Relating Hyperbolic Braids and $PSL_2(\mathbb{Z})$

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Abstract

We focus on hyperbolic braids in B_3. In particular we find properties belonging to hyperblic and non-hyperbolic mapping tori when mapped to $PSL_2(\mathbb{Z})$. Using a map $\pi : B_3 \to PSL_2(\mathbb{Z})$, we compare the hyperbolic geometry of mapping tori to their closed braids. We show for every matrix $M \in PSL_2(\mathbb{Z})$ that is non-hyperbolic $\hat{\beta}$ is non-hyperbolic where $\pi(\beta) = M$.

1 Braids

Definition 1. *A **braid** is a set of n strings, all of which are attatched to a horizontal bar at the top and at the bottom. Each string intersects any horizontal plane between the two bars exactly once.*

Braids are considered equivalent if the strings of the braid can be rearranged without passing any strings through one another or themselves and keeping the top and bottom bars fixed with the strings attached. For example, the braids in Figure 1 are equivalent.

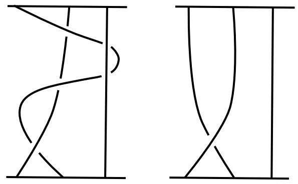

Figure 1: two equivalent braids

Definition 2. *The **closure** of a braid is when the bottom bar of the braid is glued to the top bar.*

In turn, this lets every single braid correspond to a knot or link. We also think of the closure wrapping around an axis that is perpendicular to the plane containing the braid. This puts an orientation on the braid, as we always travel clockwise around the braid axis. This gives us a **closed braid representation** of the braid. This is represented in Figure 2.

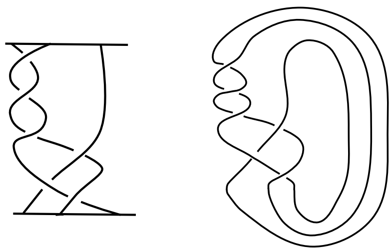

Figure 2: A braid and its closure

1.1 Braid Generators

An $n - braid$ is a braid with n strands. We will focus on the group of $3 - braids$, which we denote by B_3. The group B_3 can be described with two generators, σ_1 and σ_2. A generator refers to different crossings on certain strands of the braid. We will call σ_1 the generator where the strand on the left crosses over the strand in the middle while the right most strand stays straight. If the left strand crosses under the middle strand the generator is called σ_1^{-1}. If the middle strand crosses over the right strand, we call it σ_2. Similar to σ_1^{-1}, if the right strand crosses over the middle strand we denote it σ_2^{-1}.

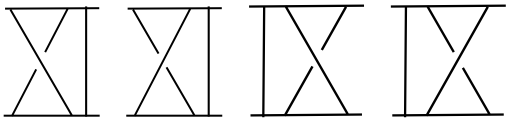

Figure 3: ordered from left to right: σ_1, σ_1^{-1}, σ_2, and σ_2^{-1}.

We can completely describe B_3 with these generators. We list the generators in order from top to bottom of the braid it represents. We call this list of generators the **braid word**.

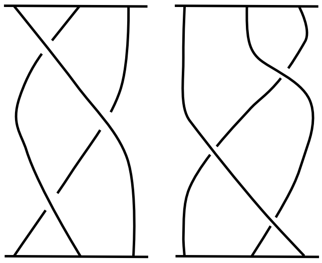

Figure 4: The braids $\sigma_1\sigma_2\sigma_1$ and $\sigma_2\sigma_1\sigma_2$.

Many braid words are equivalent to one another. For example, the braid corresponding to $\sigma_1\sigma_1^{-1}$ is equivalent to the identity element of the group. Similarly, the braid $\sigma_1\sigma_2\sigma_1$ is equivalent to $\sigma_2\sigma_1\sigma_2$ as illustrated in Figure 4.

One important braid for our results is the full twist. Full twists can be represented with multiple braid words but we will use $(\sigma_1\sigma_2)^3$. This is illustrated in Figure 5.

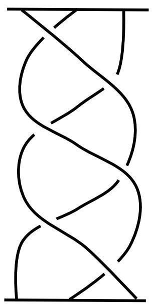

Figure 5: The full twist : $(\sigma_1\sigma_2)^3$.

