The Study of the Constant Vector Curvature Condition for Model Spaces of Three Dimensions in the Lorentzian Setting

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Abstract

The purpose of this research is to examine the properties of constant vector curvature $(cvc(\epsilon))$ on model spaces of three-dimensions in the Lorentzian setting. Constant vector curvature(ϵ) on three dimensional model spaces in the Riemannian setting is already known to be well-defined and it is possible to find the value of ϵ given the curvature tensor values. [3] Results from this research prove that Lorentzian model spaces in three-dimensions in which you can diagonalize the Ricci tensor with respect to the metric has $cvc(\epsilon)$ for some ϵ .

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1 INTRODUCTION AND BACKGROUND

As three dimensional beings, is it possible to intrinsically determine the the curvature of the surface we inhabit? Differential geometry is a branch of mathematics that uses methods of linear algebra and calculus to study the geometric properties of topological spaces that locally resemble Euclidean space known as manifolds. If a metric for a given manifold is known it is possible to study to the curvature of that space.

Let $\langle \cdot, \cdot \rangle$ be a symmetric bilinear form on a finite dimensional real vector space *V*. Assume $\langle \cdot, \cdot \rangle$ is nondegenerate, hence given a nonzero vector $v \in V$, there is some vector *w* in *V* such that $\langle v, w \rangle \neq 0$. We can then choose an orthonormal basis $\{e_i\}$ for *V* so that

$$\langle e_i, e_j \rangle = \begin{cases} -1 & \text{if } i = j, \\ 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Let $\delta_i = \langle e_i, e_i \rangle$. Let *p* be the number of *i*'s such that $\delta_i = -1$ and let *q* be the number of *i*'s such that $\delta_i = 1$. *Remark.* p + q = dimV.

Definition 1.1. The *signature* of a metric is defined to be (*p*, *q*).

Remark. A vector in *V* is said to be spacelike if $\delta_i = 1$, timelike is $\delta_i = -1$ and lightlike if $\delta_i = 0$. A lightlike vector is sometimes also referred to as degenerate or null. The zero vector is null, hence a time or spacelike vector will be non-degenerate.

If p = 0 then $\langle \cdot, \cdot \rangle$ is positive definite and the model space is defined to be *Riemannian*. If p = 1 there is one timelike direction and the manifold is defined to be *Lorentzian*.

Definition 1.2. Let $x, y, z, w \in V$ where V is a finite-dimensional real vector space. An *algebraic curvature tensor* is a function $R: V \times V \times V \to \mathbb{R}$ that measures the curvature of a manifold at a single point and satisfies the following equalities:

1. R(x, y, z, w) = -R(y, x, z, w)

2.
$$R(x, y, z, w) = R(z, w, x, y)$$

3. R(x, y, z, w) + R(y, z, x, w) + R(z, x, y, w) = 0.

The first property means that *R* is skew-symmetric in the variables (x, y) and (z, w). The second property means that *R* is symmetric in the pairs (x, y) and (z, w). And the third is the first Bianchi identity. [1]

Definition 1.3. Let $\langle \cdot, \cdot \rangle$ be an inner product on *V* and *R* be an algebraic curvature tensor. Then *M* = $(V, \langle \cdot, \cdot \rangle)$ is called a *model space*.

Remark. A manifold observed at a single point is an object known as a model space.

Definition 1.4. Let $M = (V, \langle \cdot, \cdot \rangle, R)$ be a model space where $v, w \in V$. And let π be a non-degenerate 2 plane equal to *span*{v, w}. The *sectional curvature* denoted as $k(\pi)$ is defined to be

$$k(\pi) = \frac{R(v, w, w, v)}{\langle v, v \rangle \langle w, w \rangle - \langle v, w \rangle^2}$$

Remark. The sectional curvature of a 2 plane is independent of the basis chosen for V.

Definition 1.5. Let $M = (V, \langle \cdot, \cdot \rangle, R)$ be a model space. If for all non-degenerate 2 planes, π , it follows that $k(\pi) = \epsilon$ then *M* is said to have *constant sectional curvature* ϵ denoted as $csc(\epsilon)$.

Definition 1.6. Let $M = (V, \langle \cdot, \cdot \rangle, R)$ be a model space. If for all $v \in V$ where $v \neq 0$ there exists a vector $w \in V$ such that $k(span\{v, w\}) = \epsilon$ then M is said to have *constant vector curvature* ϵ denoted as $cvc(\epsilon)$.

Constant sectional curvature implies constant vector curvature. Constant vector curvature is less stringent condition for a model space in comparison with constant sectional curvature, consequently constant vector curvature more common.

