Fully Augmented Links

John Harnois

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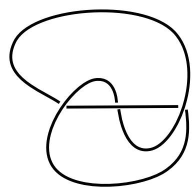
Abstract

This paper is meant to establish basic background knowledge for incoming REU students to be able to work on hyperbolic fully augmented links. Then it dwells into some new results dealing with a generalaztion of Purcell's method of viewing fully augmented links through triangulations. It wraps up with possible future topics for students to be able to research.

1 Knots

Imagine a piece of vertical string laid flat out on a table from top to bottom. Jumble it all up, then glue the end points together. There is a chance you are knot imaginative and your result looks just like a circle lying on the table know. What you have created is the *uknot*. If however you have managed to cross this little guy all around there is a good chance you do not have the unknot. Either way, what you have just created is known as a *knot*. At certain points the knot will overlap itself (unless you just have a circle). We call these points *crossings*. In the figure drawn below you should be able to identify three crossings.



We say that knots are equivalent to one another if they can bend and twist without breaking to form one another. More technically this means there exists a *homeomorphism* (bijective, bicontinous function, wiki topology for more info) between the knots. Notice how this allows for wildly different looking representations of the same knot. For example we could create an uknot with one thousand crossings. These different represantations are called the *diagrams* of the knots.

If you were to pick a point on the knot and trace the rest of the knot out, starting and ending at that point, you will notice that you go over and under the knot at the crossings. If there exists a diagram of a knot such that you alternate between going, once over, once under, once over, etc., then the knot is said to be *alternating*. It is important to realize that alternating knots can have non-alternating diagrams, the converse of this statement is false. Notice how the previously drawn knot is alternating. Can you manage to find a diagram of it that is not alternating?

Knots are not where knot theory ends, however. When you think about more than one closed path at a time (multiple knots) then what you are thinking about are *links*. Links play just as heavy a role in knot theory as knots. It should be noted that even if two knots are not intertwined together, if you think about them together in the same diagram, then they together, still form a link. In the remaining paper, if I am not to explicitally state knot or link, it is safe to assume I will be talking about both. Finally all previously stated vocab (excluding the uknot) for knots also applies to links in an intuitive fashion.

2 Invariants

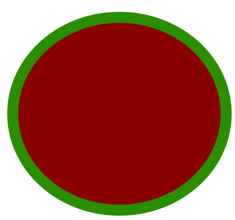
If a women was to emerge from a forest with a wolf over her shoulder and hand you two tangled messes following with a question of, "are these knots the same or are they different," what would you do? Sans the wolf lady, this is one of the most important questions in knot theory. How do you distinguish knots? Attempts at answering it has resulted in *invariants*. We have already covered one invariant, alternating knots. Knowing a knot is alternating allows one to distinguish it from all non-alternating knots. This is what invariants do, if you say one knot has one invariant and another knot lacks that invariant, then they cannot be the same knot. A concrete definition of invariants is overly technical and does not aid intuition, so I shall emit it here.

There are other invariants for knots such as arc index, stick number, linking number, jones polynomial, twist number, and hyperbolic volume. *Hyperbolic volume* and *twist number* will be of special intrest for this paper.

3 Hyperbolic Volume

Knots are often broken up into three categories; *hyperbolic*, *satelite*, and *torus*. Out of these three only hyperbolic have well defined hyperbolic volume. What is hyperbolic volume you ask? Before going through that one must first understand what is meant by $\mathbb{S}^3$.

$\mathbb{S}^3$ refers to the three sphere (sphere in three dimensions). A sphere in this context is the boundary of a ball. For example the following circle along with its interior is a 2-ball (red and green), while its boundary (green) is a 1-sphere.



The analogue of this will be a four dimensional ball, $\mathbb{B}^4$, along with its three dimensional boundary, $\mathbb{S}^3$. Thinking about the boundary of anything four dimensional is not easy. Luckily $\mathbb{S}^3$ can also be thought of as the upper half of $\mathbb{R}^3$ with ∞ , or simply, $\mathbb{H}^3$. This space does not use the usual euclidean metric, instead it takes on a hyperbolic metric.

You now have the tools to be able to understand the basics of hyperbolic volume. Suppose you are given a hyperbolic link, L. If you were to look at $\mathbb{S}^3 - L$ the resulting structure will be a hyperbolic polyhedra with finite volume. This is the volume associated to our link L.

4 Braids

There are certain representations of knots that can be thought of as vertical strings that connect to themselves, these are known as *braids*. Any knot can be represented in a braid form. We will specifically discuss three braids in this section.

Three braids can be constructed by placing three vertical strands flat from top to bottom.

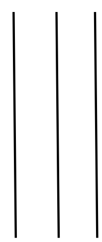
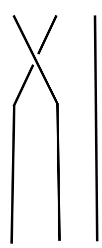


Figure 1: captiontest

Then taking a strand and placing it over an adjacent strand. For example we could take the first strand and cross it over the second (σ_1). See figure 1 (this is a test)

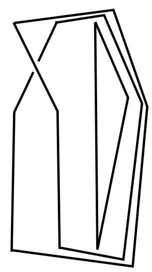


There are also three more crossings we can make; the second over the first (σ_1^{-1}), the second over the third (σ_2), and the third over the second (σ_2^{-1}). It should be noted that some identify taking a strand under its right neighbor as σ_n instead of σ_n^{-1}. Along with the *identity* (σ_1^0) these over lappings form a group. Using products of these operations, *words*, you can describe any three braid. It has been shown that all three braids can be formed with the words:

$$(\sigma_1\sigma_2)^{3n}\sigma_1^{a_1}\sigma_2^{b_1}\sigma_1^{a_2}...\sigma_1^{a_j}\sigma_2^{b_j},$$

where $n, a, b \in \mathbb{Z}$.