Definition 3. *The braids α and β are* ***conjugate*** *if there exists a braid γ such that $\gamma^{-1}\beta\gamma = \alpha$.*

Definition 4. *The conjugacy class* $[\beta]$ *of the braid* β *is the set of all braids conjugate to* β*. In other words,* $[\beta] = \{\alpha | \alpha = \gamma^{-1}\beta\gamma$ *for some* $\gamma\}$*.*

Note that conjugate braid words close to the same link. In Figure 6 we can think of sliding γ around to the top of the closed braid to cancel with γ^{-1}.

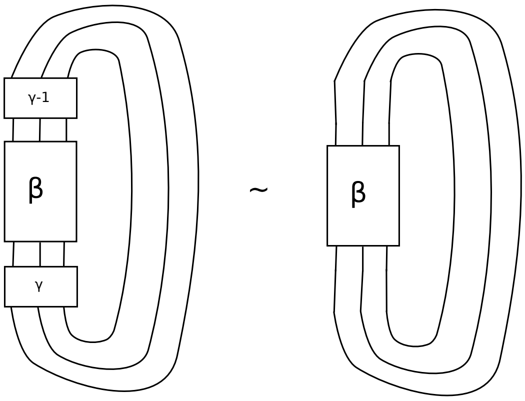

Figure 6: β and $\gamma^{-1}\beta\gamma$ close to the same link.

1.2 Hyperbolic and Non-Hyperbolic Braids

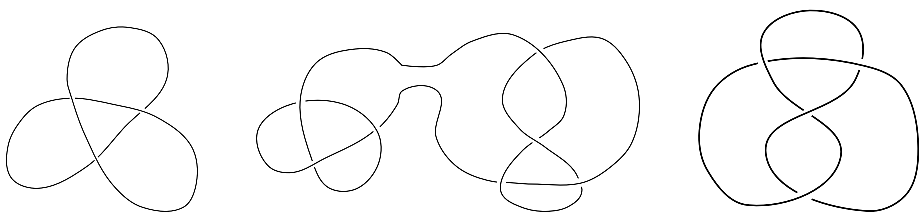

Figure 7: A torus knot, a satellite knot, and a hyperbolic knot.

A knot (link, or closed braid) can be a satellite knot, a torus knot, or a hyperbolic knot. A satellite knot contains an incompressible, non-boundary parallel torus in its complement, and a torus knot sits on the surface of a torus in $\mathbb{R}^3$.

Definition 5. *A* ***hyperbolic knot*** *is a knot with a complement that can be given a metric of constant curvature* -1.

All non hyperbolic braids in the braid group are one of the following forms:
i. σ_1^p ii. σ_2^p iii. e (the identity element) iv. $(\sigma_1\sigma_2)^p$ v. $\sigma_1^p\sigma_2$
vi. $\sigma_2^p\sigma_1$ vii. $\sigma_1^p\sigma_2^q$ viii. $\sigma_2^p(\sigma_1\sigma_2^2\sigma_1)^q$ shown in [2] ix. $(\sigma_1\sigma_2)^{3p}\sigma_1\sigma_2\sigma_1$
We will examine non-hyperbolic mapping tori of braids in B_3.

Definition 6. *Let ϕ be a mapping of the surface S to itself. A **mapping torus** is a 3-manifold of any ϕ from the homeomorphism ϕ of a surface S to itself. The mapping torus can be constructed by first finding the product $S \times I$, then gluing $S \times \{1\}$ to $S \times \{0\}$ using ϕ.*

Example 1. *Let ϕ be the identity map $\phi : \mathbb{D} \to \mathbb{D}^2$. Then M_ϕ is the solid torus shown in Figure 8.*

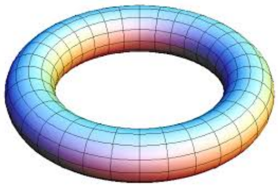

Figure 8

A braid $\beta \in B_3$ can be thought of as a homeomorphism of a thrice punctured disc $D = \mathbb{D}^2 - \{p_1, p_2, p_3\}$. Thus we can form the mapping torus M_β of the braid $\beta \in B_3$. The mapping torus of a braid is illustrated in Figure 9 where the disk enclosing the braid axis is D. Note that there is a close relationship between the mapping torus M_β and the closed braid $\hat{\beta}$. In fact, let A_β be the braid axis of β, and $L = \hat{\beta} \cup A_\beta$. Then $M_\beta = S^3 - L$, and to get $\hat{\beta}$ from M_β you perform a $(1, 0)$ Dehn filling on the axis A_β.