In differential geometry, constant vector curvature is a fairly new area of study in the realm of curvature conditions on manifolds. The limited research of this topic has been studied previously in 2013 by Kelci Mumford and in 2014 by Albany Thompson. Mumford proved that model spaces of three dimensions with a positive definite inner product which have $cvc(\epsilon)$, that such constant vector curvature ϵ is well-defined. The following year, Thompson proved that not only does every model space in three dimensions with a positive definite inner product have $cvc(\epsilon)$ for some ϵ , but that you can find the value of ϵ given the curvature tensor values. Additionally, Thompson addressed some cases with stronger curvature conditions such as constant sectional curvature and extremal constant vector curvature. Extremal constant vector curvature is a property of a model space with $cvc(\epsilon)$ such that ϵ is a bound on all of the sectional curvature tensor values. Results discovered by Mumford and Thompson provided much of the motivation for this research in which we will similarly study the condition of constant vector curvature, but under the Lorentzian setting.

The difference between this research and the past research done is that we will be working with an inner product in the non-positive definite case. It should be noted that we will be working with a metric of signature (1,2) and assuming that there exists an orthonormal basis with respect to the inner product that diagonalizes the Ricci tensor. These conditions allow us to study a Lorentzian model space and assume there are three nonzero curvature entries, which include R_{1221} , R_{1331} and R_{2332} .

Remark. The Ricci tensor is independent of the basis chosen for V. [2]

Note that in this paper we will refer to e_1 as the timelike vector and e_2 , e_3 as the spacelike vectors.

2 **Results**

Theorem 2.1. Let $M = (V, \langle \cdot, \cdot \rangle, R)$ be a three-dimensional Lorentzian model space and let $\{e_1, e_2, e_3\}$ be an orthonormal basis on V with respect to $\langle \cdot, \cdot \rangle$. And let $R_{1221} = \alpha$, $R_{1331} = \beta$, $R_{2332} = \gamma$ such that $\alpha > \beta \ge \gamma$, $\gamma \ge 0$. If M has $cvc(\epsilon)$, then $\epsilon = -\alpha$.

Proof. Let $v = xe_1 + ye_2 + ze_3$ and $w = ae_1 + be_2 + ce_3$ where $x, y, z, a, b, c \in \mathbb{R}$. And consider the following cases.

Case 1: Let $v = e_1$. Then,

$$\begin{split} k(span\{v,w\}) &= \frac{R(e_1,w,w,e_1)}{\langle e_1,e_1 \rangle \langle w,w \rangle - \langle e_1,w \rangle^2} \\ &= \frac{R(e_1,ae_1 + be_2 + ce_3,ae_1 + be_2 + ce_3,e_1)}{\langle e_1,e_1 \rangle \langle ae_1 + be_2 + ce_3,ae_1 + be_2 + ce_3 \rangle - \langle e_1,ae_1 + be_2 + ce_3 \rangle^2} \\ &= \frac{R(e_1,be_2,be_2,e_1) + R(e_1,ce_3,ce_3,e_1)}{(-1)(-a^2 + b^2 + c^2) - (a^2)} \\ &= \frac{b^2 R_{1221} + c^2 R_{1331}}{-b^2 - c^2} \\ &= \frac{\alpha b^2 + \beta c^2}{-b^2 - c^2}. \end{split}$$

Now solving for ϵ we get,

$$\frac{\alpha b^2 + \beta c^2}{-b^2 - c^2} = \epsilon$$
$$\alpha b^2 + \beta c^2 = -\epsilon b^2 - \epsilon c^2$$
$$b^2(\alpha + \epsilon) + c^2(\beta + \epsilon) = 0.$$

Thus, $\epsilon \in [-\beta, -\alpha]$.

Case 2: Let $v = e_2$. Then,

$$k(span\{v, w\}) = \frac{R(e_2, w, w, e_2)}{\langle e_2, e_2 \rangle \langle w, w \rangle - \langle e_2, w \rangle^2}$$

=
$$\frac{R(e_2, ae_1 + be_2 + ce_3, ae_1 + be_2 + ce_3, e_2)}{\langle e_2, e_2 \rangle \langle ae_1 + be_2 + ce_3, ae_1 + be_2 + ce_3 \rangle - \langle e_2, ae_1 + be_2 + ce_3 \rangle^2}$$

=
$$\frac{R(e_2, ae_1, ae_1, e_2) + R(e_2, ce_3, ce_3, e_2)}{(1)(-a^2 + b^2 + c^2) - (b^2)}$$

=
$$\frac{a^2 R_{2112} + c^2 R_{2332}}{-a^2 + c^2}$$

=
$$\frac{\alpha a^2 + \gamma c^2}{-a^2 + c^2}.$$

Now solving for ϵ we get,

$$\begin{aligned} \frac{\alpha a^2 + \gamma c^2}{-a^2 + c^2} &= \epsilon \\ \alpha a^2 + \gamma c^2 &= -\epsilon a^2 + \epsilon c^2 \\ a^2 (\alpha + \epsilon) + c^2 (\gamma - \epsilon) &= 0. \end{aligned}$$

And combing the two cases gives us, $e \in [-\alpha, -\beta] \cap [(-\infty, -\alpha] \cup [\gamma, \infty)] = \{-\alpha\}$. Therefore, if *M* has cvc(e) it must be $cvc(-\alpha)$.

