Journal 1

Karina Novoa

August 20, 2015

1 Introduction

For purposes of this paper, we define a *knot* to be a closed loop in R^3. A collection of knots which may or may not be interlaced is called a *link*; a knot is considered a one component link. Knot and link diagrams are represented by a projection onto the plane where over-crossings are denoted by a solid line and under-crossings are denoted by a break in the line. A knot with no crossings is called the *unknot*. Two knots are *equivalent* if we can wiggle, bend, and untwist the knots until they match but never cut and reglue the knot.

This paper focuses on reduced alternating links. A knot diagram is *reduced* if it is drawn without any nugatory crossings in the knot as shown in Figure 1. A diagram is *alternating* if going along a strand of a knot, each consecutive crossing changes from over to under crossings and vice versa. Conversely, if there is at least one pair of consecutive crossings which are both over or under crossings, then the diagram is referred to being *non-alternating*. A link which has an alternating projection is called an *alternating link*. See Figure 2 for an example of alternating and non-alternating diagrams.

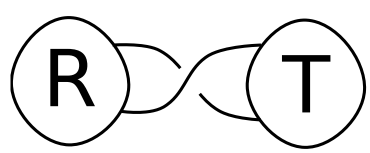


Figure 1: Nugatory crossing

A *twist* is a region of a knot involving two strands which intertwine with each other. The *twist number* is the fewest number of twists for any given diagram of a link. We define a *tangle* to be a region of the link in which we draw a circle around it so that the two strands entering the circle are the two strand exiting the circle. Figure 3 demostrates two special tangles which are the *zero-tangle* and ∞*-tangle*. A *flype* is when we flip a region of the knot forcing the crossings of that region to change position. Note that flypes can

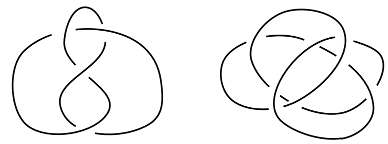


Figure 2: Alternating vs Non-Alternating diagram

diminish the number of twists in a link. With this, we mention that a diagram is *twist reduced* if we can flype all crossings in a twist to one section as shown in Figure 4.

Figure 3: Zero tangle (left), Infinity tangle (right)

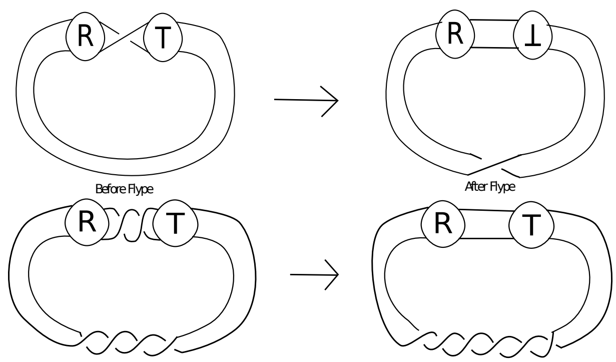


Figure 4: On top we see an example of a flype with tangle R and T. On the bottom we see how two flypes reduces the number of twists.

To talk about smoothings, we define an A region as the area to the left of an under-crossing and the B region as the area to the right of an under-crossings. When implementing a *smoothing*, we cut the crossing of a knot and glue the ends of the strands such that when joining the A regions, we have an A-smoothing and similary is true of the B-smoothing. In Figure 5, we illustrate A and B regions as well as their smoothings.

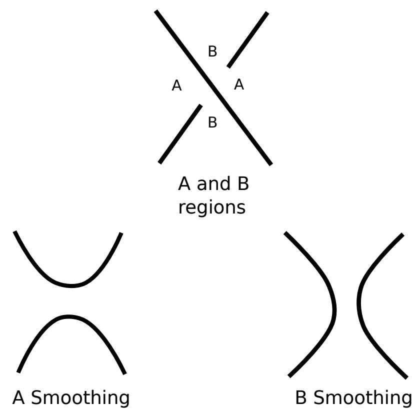


Figure 5: A and B regions, before and after smoothing.

2 The Three-Variable Bracket Polynomial

The *three-variable bracket polynomial*, in this paper will be referred to as the bracket polynomial, is an invariant of some links; specifically, it is an invariant for reduced alternating links (Lafferty). One can tell whether two links are different if they do not have the same bracket polynomial; however, if two links do have the same bracket polynomial, that doesn't necessarily imply both links are the same. There are two appraoches to derive this polynomial: the Recursive and State Model methods.