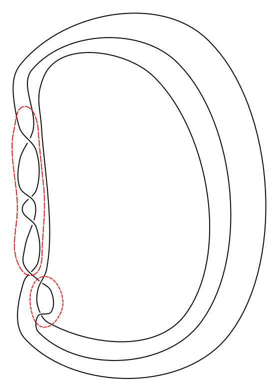
Once the braids are formed they may be made into their corresponding links by attaching bottoms to tops.



Of course, there also exist n braids. The algebras on these braids is extended canonically, where any n braid can be described using words consisting of: (σ_1^{-n+1}),...,(σ_1^0),...,(σ_1^{n-1}).

5 Twists

Links often form *twists*. A twist is intuitively just a sequence of crossings between two strands. In the following diagram the two circled regions are twists, for every region we add one to something called the *twist number*, thus we say the diagram has twist number two.

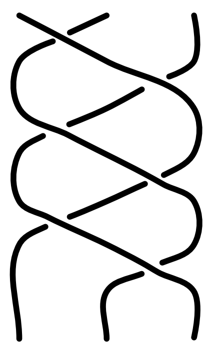


A more rigorous way to define twists is through *bigons*. A bigon is the region that is formed between two crossings. Notice how the top twist has a sequence of three bigons while the bottom only has one. Twists are formed by maximal sequences of bigons. That is to say, if you look at the section of the top twist that has a sequence of just two bigons, then you will not be looking at the whole twist because the sequence can be made bigger.

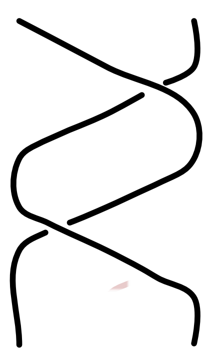
Note however, that this definition can be tricky at times. If it is the case that a twist has only one crossing, then there will be no bigons. In this scenario the twist can be thought of as a sequence of zero bigons. We call this a *half twist*, contrary to a possible interpretation, it still adds one to the twist number.

There exist representations of knots called *twist reduced diagrams*. They are diagrams which have the smallest number of twists for any represantation of a knot. We say that the twist number of the knot is the twist number of its twist reduced diagram. The twist number of a knot is an invariant, not the twist number of a diagram.

There is a notion of *generalized twists* as well. This is were we look at crossings of more than two strands. Notice how the braid word $(\sigma_1\sigma_2)^3$ describes such a twist.



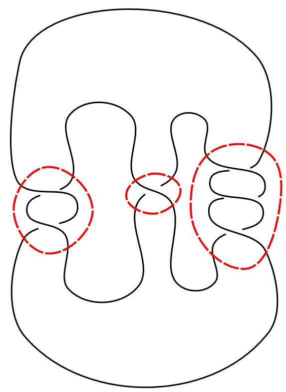
By removing the middle strand, we derive a twist.



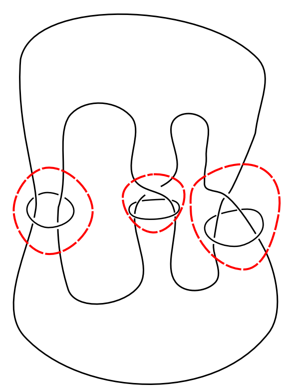
Thus we define *general twists* to be twists with excessive strands in the center. We can have twists of any number of strands greater than one. General half twists and bigons extend in an obvious manner.

6 Augmentation

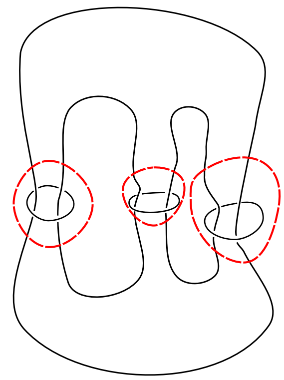
Augmenting is a process in which crossings in twists are removed and replaced with a ring (there is an inverse to this process known as dehn filling). Augmenting assumes a twist reduced diagram. If the twist is comprised in an odd number of half twists then then all but one is removed, if the twist is comprised of an even number, then all half twsts are removed. After this a ring is placed around the place of removed half twists. This idea can be more easily communicated through diagrams. Note how the following diagram's first twist has an even number of half twists, and the remaining twists have an odd number.



We perform augmentation in the previously described way, deleting all half twists on the first twist and all but one on the remaining two, then add rings.



This completes the augmenting process. Augmenting changes the link, it necessarily increases its hyperbolic volume. We call the resulting link *fully augmented*. One may further remove the remaining crossings without change to the volume.

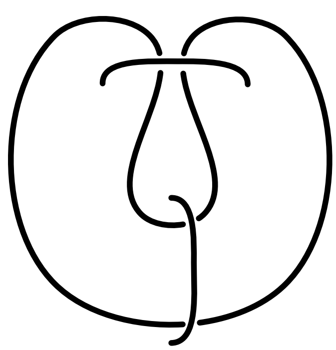


The original link is refered to as a *child* and the fully augmented link, once all crossings are removed, its *parent*.