Theorem 2.2. Let $M = (V, \langle \cdot, \cdot \rangle, R)$ be a Lorentzian model space where dim(V) = 3 and let $\{e_1, e_2, e_3\}$ be an orthonormal basis with respect to the inner product and such that $R_{1221} = \alpha, R_{1331} = \beta, R_{2332} = \gamma$ where $\beta > \alpha > \gamma, \gamma \ge 0$ then if M has $cvc(\epsilon)$ then $\epsilon = -\beta$.

Proof. **Case1:** Let $v = e_1$. Then,

$$\begin{split} k(span\{v,w\}) &= \frac{R(e_1,w,w,e_1)}{\langle e_1,e_1 \rangle \langle w,w \rangle - \langle e_1,w \rangle^2} \\ &= \frac{R(e_1,ae_1 + be_2 + ce_3, ae_1 + be_2 + ce_3, e_1)}{\langle e_1,e_1 \rangle \langle ae_1 + be_2 + ce_3, ae_1 + be_2 + ce_3 \rangle - \langle e_1,ae_1 + be_2 + ce_3 \rangle^2} \\ &= \frac{R(e_1,be_2,be_2,e_1) + R(e_1,ce_3,ce_3,e_1)}{(-1)(-a^2 + b^2 + c^2) - (a^2)} \\ &= \frac{b^2 R_{1221} + c^2 R_{1331}}{-b^2 - c^2} \\ &= \frac{ab^2 + \beta c^2}{-b^2 - c^2}. \end{split}$$

Now solving for ϵ we get,

$$\frac{\alpha b^2 + \beta c^2}{-b^2 - c^2} = \epsilon$$
$$\alpha b^2 + \beta c^2 = -\epsilon b^2 - \epsilon c^2$$
$$b^2(\alpha + \epsilon) + c^2(\beta + \epsilon) = 0.$$

Thus, $\epsilon \in [-\beta, -\alpha]$.

Case 2: Let $v = e_2$. Then,

$$\begin{aligned} k(span\{v,w\}) &= \frac{R(e_2, w, w, e_2)}{\langle e_2, e_2 \rangle \langle w, w \rangle - \langle e_2, w \rangle^2} \\ &= \frac{R(e_2, ae_1 + be_2 + ce_3, ae_1 + be_2 + ce_3, e_2)}{\langle e_2, e_2 \rangle \langle ae_1 + be_2 + ce_3, ae_1 + be_2 + ce_3 \rangle - \langle e_2, ae_1 + be_2 + ce_3 \rangle^2} \\ &= \frac{R(e_2, ae_1, ae_1, e_2) + R(e_2, ce_3, ce_3, e_2)}{(1)(-a^2 + b^2 + c^2) - (b^2)} \\ &= \frac{a^2 R_{2112} + c^2 R_{2332}}{-a^2 + c^2} \\ &= \frac{aa^2 + \gamma c^2}{-a^2 + c^2}. \end{aligned}$$

Now solving for ϵ we get,

$$\frac{\alpha a^2 + \gamma c^2}{-a^2 + c^2} = \epsilon$$
$$\alpha a^2 + \gamma c^2 = -\epsilon a^2 + \epsilon c^2$$
$$a^2(\alpha + \epsilon) + c^2(\gamma - \epsilon) = 0.$$

And combing the two cases gives us, $\epsilon \in [-\beta, -\alpha] \cap [(-\inf, -\alpha] \cup [\gamma, \inf)] = [-\beta, -\alpha].$

Case 3: Let $v = e_3$. Then,

$$\begin{aligned} k(span\{v,w\}) &= \frac{R(e_3,w,w,e_3)}{\langle e_3,e_3 \rangle \langle w,w \rangle - \langle e_3,w \rangle^2} \\ &= \frac{R(e_3,ae_1 + be_2 + ce_3,ae_1 + be_2 + ce_3,e_3)}{\langle e_3,e_3 \rangle \langle ae_1 + be_2 + ce_3,ae_1 + be_2 + ce_3 \rangle - \langle e_3,ae_1 + be_2 + ce_3 \rangle^2} \\ &= \frac{R(e_3,ae_1,ae_1,e_3) + R(e_3,be_2,be_2,e_3)}{(1)(-a^2 + b^2 + c^2) - (c^2)} \\ &= \frac{a^2 R_{3113} + b^2 R_{3223}}{-a^2 + b^2} \\ &= \frac{\beta a^2 + \gamma c^2}{-a^2 + c^2}. \end{aligned}$$