In the recursive approach, the *three-variable bracket polynomial* of an unoriented link L is denoted by $\langle L \rangle$ such that at each crossings of L, we satisfy

$$\left\langle \times \right\rangle = A \left\langle)\,(\right\rangle + B \left\langle \asymp \right\rangle \tag{1}$$

$$\left\langle L \bigcirc \right\rangle = d \left\langle L \right\rangle \tag{2}$$

$$\left\langle \bigcirc \right\rangle = 1 \tag{3}$$

This paper focuses on the state model approach. A *state* **S** is a choice of A or B smoothings for each crossing. It contributes $A^m B^n d^p$ where m is the number of A smoothings, n is the number of B smoothings, and p the number of disjoint loops minus one. Since each crossing has two choices for smoothing, there are a total of 2^x states where x is the total number of crossings of a link. Considering all the possible states **S** contribution, the bracket polynomial of the link is

$$\left\langle L \right\rangle = \sum_{\mathbf{S}} A^m B^n d^p.$$

This procedure is exhibited for the reader using the Hopf Link in Figure 6.

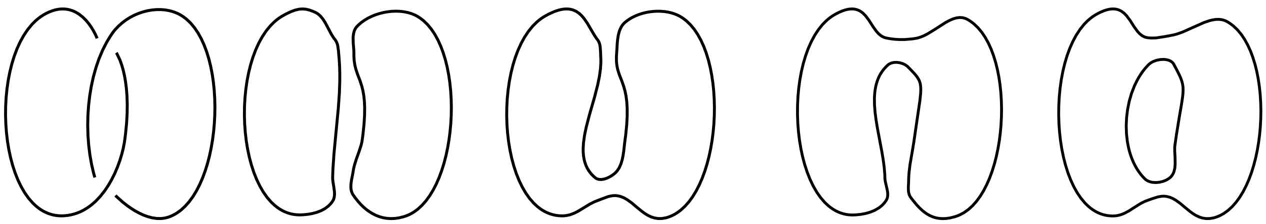


Figure 6: The bracket polynomial of the Hopf Link is $A^2d + 2AB + B^2d$

3 Surfaces and Knots

We begin this section by describing a *surface* as a set of points with property that for any point on the surface, there is a neighborhood around that point which is a disc. A surface is *orientable* if we can paint two sides of the surface in different colors such that the different colors only meet along the boundary of the surface. However, suppose we start painting one side of the surface and paint along that same side, if we find that the surface was all painted the same color, that means the surface is *non-orientable*. Figure 7 illustrates the difference of this with the two-sided torus and one sided Mobius strip.

The Euler characteristic is a topological invariant that can be computed by a cell decomposition of a surface in the following way. Let V denote the number of vertices, E the number of edges and F the number of faces in a cell decomposition of a surface S. The Euler characteristic $\chi(S)$ of S is

$$\chi(S) = V - E + F.$$

Examples of cell decompositions for a disc are shown in Figure 8. Since both have the same Euler characteristic, they're considered the same topologically. The image for (a) is a one vertex, one edge, one face cell decompoistion of a disc with Euler characteristic equal to one. Similarly, image (b) is a rectangle with four vertices, fours edges and one face

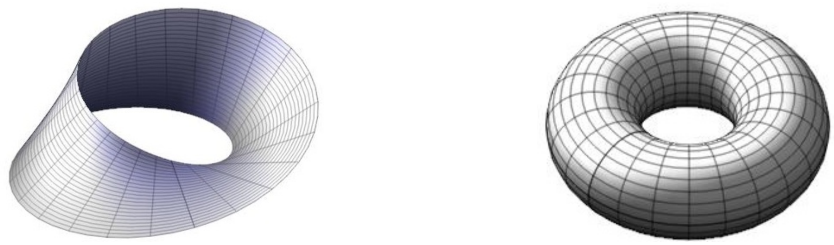


Figure 7: On the left we have a one sided Mobius strip. On the right we have a two-sided (outside and inside) torus.

that also has Euler characteristic equal to one. This paper focuses on orientable surfaces with boundary so to determine the Euler characteristic of these surfaces we must first talk about Seifert's algorithm.

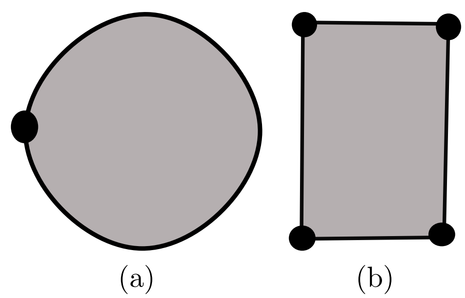


Figure 8: Cell decompositions of a disc.