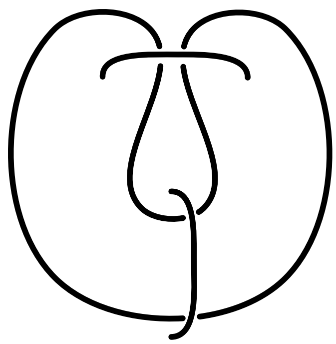
Just like twists, we can generalize augmentation. To do this we simply apply the same rules to links with general twists. We look at how many generalized half twists the twists are comprised of, then remove all but one, if odd, or all, if even. The resulting link is a *generalized fully augmented link*.

7 Polyhedral Decomposition

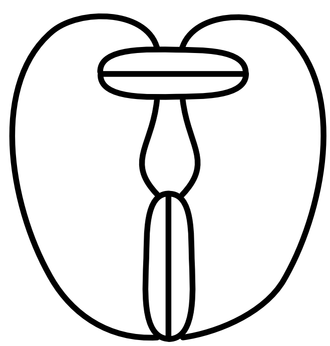
As mentioned previously there is a hyperbolic polyhedra associated with a hyperbolic link. The following is a step by step process showing how to decompose the borromean rings into two ideal regular octehedra. Begin by placing added circles perpindicular to the plane, leaving the remaining part of the link parallel to the plane.



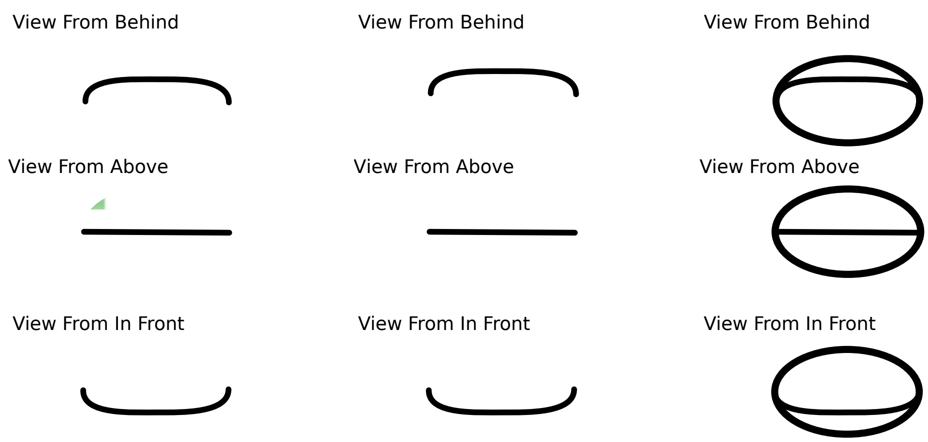
Now we have that the link is symmetric about the plane. Since we are looking $\mathbb{S}^3$ without the link, which can be thought of as $\mathbb{R}^3$ with ∞, we now cut off the bottom half of the link and only focus on the top. Eventually we will view the bottom half again, but since it will be the same as the top we need not think about it. So currently we have the upper half of $\mathbb{R}^3$ with ∞ without the top half of the borromean rings.



Imagine our remaining link as being completely rigid, and covered with a black ink. Now Imagine that you're a giant with a balloon and you're floating somewhere high up in $\mathbb{R}^3$. The balloon you have is magical, for you have traded it for your family's cow, thusly, it will never pop. You, being a smart mathematician, blow it bigger and bigger until it covers all of the upper half of $\mathbb{R}^3$. Then you shrink it back down into a reasonable sized giant balloon such that it fits in your giant size hands. As you analyze the balloon you notice its surface is covered in ink.

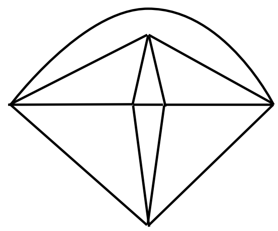


Parts of the link that were above the plane imprint pill shaped regions. Notice how the white space in these regions is identified in $\mathbb{R}^3$, but not on the balloon. As the balloon was blown up over these parts of the link it eventually creased and touched itself in these regions. Then as the balloon was shrunk back the balloon unfolded and created two regions instead of one. The following pictures are meant to isllustrate a section of the balloon that approaches these parts of the link; the first picture shows the balloon being just above the link, the next shows the expanding of this part of the balloon until it touches itself, and finally the balloon retracts and the creation of the pill like regions is shown.



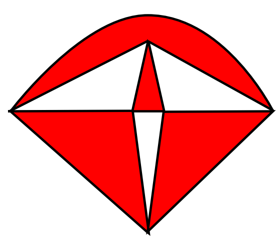
We can think of the seemingly additional edges that these pill regionas have as being the boundary of $\mathbb{R}^3$ that lives directly under the vertical parts of the link. For conviniences sake, let us refer to them as additional edges.

You now have been told that your balloon has the ability to mold into isotopic shapes, for the man whom was traded your cow found it to be exceptional. With your newfound ability, you shrink all parts of the balloon covered in ink, besides the additional edges, into singularities. As you do this the additional edges stretch outwards towards the singularities.



As an estute geometer, you see that this is in fact an octehedran. Once the octehedran made from the same process using the bottom half of $\mathbb{R}^3$ is included the process of polyhedral decomposition is complete. This is the main process used for finding the hyperbolic polyhedra. It may not be clear yet, why these polyhedra are hyperbolic. This will be covered later on.

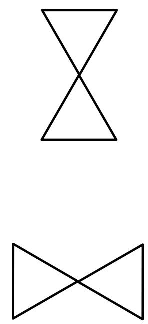
An intresting property that the resulting polyhedra will always have is that they can be checkerboard colored.