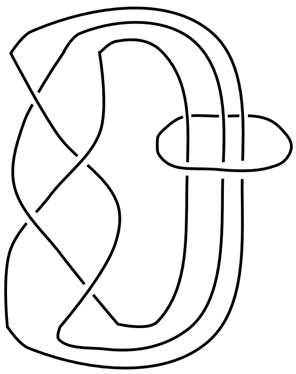

Figure 9: mapping torus of the closed braid $\hat{beta} = \sigma_1 \sigma_2^{-1} \sigma_1$.

1.3 $PSL_2(\mathbb{Z})$

Definition 7. *$PSL_2(\mathbb{Z})$ is the projective speical linear group of 2 by 2 matricies with integer entries and determinant ± 1.*

A matrix $M \in PSL_2(\mathbb{Z})$ is **hyperbolic** when the $tr(M) >\mid 2 \mid$.

A matrix $M \in PSL_2(\mathbb{Z})$ is **parabolic** when the $tr(M) =\mid 2 \mid$.

A matrix $M \in PSL_2(\mathbb{Z})$ is **elliptic** when the $tr(M) \leq\mid 1 \mid$.

There is a natural epimorphism from $PSL_2(\mathbb{Z})$ with generators $\pi(\sigma_1^{-1})$ and $\pi(\sigma_2)$, where $\pi : B_3 \to PSL_2(\mathbb{Z})$ is the epimorphism from the 3-braid group B_3 to $PSL_2(\mathbb{Z})$ given by:

$$\pi(\sigma_1^{-1}) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \pi(\sigma_2) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

The kernel of π is the cyclic group generated by the full twist, or $N = (\sigma_1\sigma_2)^{3k}$.

If there exists an $M \in PSL_2(\mathbb{Z})$ where $\pi(\beta) = M$, then $\pi(\beta)$ really corresponds to the mapping torus M_β of the braid β. From [4], the mapping torus M_β of the braid β is hyperbolic if and only if $\pi(\beta)$ is a hyperbolic matrix. Our goal is to compare the hyperbolic geometry of the mapping torus M_β and the closed braid $\hat{\beta}$. We will think of $PSL_2(\mathbb{Z})$ as the subgroup of isometries of $\mathbb{H}^2$ consisting of all linear fractional maps

$$\varphi(z) = \frac{az+b}{cz+d},$$

where $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$.

2 Results

The following is from [3], but for completeness we will restate the definition.

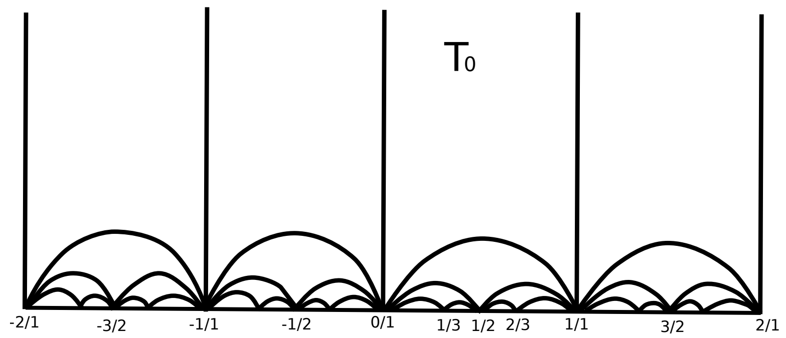

Figure 10: The Farey Tessellation

Definition 8. *If we erase the circles $C_{\frac{p}{q}}$ and connect the two points $(\frac{p}{q}, 0)$ and $(\frac{p'}{q'}, 0)$ by a semi-circle centered on the x-axis exactly when the circles $C_{\frac{p}{q}}$ and $C_{\frac{p'}{q'}}$ are tangent. The resulting collection of hyperbolic geodesics is the* ***Farey tessellation*** *of the hyperbolic plane.*

Given ant triangle T in the Farey Tessellation, there is a matrix $M \in PSL_2(\mathbb{Z})$ mapping $T \to T_0$.

2.1 A result on non-hyperbolic closed braids

Theorem 1. *If $M \in PSL_2(\mathbb{Z})$ and is non-hyperbolic, then $\hat{\beta}$ is non-hyperbolic where $\pi(\beta) = M$.*

Recall that if M is non-hyberbolic, then $tr(M)^2 \leq 4$. Since $tr(M) \in \mathbb{Z}$, $tr(M) = 0, \pm 1, \pm 2$. Moreover, in $PSL_2(\mathbb{Z})$, a matrix and its negative are the same, so we assume $tr(M) \geq 0$.