The goal when using Seifert's algorithm is to obtain a Seifert surface. A *Seifert surface* is a surface with boundary where the boundary is a link. To apply Seifert's algorithm, we first arbitrarily orient each component of a given link. To smooth the link, remove the crossing and smooth compatibly with orientation. What is left are called Seifert circles and let each bound a disc. Some of the Seifert circles might be inside of others. In this case, one can think of inner discs as being higher than other ones. Next we insert half twisted bands to attach the Seifert circles corresponding to the original crossing. Figure 9 illustrates this procedure on the figure-eight knot.

The *genus* g for a closed surface is the number of holes it has. See Figure 10 for examples. It can be shown that the genus and the Euler characteristic $\chi(S)$ of a closed orientable surface S is related by

$$\chi(S) = 2 - 2g$$

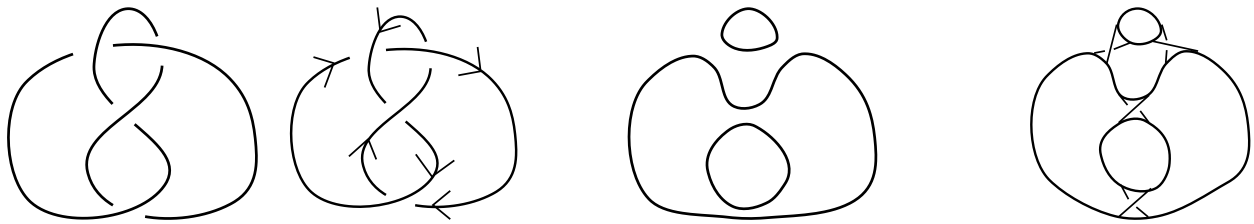


Figure 9: Procedure of Seifert's algorithm

If a surface has b boundary components, then we have

$$\chi(S) = 2 - 2g - b.$$

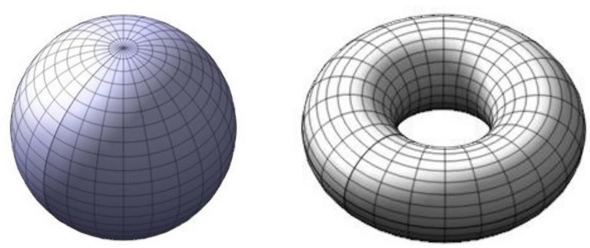


Figure 10: The sphere has genus 0. The torus has genus 1

Note that all crossings in a diagram are smoothed while applying Seifert's algorithm, so the Seifert circles arise from a state of the diagram in the context of the bracket polynomial. For this reason we make the following definition.

Definition 1. *A* **Seifert state** ***S*** *for an unoriented alternating diagram D is obtained by orienting the link and choosing the oriented smoothing at each crossing.*

See Figure 11 for an example of obtaining a Seifert state from a diagram. The reader can check that this Seifert surface results from performing all B smoothings.

Remark 1.

If the orientations on both strands of the twist follow the same direction then that is a *compatible orientation*. When the orientation on both strands of the twist are in opposite directions then that is an *incompatible orientation*. This is illustrated below.