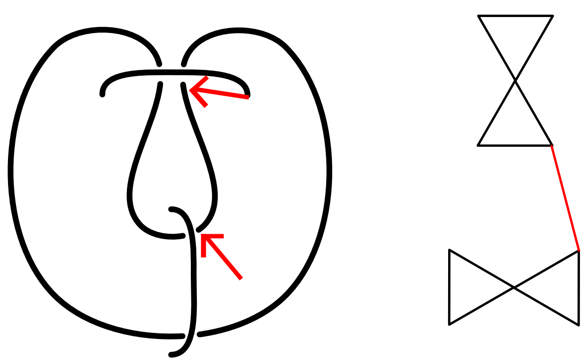
Note how the colored part of the polyhedra corresponded to the region bounded by the additional edges. Amazingly, this will be the case no matter what link you are decomposing!

This process is rather cumbersome. We will eventually see many more ways to get the same polyhedra from augmented links. I'd like to show now a method that is simply a sped up process of the recently shown method.

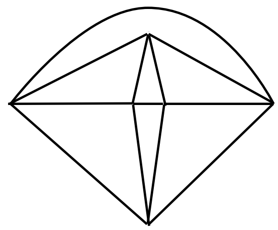
For every perpindicular circle add five vertices to the plane, connect them so that each pair of five vertices creates two triangles.



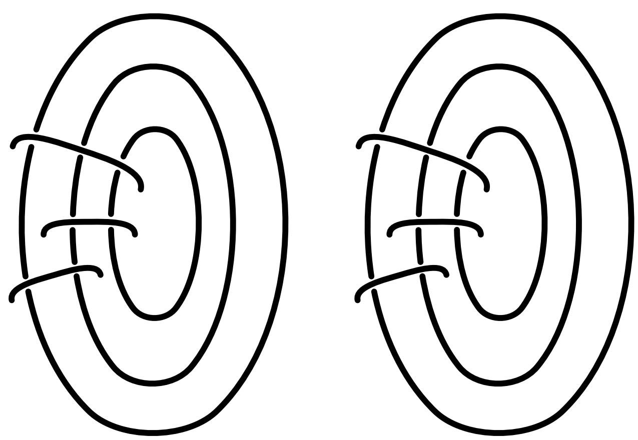
Then identify points if they are connected they represent components of the link that were the same.

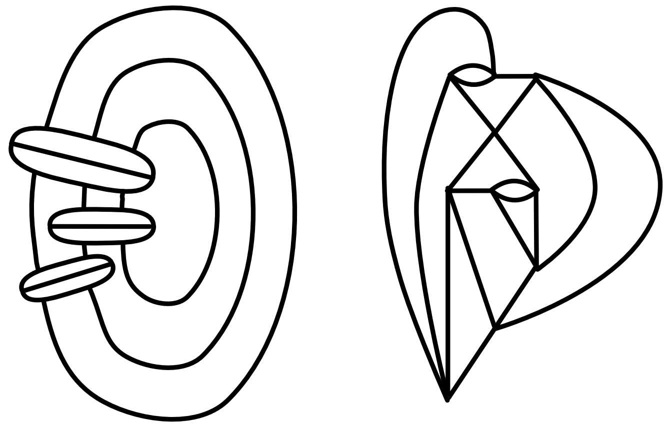


Once you've done this with all vertices the result should be the same as before, an octahedron.

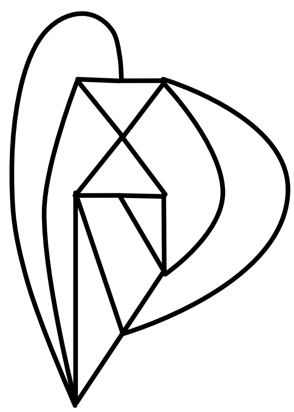


Finally, there is an analogous process one can take to find the polyhedra of hyperbolic generalized fully augmented links. Things can become somewhat strange here. The following is a detailed example to illustrate a possible problem that can occur. It is the decomposition of the parent of three braids with words, $(\sigma_1\sigma_2)^{3n}\sigma_1^a\sigma_2^b$. We begin the same way; cut off the bottom half of the link, blow up a balloon, analyze ink stains, shrink ink excluding additional edges.





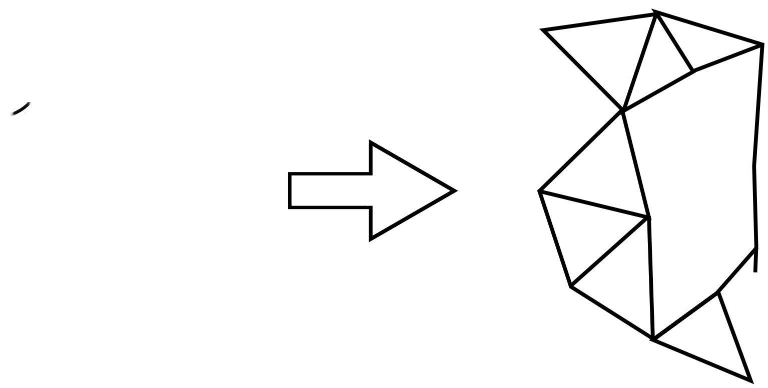
Notice the bigonal regions that form where pairs of vertices have multiple edges. When this happens, we simply identify the two edges together. The resulting graph will be the polyhedra lying in the upper half plane. Finish up in the same way as before and you are done.



You should be able to construct a quicker way to do this that is similiar to the quick method for fully augmented links that was shown earlier. The checkboard coloring for these polyhedra remains.