Proof. **Case 1:** $tr(M) = 0$

Let $M = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ where $-a^2 - bc = 1$. The fixed point p of $\varphi(M)$ is $p = \frac{a}{c} + \frac{1}{c}i$. The point p lies in a Farey triangle T. There exists an $A \in PSL_2(\mathbb{Z})$ such that $T \to T_0$ where T_0 is the Farey Triangle with vertices at $x = 0, 1$, and ∞.

Then the fixed point of AMA^{-1} is $A(p)$. Indeed, $AMA^{-1}(A(p)) = AM(p) = A(p)$, justifying the claim. Now we have that M is conjugate to an $M' \in PSL_2(\mathbb{Z})$ fixing a point in T_0.

Lemma 1. *If $M' \in PSL_2(\mathbb{Z})$ and $tr(M') = 0$, and M' fixes a point p in T_0, then $p = i, \frac{1}{2} + \frac{1}{2}i$, or $1 + i$.*

Proof. Let M' satisfy the hypotheses of the Lemma. We know $p = \frac{a}{c} + \frac{1}{c}i$, and since p is in T_0, $0 \leq \frac{a}{c} \leq 1$ and $(\frac{1}{c})^2 \geq \frac{1}{4} - (\frac{a}{c} - \frac{1}{2})^2$. Solving this system of inequalities gives $p = i, \frac{1}{2} + \frac{1}{2}i$, or $1 + i$. $\square$

The points $i, \frac{1}{2} + \frac{1}{2}i$, and $1 + i$ are all related by a rotation $R = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$. Therefore, adjusting A by a power of R if necessary, M is conjugate to an isometry that fixes i, and any traceless matrix M is conjugate to $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in PSL_2(\mathbb{Z})$ who's fixed point is i.

Claim: If $\pi(\beta) = M$ and $tr(M) = 0$, then $\hat{\beta}$ is non-hyperbolic.

Proof. The claim is true for $\beta_0 = \sigma_1\sigma_2\sigma_1$, since $\pi(\beta_0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and $\hat{beta}_0$ is the Hopf Link which is not hyperbolic. Now let $\beta' \in B_3$ be such that $\pi(\beta) = M$ is traceless. By Lemma 1 M is conjugate to $\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$, so β' is conjugate to a β such that $\pi(\beta) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, which is of the form $\beta = N^p\sigma_1\sigma_2\sigma_1$. Thus, $\hat{\beta}' = \hat{\beta} = \widehat{N^p\sigma_1\sigma_2}\sigma_1$, and we show the last link is non-hyperbolic. Figure 11 illustrates the projection of $\widehat{N\beta}$. This link has an essential annulus in the complement, and therefore is non-hyperbolic. Indeed, the solid component is a trefoil which lies on a torus. The second component lies entirely inside the torus. Cutting the torus along the trefoil gives the desired annulus.

Therefore, if $M \in PSL_2(\mathbb{Z}), tr(M) = 0$, then for all $\hat{\beta}$ such that $\pi(\beta) = M$, $\hat{\beta}$ is non-hyperbolic. $\square$

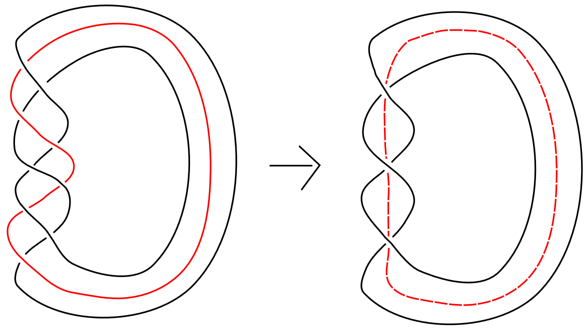

Figure 11: essential annulus of $\widehat{N\beta}$.

Case 2: $tr(M) = 1$

Let $M = \begin{pmatrix} a & b \\ c & 1-a \end{pmatrix}$ where $a(1-a) - cb = 1$. The fixed point p of $\varphi(M)$ is $p = \frac{-1+2a}{2c} + \frac{\sqrt{3}}{2c}i$. By the same argument as in case 1, p lies in a Farey Triangle T_i and there is an $A \in PSL_2(\mathbb{Z})$ such that $T_i \to T_0$. The fixed point of AMA^{-1} is $A(p)$, and M is conjugate to an $M' \in PSL_2(\mathbb{Z})$ fixing a point in T_0.

