessarily invertible, if $A \in G_R$ for some R and $Ker(R) = 0$, must it be that $Ker(A) = 0$?

2. For $A \in GL(n)$, does the Jordan normal form of A determine or constrain the Jordan normal form of K_A?

3. Given some number of elements from a group $G \in GL(n)$, can the study of Kaylor matrices determine whether or not G is the structure group for some algebraic curvature tensor R?

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6 References

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