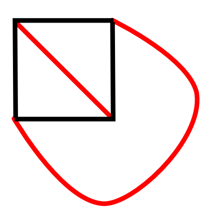
8 Circle Packings and Triangulations

If you are familiar with graph theory, you may be familiar with *circle packing*. If you're not, it's a collection of connected circles in the plane. The circle packing theorem states that for every simple, connected, planar graph there exists an circle packing whose *nerve* is isotopic to the graph. A nerve of a circle packing is the graph resulting from replacing a circle with a vertex and points of tangency with edges.

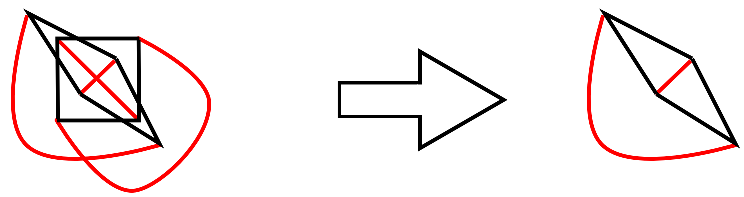


It's been shown by Purcell[1] that if you begin with any triangulation of the plane (this assumes no two triangles share more than one edge) and put a *dimer* on the triangulation, then the circle packing whose nerve is that triangulation is associated to a hyperbolic fully augmented link. Furthermore, every hyperbolic fully augmented link is realizable in this manner. I will demonstrate how to see the associated link with either directly from the triangulation, or from the circle packing.

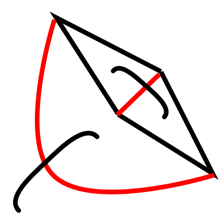
First begin with your triangulation, then choose a dimer. A dimer is a coloring of a tesselation such that every facet has exactly one boundary component colored.



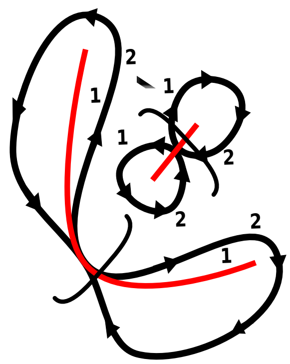
After this construct the *dual* of the triangulation, this means replace facets with vertices and vertices with facets.



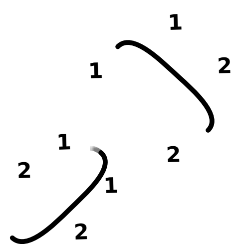
Place circles that are perpendicular to the plane around colored edges.



Place directed figure eight's over colored edges. Then label the edges that are hit by the figure eight in order.

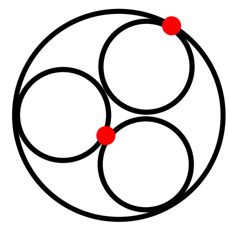


Now identify the vertices that are matched by colored edges. After this you will have a four valent graph. Split the vertices such that it is two valent and that the edges of the vertices have the same numbering from the figure eight.

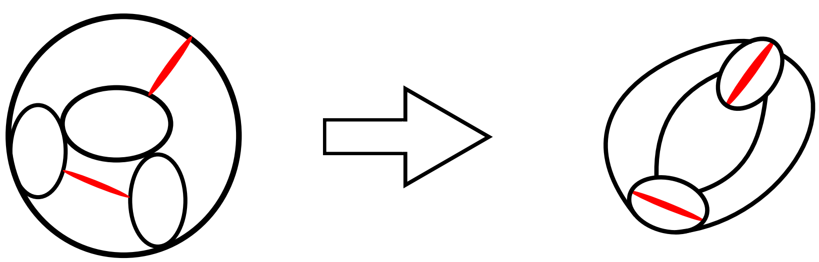


The procedure is now complete! The triangulation corresponded to the Borromean rings.

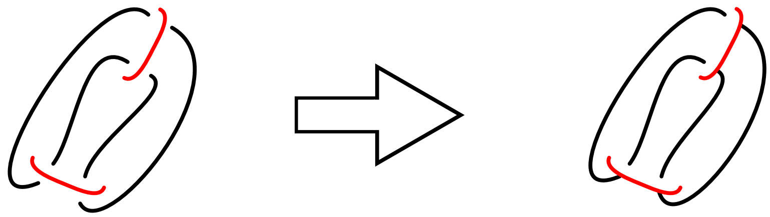
Now create the circle packing of the original triangulation.



These circle packing are representing the hyperbolic polyhedral structure of the links complement. In the polyhedra decomposition section we went over splitting of the link to create a hyperbolic structure. Here in order to get back to the link, we must do the reverse process. Let us assume that the current circle packing represents the polyhedra that comes from the upper half of $\mathbb{R}^3$. Reversing the process starts by smoothing the colored vertices out towards circles, then splitting each vertex back into line segments.



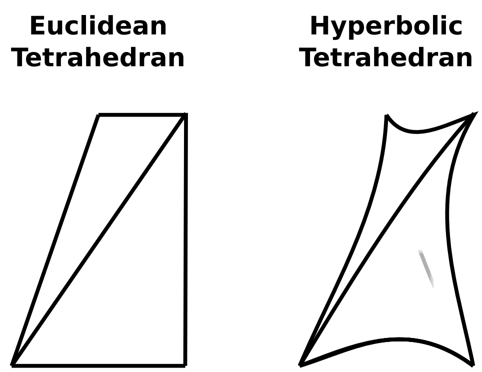
Finally identify sides opposite the colored edge, then do the same for the circle packing on the bottom half of $\mathbb{R}^3$, and then glue the resulting structures where they meet at the plane.