Lemma 2. *If $M' \in PSL_2(\mathbb{Z})$ and $tr(M') = 1$, and M' fixes a point p in T_0, then $p = \frac{1}{2} + \frac{\sqrt{3}}{2}i$.*

Proof. Let M' satisfy the hypotheses of the Lemma. We know $p = \frac{-1+2a}{2c} + \frac{\sqrt{3}}{2c}i$, and since p is in T_0, $\frac{1}{2} < a < c + \frac{1}{2}$ and $3 \geq (-1+2a)(2c-2a+1)$. Solving these inequalities gives $a = c = 1$, so the fixed point is $p = \frac{1}{2} + \frac{\sqrt{3}}{2}$. $\square$

There are two such matrices with a fixed point at $p = \frac{1}{2} + \frac{\sqrt{3}}{2}$. So, M is conjugate to an isometry that fixes $\frac{1}{2} + \frac{\sqrt{3}}{2}$. Any matrix M with a trace of ± 1, is conjugate to $\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$ who's fixed point is $\frac{1}{2} + \frac{\sqrt{3}}{2}$.

Claim: If $\pi(\beta) = M$ and $tr(M) = 1$, then $\hat{\beta}$ is non-hyperbolic.

Proof. The claim is true for $\beta_0 = \sigma_1^{-1}\sigma_2^{-1}$ since $\pi(\beta_0) = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$, and $\hat{\beta}_0$ is the unknot. Similarly, if $\beta = \sigma_2\sigma_1$, then $\pi(\beta_1) = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$ and $\hat{\beta}$ is not hyperbolic. Now let $\beta' \in B_3$ where $\pi(\beta') = M$ has trace 1. Then β' is conjugate to a β such that $\pi(\beta) = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$, or $\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$, which is of the form $\beta = N^p\sigma_1^{-1}\sigma_2^{-1}$ or $\beta = N^p\sigma_2\sigma_1$. $\hat{\beta}' = \hat{\beta} = \widehat{N^p\sigma_1^{-1}\sigma_2^{-1}}$ which will always be a torus knot as ilustrated in figure 12.

Therefore, if $M \in PSL_2(\mathbb{Z}), tr(M) = 1$, then for all $\hat{\beta}$ such that $\pi(\beta) = M$, $\hat{\beta}$ is non-hyperbolic. $\square$

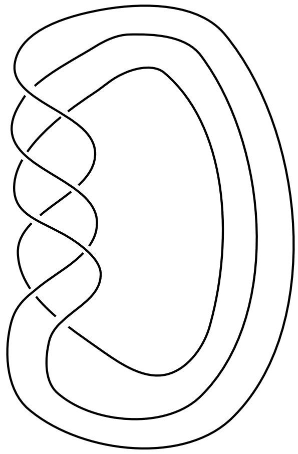

Figure 12: $N\beta_0$ is a torus link

Case 3: $tr(M) = 2$.

Let $M = \begin{pmatrix} a & b \\ c & 2-a \end{pmatrix}$ where $a(1-a) - cb = 1$. The fixed point p of $\varphi(M)$ is $p = \frac{r}{s}$ where $r, s \in \mathbb{Z}$.

Claim: There exists an $A \in PSL_2(\mathbb{Z})$ such that $A(\frac{r}{s}) = \infty$.

Proof. Let $r, s \in \mathbb{Z}$, be in reduced form. Since r and s are relatively prime, we can find an $t, u \in \mathbb{Z}$ such that $rt - su = 1$. Then $\begin{pmatrix} p & s \\ q & r \end{pmatrix} \in PSL_2(\mathbb{Z})$ and $\varphi(z) = \frac{rz+t}{sz+u}$ is such that $\varphi(\infty) = \frac{r}{s}$. $\varphi^{-1}(\frac{r}{s}) = \infty$, and let $A = \varphi^{-1}$.

□

The fixed point p of AMA^{-1} is $A(\frac{r}{s})$ so $p = \infty$. Since any isometry of $\mathbb{H}^2$ fixing ∞ is a horizontal translation, we have $AMA^{-1} = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ for some $n \in \mathbb{Z}$.

Remark: each n gives a different conjugacy class in $PSL_2(\mathbb{Z})$ and M is conjugate to exactly one matrix $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \in PSL_2(\mathbb{Z})$.

Claim: If $\pi(\beta) = M$ and $tr(M) = 2$, then $\hat{\beta}$ is non-hyperbolic.