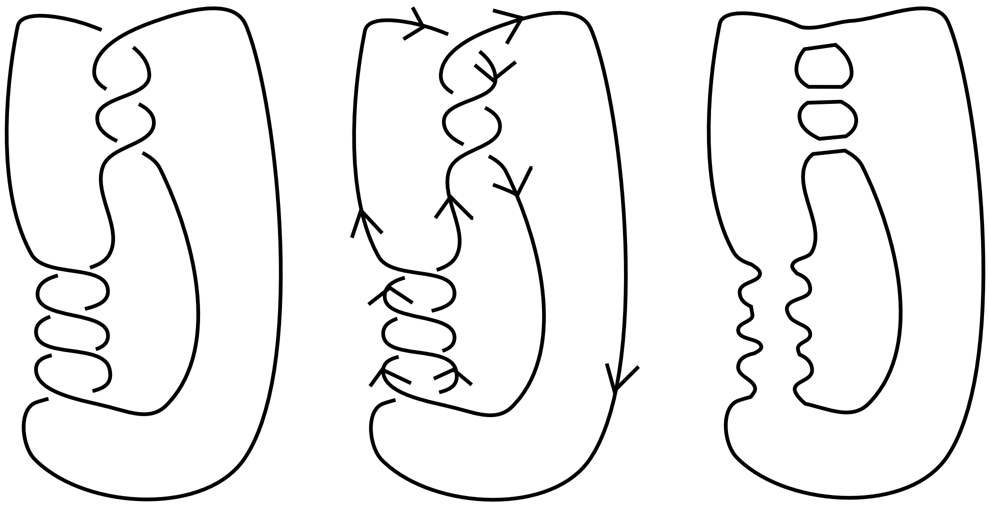
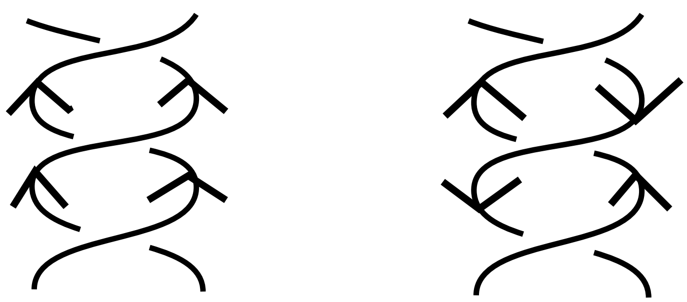


Figure 11: Example of a Seifert state



Remark 2.

Considering placing an orientation on an unoriented two-bridge link diagram with two twists. Without loss of generality, we assume the left strand is oriented "clockwise" as in Figure 12.

The rest of the orientation depends on whether there are an even or odd number of corssings in each twist. Note that if there is an even number of crossings in a twist, the strand entering on the top left leaves on the bottom left - thus the strands enter and leave on the same side of the twist. If there is an odd number of crossings, the strands enter and leave on opposite sides of the twist. We will use this observation together with our initial choice of orientation on the upper left to determine the orientation on the enire link.

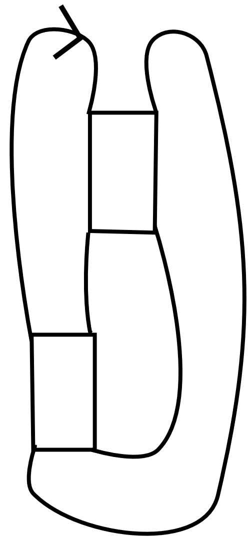
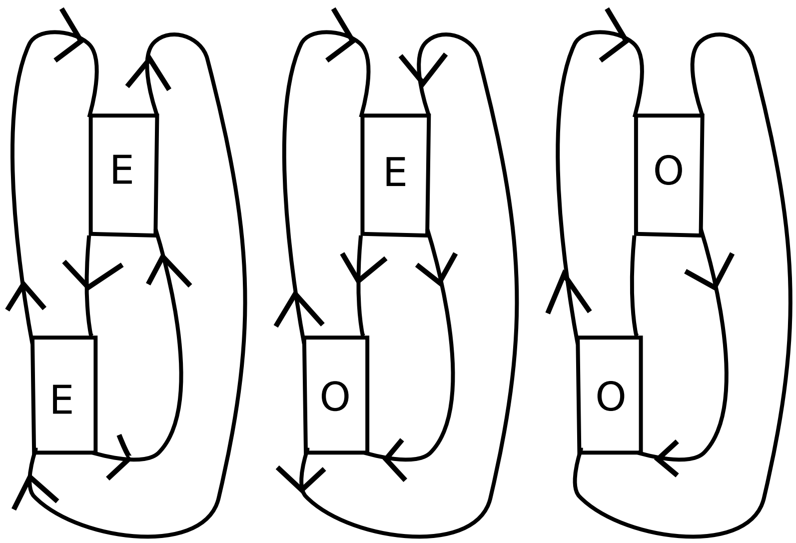


Figure 12

Remark 3.

Observe that in (a) and (b) of Figure 13, the two-bridge link consist of only one component and (c) consist of two components. Under one component links (knots), regardless of the initial choice of orientation, after smoothing the crossings, we end with the same number of Seifert circles. However, with two component links, the initial choice of orientation will determine the number of Seifert circles our link has after smoothing the crossings. The reader should observe that changing the orientation of each component doesn't change the number of Seifert circles. Figure 14 is an example on a two-bridge link with two twists where each twist has an odd number of crossings. This Figure illustrates how our choice of orientation effects the resulting number of Seifert circles $| \mathbf{S} |$. When we orient the link as in Figure 14 (a), the three crossing twist is incompatibly oriented which will lead to four Seifert circles. If we orient the link as in (b) of Figure 14 then the five twist is oppositely oriented leading to six Seifert circles. One last noteworthy observation for this case is that one twist will be incompatibly oriented and the other will be compaitbly oriented in either choice of orientation. The reader can check this by testing both orientations in the unoriented component of Figure 13 (c).