Now solving for ϵ we get,

$$\frac{\beta a^2 + \gamma c^2}{-a^2 + c^2} = \epsilon$$
$$\beta a^2 + \gamma c^2 = -\epsilon a^2 + \epsilon c^2$$
$$a^2 (\beta + \epsilon) + c^2 (\gamma - \epsilon) = 0.$$

And combing the three cases gives us, $\epsilon \in [-\beta, -\alpha] \cap [(-\infty, -\beta] \cup [\gamma, \infty)] = \{-\beta\}$. Therefore, if *M* has $cvc(\epsilon)$ it must be $cvc(-\beta)$.

The above theorems can be used as a method to narrow down what the value of ϵ must be if the model space does infact have $cvc(\epsilon)$ and such that $\alpha > \beta \ge \gamma$ or $\beta > \alpha \ge \gamma$, whichever is largest. Now we will look at a model space and prove that it has $cvc(\epsilon)$ and given the curvature values we will be able to find the value of ϵ .

Theorem 2.3. Let $M = (V, \langle \cdot, \cdot \rangle, R)$ be a three-dimensional Lorentzian model space and let $\{e_1, e_2, e_3\}$ be an orthonormal basis on V with respect to $\langle \cdot, \cdot \rangle$ and let $R_{1221} = \alpha$, $R_{1331} = \beta$, $R_{2332} = \gamma$ such that $\alpha > \beta \ge \gamma$ then M has $cvc(-\alpha)$.

Proof. Case 1: If *v* is the form of $v = xe_1 + ye_2$ where one or both *x*, *y* is nonzero. In this case, choose *w* to be any vector so that $span\{v, w\} = span\{e_1, e_2\}$. Thus,

$$k(span\{e_1, e_2\}) = \frac{R(e_1, e_2, e_2, e_1)}{\langle e_1, e_1 \rangle \langle e_2, e_2 \rangle - \langle e_1, e_2 \rangle^2}$$
$$= \frac{\alpha}{(-1)(1) - (0)^2}$$
$$= -\alpha.$$

Case 2: If $v = xe_1 + ye_2 + ze_3$ with $z \neq 0$, we scale v so that z = 1 and consider the vector $w = ae_1 + be_2 + ce_3$. Then $x, z \neq 0$ or $y, z \neq 0$ are accounted for by the following, so if $v = xe_1 + ye_2 + e_3$ let

 $w = \sqrt{\frac{\gamma + \alpha}{\alpha - \beta}} e_1 + e_2$. Hence,

$$\begin{split} k(span\{v,w\}) &= \frac{R(v,w,w,v)}{\langle v,v \rangle \langle w,w \rangle - \langle v,w \rangle^2} \\ &= \frac{\alpha x^2 + \alpha \frac{\gamma + \alpha}{\alpha - \beta} y^2 + \beta \frac{\gamma + \alpha}{\alpha - \beta} - 2\alpha \sqrt{\frac{\gamma + \alpha}{\alpha - \beta}} xy + \gamma}{(-x^2 + y^2 + 1)(-\frac{\gamma + \beta}{\alpha - \beta}) - (\frac{\gamma + \alpha}{\alpha - \beta} x^2 + y^2 - 2\sqrt{\frac{\gamma + \alpha}{\alpha - \beta}} xy)} \\ &= \frac{\alpha (x^2 + \frac{\gamma + \alpha}{\alpha - \beta} y^2 - 2\sqrt{\frac{\gamma + \alpha}{\alpha - \beta}} xy + \frac{\gamma + \beta}{\alpha - \beta})}{-x^2 - \frac{\gamma + \alpha}{\alpha - \beta} y^2 + 2\sqrt{\frac{\gamma + \alpha}{\alpha - \beta}} xy - \frac{\gamma + \beta}{\alpha - \beta}} \\ &= -\alpha. \end{split}$$

3 CONCLUSION

Given a Lorentzian model space in three-dimensions in which the Ricci tensor is diagonalized with respect to the inner product such that $R_{1221} > R_{1331} \ge R_{2332}$, then that model space must have $cvc(\epsilon)$ for some ϵ . Additionally, if the values of the curvature tensor are known then it is possible to find the value of ϵ .

4 **OPEN QUESTIONS**

- Under the condition in three dimensions in which the Ricci tensor is not necessarily diagonalized, what can we conclude about the constant vector condition?
- Is a Lorentzian model space of higher dimensions cvc anything? If so, what conditions must hold to make this true?

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