Proof. The claim is true for $\beta_0 = \sigma_1^n$ since $\pi(\beta_0) = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ and β_0 is a split link with one component a (2,n) torus link. Adding full twists wraps the unknotted component around the (2,n) torus link, while adding twists to the torus link. These are the links Birman and Menasco found to be non-hyperbolic in [2]. An illustration is shown in Figure 13.

Let $\pi(\beta) = M$ and $tr(M) = 2$, then β is conjugate to β' with $\pi(\beta') = \begin{pmatrix} 1 & n \\ 0 & n \end{pmatrix}$.

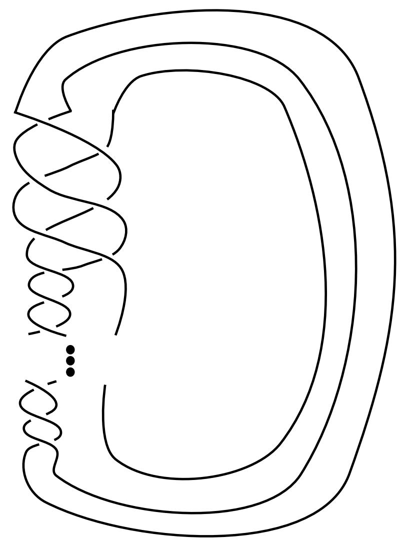

Figure 13: The braid $N^p\sigma_1^n$ is never hyperbolic.

□

□

Corollary 1. *It follows from Theorem 1 if* $\hat{beta}$ *is hyperbolic and* $\pi(\beta) = M$*, then* $M \in PSL_2(\mathbb{Z})$ *is hyperbolic.*

On the other hand, if $M \in PSL_2(\mathbb{Z})$, is hyperbolic, and $\pi(\beta) = M$, the closed braid $\hat{\beta}$ is not necessarily hyperbolic.

Example 2. *Let* $\hat{\beta} = \sigma_1^5\sigma_2$*. We know that* $\hat{\beta}$ *is non-hyperbolic.*

However, $\pi(\beta) = \begin{pmatrix} -4 & -5 \\ 1 & 1 \end{pmatrix} = M$*, and tr(M)=3, so M is not hyperbolic, meaning* $\pi(\beta)$ *maps to a hyperbolic mapping torus.*

In fact, we can go further to say that if $\hat{\beta}$= $\sigma_1^p\sigma_2$ or $\sigma_2^p\sigma_1$, and $p \geq 5$, then $tr(\pi(\beta)) > 2$, making $\pi(\beta)$ hyperbolic. This is shown in Figure 14.

3 Acknowledgements

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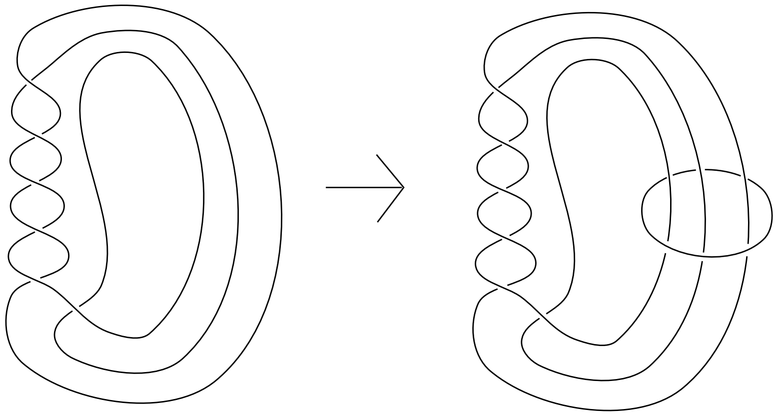

Figure 14: The non-hyperbolic closed braid $\hat{\beta} = \sigma_1^5 \sigma_2$ on the left maps to the hyperbolic mapping torus shown on the right.

References

[1] M. T. Lozano and J. H. Przytycki. Incompressible surfaces in the exterior of a closed 3-braid. *Math. Proc. Phil. Soc.* **98** (1985), 275-299.

[2] Joan S. Birman and William H. Menasco. Special Positions for Essential Tori in Link Complements. *Topology.* **33** Num. 3. (1994), 525-556.

[3] Francis Bonahon. Low-Dimensional Geometry From Euclidean Surfaces to Hyperbolic Knots. **49** (2009).

[4] William P. Thurston. *Hyperbolic Structures on 3-manifolds, II: surface groups and 3-manifolds which fiber over the circle.*