Lemma 1. *Let L be a two twist 2-bridge link with c_1, c_2 crossings in twists one and two, respectively. The number of circles $| \boldsymbol{S} |$ in a Seifert state of L is*



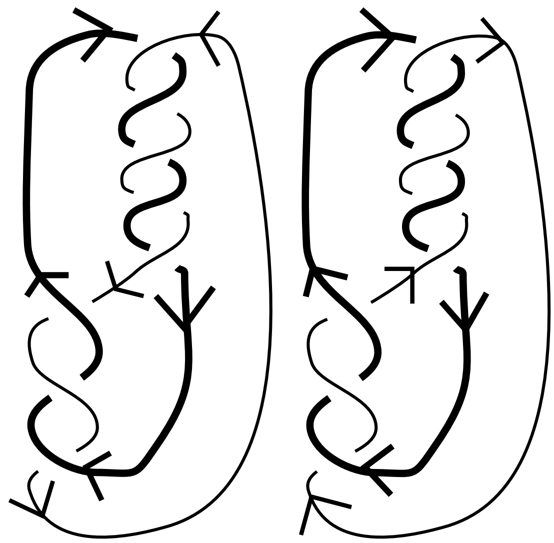
(a) (b) (c)

Figure 13

$$|S| = \begin{cases} c_1 + 1 & c_1 \text{ odd, } c_2 \text{ even} \\ c_1 + c_2 - 1 & c_1, c_2 \text{ even} \\ c_1 + 1 & c_1, c_2 \text{ odd, twist 1 is incompatibly oriented} \end{cases}$$

Proof. Case 1: Suppose we have c_1 odd number of crossings for the first twist and c_2 even number of crossings for the second twist. Remark 3 illustrates this link will have one component and thus one orientation to consider. As shown in Figure 13 (b), the twist with odd crossings has incompatible orientations at the endpoints. The reader can verify that throughout the crossings of this twist, the strands will be incompatibly oriented. The twist with even number of crossings has endpoints that are compatibly oriented which means that entire twist was compatibly oriented. When smoothing the crossings, incompatible orientations with c crossings creates $c - 1$ disjoint loops and has a zero tangle, while compatible orientations form an infinity tangle. The connectivity outside of the twists must remain the same as in Figure 13, thus creating two additional loops. Therefore, the net total of Seifert circles $| S |$ is $c_1 + 1$. See Figure 11 for a specific example of this case.

Case 2: Assume c_1 and c_2 are even number of crossings for both twists. As seen in Remark 3 and Figure 13 (a), this type of link has one component with endpoints incompatibly



(a) Orientation 1 (b) Orientation 2

Figure 14: Two-bridge link with a five and three crossing twist.

oriented at each twist. This implies our link is incompatibly oriented throughout the twist thus when smoothing, similarly to case 1, we have $c_1 + c_2 - 2$ disjoint loops created and one additional loop from connectivity. Therefore the total number of Seiferts cirlces $\mid S \mid$ is $c_1 + c_2 - 1$.

Case 3: Now suppose c_1 and c_2 have an odd number of crossings for both twists. Figure 13 (c) illustrates that this link will consist of two components. Assume twist one is incompatibly oriented. As noted in Remark 3, since one twist is incompatibly oriented, the other twist must be compatibly oriented. Once again, similarly to case 1, twist one will create $c_1 - 1$ disjoint loops and through connectivity, we get two additional loops. The net total of Seifert circles $\mid S \mid$ is $c_1 + 1$.