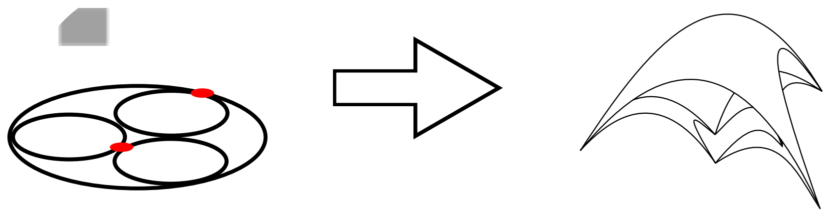
Once again we see the result being the Borromean rings. So, cool beans I guess.

9 Why Are These Polyhedra Hyperbolic?

I'm glad you asked, first though, let us ask why polyhedra we think of everyday are euclidean. Imagine a tetrahedran, and choose any two points, either on it or in it. If the tetrahedran is truly euclidean, you will be able to draw a line between these two points. We know a line as the shortest distance between two points. The only reason we about lines being the shortest distance is because we use the euclidean metric. If instead we were to use a hyperbolic metric, the shortest distance would become euclidean arcs. So a tetrahedran would be hyperbolic is there was a circular arc on/in the tetrahedran that connected any two points, on/in the tetrahedran.



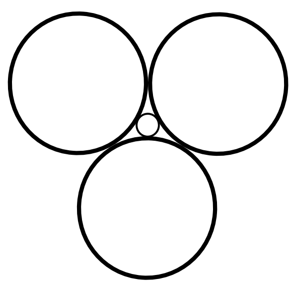
When a surface of n-dimensions contains lines like these in a metric M we say it's *totally geodisic* in M. So if our polyhedra have totally geodisic hyperbolic surfaces we know it's hyperoblic. Let's go back to our circle packing, which recall, represents the structure of the complement. Specifically let's consider the circle packing of the Borromean rings. Think of the circles as being the equator of hemispheres that are cut in half by the plane. Consider the top half of $\mathbb{R}^3$. Then connect vertices on hemispheres with hyperbolic lines.



The resulting structure will be a hyperbolic polyhedran due to process of construction. Here it is a hyperbolic octahedran. This process can always be done on our hyperbolic fully augmented links because they correspond to circle packings. One may wonder how to get circle packings for other links...

10 Extending Circle Packings and Triangulations

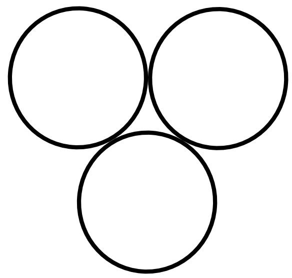
Given any three mutually tangent circles, there exists a unique circle, up to Mobius transformation, that lay tangent to all three.



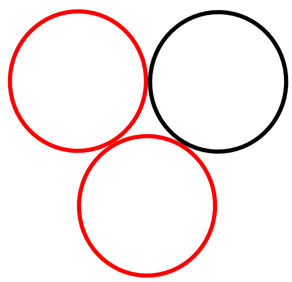
Together these four circles create six points of tangency.

Lemma 1. *Choose any four points of tangency such that each circle contains two, then there exists a circle, C, that contains these four points. Furthermore, C meets the four circles at angles of* $\pi/4$*.*

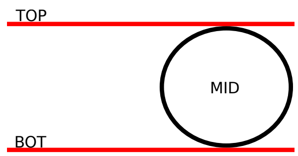
Proof. Let three circles lay tangent to one another in the plane.



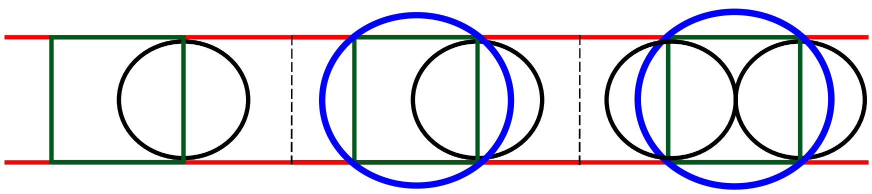
Pair two circles together.



Send the point of tangency between the focused pair to infinity. We refer to, for sake of clarity, the circles based on their placement; TOP, MID, and BOT.

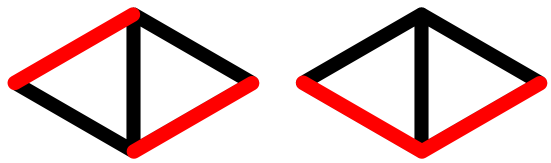


Construct a square, S, with perimeter four times the diameter of MID, and two vertices on points of tangency of MID. Circumscribe a circle, C, about S. Note that the center of C is on MID. Construct a circle, C', which is tangent to TOP and BOT at the vertices of S not on MID.

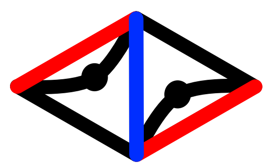


By construction, C' is the unique circle that is tangent to TOP, MID, and BOT. Furthermore, since C was circumscribed about a square, it meets TOP, MID, BOT, and C' at angles of $\pi/4$. □

This lemma will allow us to begin to describe generalized fully augmented links using circle packings. Suppose you are given some triangulation with a dimer. If we choose any edge that is not colored, the two triangles that it is a part of will look one of these two.



We call the edge on the left a *connector* and the edge on the right a *splitter*. We refer to the diagrams as connector and splitter diagrams, respectively. For now we focus on the connector. Add additonal edges to colored edges, place vertices in the middle of these additional edges, then color the connector.



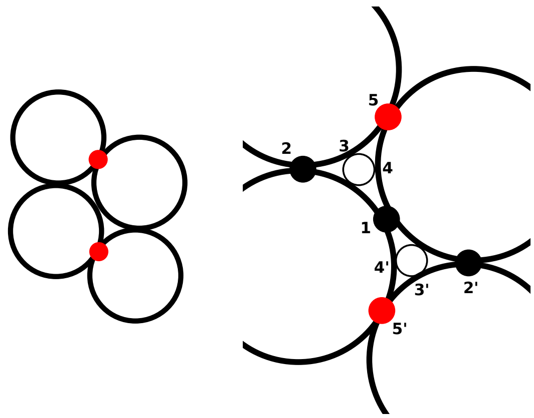
This process is referred to as *adding a connector*.

The circle packings whose nerves are triangulations with a modified dimer have useful properties for understanding hyperbolic generalized fully augmented links.

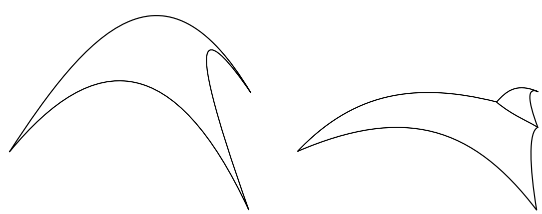
Theorem 2. *Every triangulation with an added connecter corresponds to a hyperbolic generalized fully augmented link, furthermore, this link has exactly the volume of two regular ideal hyperbolic octehedra plus the volume of the link which corresponded to the original dimer.*

Proof. Construct an arbitrary triangulation of the plane with a dimer such that there exists a connecter diagram in it. Focus on the connecter diagram, construct its circle packing.

Construct the two unique circles that lay tangent to three of the four circles. Label the following points of tangency as in the diagram.



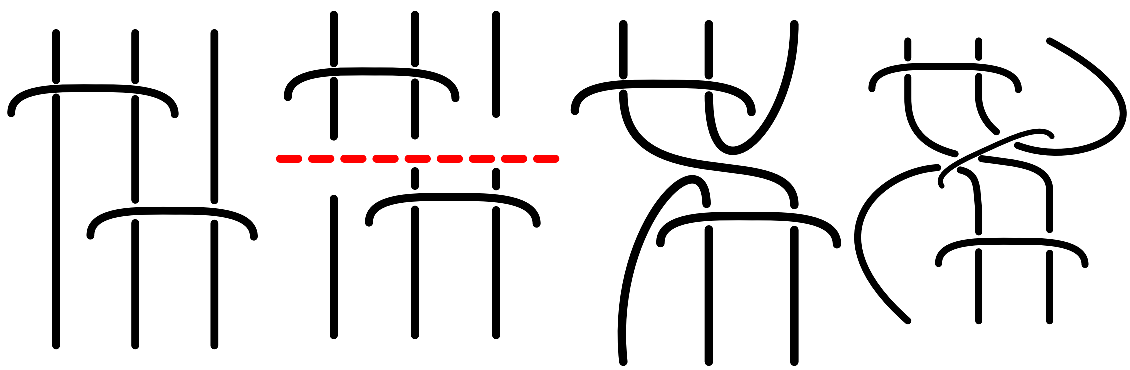
Lemma 1 tells us that points 2,3,4 (2',3',4') and 1 must lay on a circle. This allows for the original triangular face on the polyhedra with vertices 1,2,5 (1',2',5') to be pushed down on its edges at the two points 3,4 (3',4'). The resulting addition will be a hyperbolic pyramid of the square. Since this takes place four times in total, twice on the upper half of $\mathbb{R}^3$ and twice on the bottom, we get four times the volume of this structure added onto our original polyhedra.



Lemma 1 also tells us that the square face will have dihedral andgles of $\pi/4$, so in total we will have the same angles that would correspond to four halfs of regular ideal octehedran. Thus this process corresponds to adding two octehedran to the original volume. Furthermore since we only locally changed part of the structure and kept it hyperbolic, the whole structure itself must remain hyperbolic. Color 1. Take the nerve of the new circle packing, erase the edges that correspond to unlabeled points of tangency in the local picture. The resulting structure will be a triangulation with an added connecter. □

What's great about these guys is you can view the corresponding link of the triangulation with an added connector the same way as you could a normal triangulation. Simply take the dual! Perhaps even cooler is the fact that if you can see the adding of a connecter directly on a link. The following illustrates just this.

A connecter diagram in a link looks locally like two circles and three vertical lines. The first circle has a disk which is punctured by the first two lines, the second circle has a disk which is punctured by the second two. The first circle is placed above the second. Adding a connecter takes the three lines and cuts them where they lie in between the two circles. Then the far left line from the top circle is attached to the far right line of the bottom. The middle line on the top circle is connected its closest neighbor on the right, then the middle line on the bottom is connected to the remaining line. Finally a perpendicular circle is placed around the threelines in between the two circles.



We have now begun our transcending into an unexplored world of relationships between tesselations of the plane and generalized fully augmented links.