4 Genus and The Three-Variable Bracket Polynomial

Definition 2. *A* **maximal state** ***S*** *of an alternating diagram D is one that contributes to the term of the highest degree of d in the three-variable bracket polynoimal.*

Theorem 1. *Let D be a twist reduced diagram with at least three crossings per twist, then the unique maximal state for D is obtained by smoothing across the twists.*

Proof. To see this, we will consider a single twist and show that smoothing across the twist is the unique smoothing that maximizes the number of components. Consider a B twist with three crossings as in Figure 15. In the different cases we choose one, two, and three A smoothings, respectively and in case four we have no A smoothings/ all B smoothing. Observe that in each case, the number of A and B smoothing determine the state. Regardless of which crossing we choose for the different smoothings, the result is the same. Outside of the twist, no matter what happens to the rest of the link, it is clear that case 3 creates the maximum number of loops. Whether we connect the top strands together and the bottom strands together or if we connect the right hand strands together and similarly for the left side, case 3 still creates the maximum number of loops. Thus, replacing each twist with case 3 is the only way to generate the maximal state for D.

In general, suppose we have a B twist with n crossings. Smoothing across the twist (A smoothings in this case) will create the maximum number of loops generating n-1 loops and a zero tangle. Each time we change an A smoothing to a B smoothing, the number of loops drops down by one. By connecting the top strands together and the bottom strands together, for all but one state, the number of loops increases by two. The state with this exception is the state with no A smoothings which only increases by one loop. When connecting each of the sides together, in all but one state, the number of loops created is equal to the number of A smoothings applied. Again, the exception is for the state with no A smoothing; in that case, when connecting the strands of each side, we gain two loops. Regardless of which way we chose to close the link, there is only one state that will give us the highest degree of d. Without loss of generality, we can apply the same procedure for an A twist with n crossings.

Definition 3. *The bracket polynomial of a link L $\langle \mathsf{L} \rangle$ gives the genus g if we get equality in Lemma 3.*

Theorem 2. *Given D, a twist reduced alternating diagram with at least three crossings per twist, we can orient D so that every twist is incompatibly oriented if and only if we can determine the genus g of the corresponding link L through the three-variable bracket polynomial.*

Proof. Suppose we are given a twist reduced alternating diagram D where each twist has at least three crossings. Also, suppose we can orient D in a way such that each twist is incompatibly oriented. Proceed with Seifert's algorithm and smooth each crossing based on orientation. Doing so implies we are smoothing across the twist. By theorem 1, this means we have a unique Seifert state which gives the maximal degree of d. Lemma 3 indicates how we can find the genus of L using the maximal degree of d.

Conversely, suppose we are given the bracket polynomial for a link L with corresponding diagram D such that each twist has at least three crossings and suppose we know the genus g. Knowing the genus implies the link is orientable. Since the genus is found from the

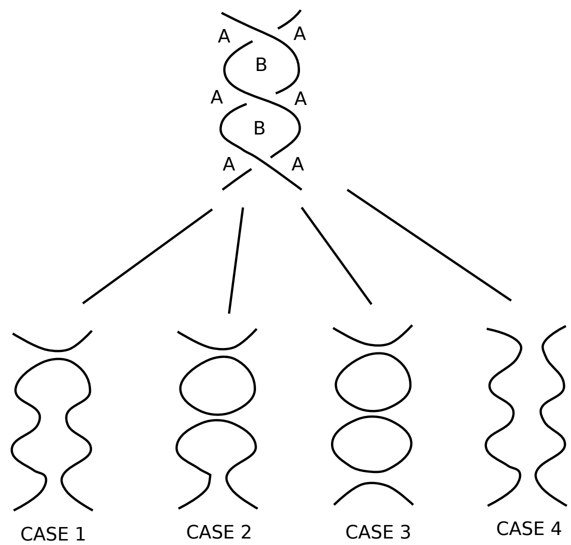


Figure 15

highest degree of d and the link is orientable, that means there exists a Seifert state **S** with the maximum number of circles. Recall that each twist in the maximum state is smoothed across by theorem 1. **S** has each twist smoothed across which means that each twist is incompatibly oriented.

5 Lemmas with proofs to be completed.

Lemma 2. *The exponent of d in each term of a bracket polynomial for a link L with c crossings can be rewritten in terms of the Euler characteristic of its surface S,*

$$\sum_{S} A^m B^n d^{\chi(\boldsymbol{S})+c-1}.$$

Proof. Not yet complete.

Lemma 3. *Let M be the maximum degree of d in a bracket polynomial for link L with b boundary components and c crossings. The relationship between the genus g and the maximum degree M is*

$$g(L) \geq \frac{c(L) - b + 1 - M}{2}.$$

Proof. Not yet complete.

Lemma 4. *The bracket polynomial gives the genus if and only if a Seifert state $\boldsymbol{S}$ has the maximum number of Seiert circles $|\ \boldsymbol{S}\ |$.*

Proof. Not yet complete.