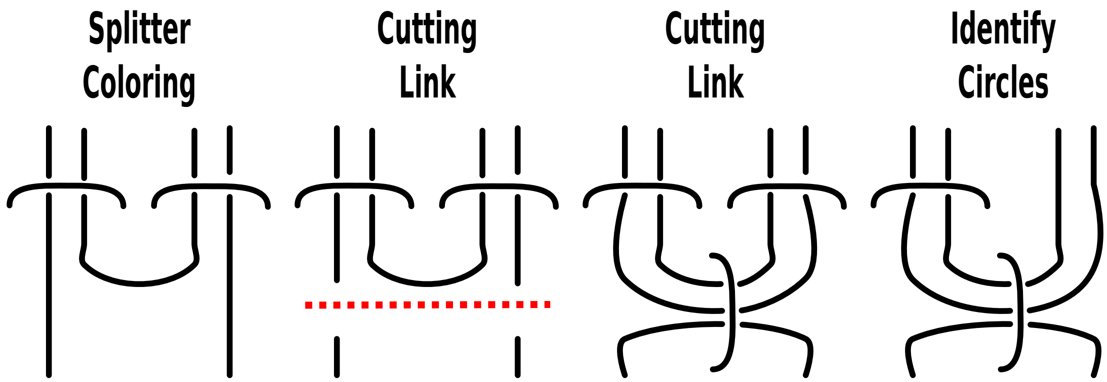
11 Creating Three Braids*

This will be an extra result after REU. The conjecture is that a specific class of dimers on triangulations associated with octehedral hyperbolic fully augmented links give rise to all three braid parents once a connecter edge is added. Since adding a connecter edge adds an octehedral amount of volume we gain the corallary that all three braid parents are octehedral.

12 Splitter Edges*

This will be an extra result after REU. The conjecture is that by adding a splitter edge a fully augmented link can be turned into a hyperbolic generalized fully augmented link in which volume is preserved.

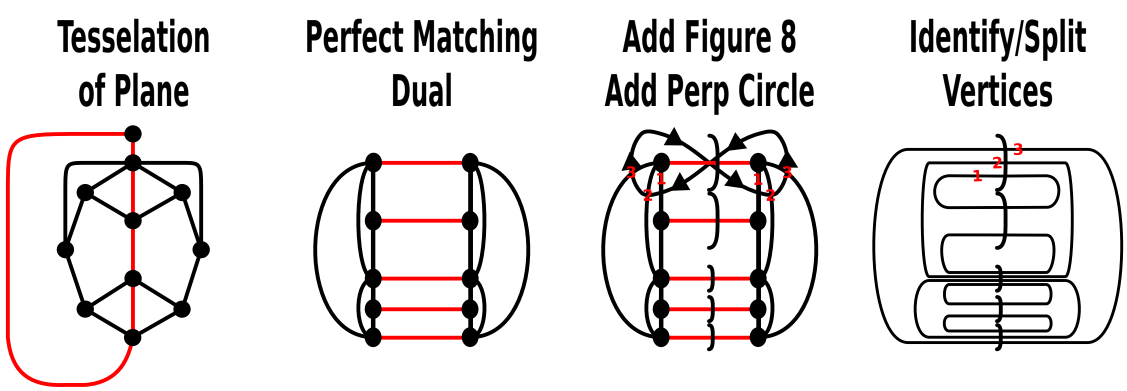
The following is a local example of what happens to a link when a splitter edge is added.



13 Tesselations*

This will be an extra result after REU. The conjecture is that the idea of triangulations for fully augmented links was just a subset of possible tesselations of the plane that correspond to generalized augmented links. If a tessalation of the plane has the property that its dual is a simple, connected, planar graph in which all vertices are of degree at least three and there exists a perfect matching on the graph such that matched vertices are of the same degree then in an analagous fashion to purcell, one can construct the generalized fully augmented link from the tesselation. The matching will corespond to the dimer in the tesselation.

The following is an example of the tesselation conjecture.



14 Open Questions

Here I enumerate some possible avenues of research for the future. I give a rating of difficulty based on my intuition.

1. *May constitute open questions at the time of REU 2016, though I hope to be able to prove them by then.

2. Using the idea of tesselations for generalized augmented links, is it possible to extend connecter and splitter edges in an intuitive way? If so, what is the change to volume? [Easy]

3. I've intuition that non-simple graphs (duals of non standard tesselations with bigonal like regions) somehow play a role in finding generalized augmented links. When are they allowable? What kind of graphs do they produce? [Medium]

4. It is well known, thanks to C. Adams, that all thrice punctured disks are totally geodisic. Is it possible to say when, in a generalized fully augmented link, an n-punctured sphere is totally geodisic? I suggest starting with looking at four punctured spheres. I feel the answer may be able to be found out by looking at patterns of tessselations of the plane. [Hard]

5. Purcell found all triangulations of the plane that lead to octehedral fully augmented links. Is it possible to find this when using square tesselations of the plane? Pentagonal tessalations of the plane? A general tesselation on the plane? [Medium]

6. Is it possible to use dehn fillings on generalized fully augmented links to give bounds of volumes of children of specific parents? [Medium]

7. I'm not quite certain about this one so ask Dr.Trapp about the volume conjecture. Apparently this would give a really juicy paper if you could do stuff with it. [Hard]

8. Notice how in the example of tesselations of the plane, the tesselations is completely symmetric about the dimer. Attempt to find patterns of these special constructions. What can be said about there links? [Easy]

9. How do belt sums play into the idea of tessalations of the plane? [Medium]

10. Where is the wizard? [Impossible]

15 Acknowledgements

I would like to thank Dr. Rolland Trapp for providing continous support, both mathematical and emotional. Dr. Corey Dunn has also been an invaluable aspect to the REU.

Thanks to his extreme detective skills, the wizard has yet to be lost. This project was funded both by NSF grant DMS-1461286 and California State University, San Bernardino.

16 References

1. Jessica S. Purcell, *An Introduction to Fully Augmented Links*, Contemporary Mathematics, **541** (2011)