Linear Dependence of Algebraic Curvature Tensors with Associated Chain Complexes

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August 11, 2015

1 Introduction

Definition 1. *Let V be a real, finite-dimensional vector space. Let $R : V^4 \to \mathbb{R}$ be a multilinear function. Then, we call R an* ***algebraic curvature tensor*** *if it satisfies the following properties:*

1. $R(x, y, z, w) = -R(y, x, z, w)$
2. $R(x, y, z, w) = R(z, w, x, y)$
3. $R(x, y, z, w) + R(z, x, y, w) + R(y, z, x, w) = 0$

We call the last property the Bianchi Identity. We denote the vector space of all algebraic curvature tensors on V as $A(V)$.

In an inner product space, there is a natural way to construct an algebraic curvature tensor from the inner product.

Theorem 1. *Let V be a vector space with positive-definite inner product ϕ. Then,*

$$R_\phi(x, y, z, w) = \phi(x, w)\phi(y, z) - \phi(x, z)\phi(y, w)$$

is an algebraic curvature tensor.

We can extend this construction to certain linear operators and matrices on V as follows.

Definition 1. *Let V be a vector space with positive-definite inner product ϕ. Let τ be a bilinear form on V. Let $A : V \to V$ be a linear operator. Then, we say that the following are* ***canonical algebraic curvature tensors of symmetric build:***

$$R^S_A(x, y, z, w) = \phi(Ax, w)\phi(Ay, z) - 7\phi(Ax, z)\phi(Ay, w)$$
$$R^S_\tau(x, y, z, w) = \tau(x, w)\tau(y, z) - \tau(x, z)\tau(y, w)$$

Additionally, we say that the following are ***canonical algebraic curvature tensors of anti-symmetric build:***

$$R_A^\Lambda(x,y,z,w) = \phi(Ax,w)\phi(Ay,z) - \phi(Ax,z)\phi(Ay,w) - 2\phi(Ax,y)\phi(Az,w)$$
$$R_\tau^\Lambda(x,y,z,w) = \tau(x,w)\tau(y,z) - \tau(x,z)\tau(y,w) - 2\tau(x,y)\tau(z,w)$$

This idea of *building* an algebraic curvature tensor out of a matrix or bilinear form is central to our study the tensors themselves. Throughout this paper, we will say two canonical algebraic curvature tensors R_A and R_B have the "same build" if A and B are either both symmetric or both antisymmetric.

By citeDiroff , we have that $R_A^S, R_\tau^S \in A(V) \iff A = A^*$ and τ is a symmetric form. Similarly, we have that $R_A^\Lambda, R_\tau^\Lambda \in A(V) \iff A = -A^*$ and τ is an anti-symmetric form. This justifies our use of the phrase "algebraic curvature tensor" in the second definition.

It is important to note that if R_A is an algebraic curvature tensor, then $R_{\lambda A}$ is also an algebraic curvature tensor for $\lambda \in \mathbb{R}$, with $R_{\lambda A}(x,y,z,w) = R(\lambda Ax, \lambda Ay, z, w) = \lambda^2 R(Ax, Ay, z, w) = \lambda^2 R_A$ by the multilinearity of R. Thus, if $\{R_{\lambda_1 A_1}, ..., R_{\lambda_k A_k}\}$ is a linearly dependent set, so is $\{R_{A_1}, ..., R_{A_k}\}$ for $\lambda_i \in \mathbb{R}$. Therefore, when we pick coefficients for a linear dependence equation, we will only choose ± 1, as any positive scalar multiple of ± 1 can be "absorbed" into the operator in the subscript.

We wish to consider when a set of canonical algebraic curvature tensors (of either build) is linearly independent. When the build of a particular tensor is unimportant or unknown, we may omit the superscript: R_A or R_τ. Furthermore, we will assume throughout that V denotes a real, finite-dimensional vector space, so that M is a symmetric operator on V if and only if $M = M^*$, and M is an antisymmetric operator on V if and only if $M = -M^*$.

To place restrictions on our different sets of tensors, we borrow an idea from algebraic topology: the chain complex. For our purposes, we will consider only complexes of operators that all act on the same vector space.

Definition 1. *Let V be a vector space, and let $A_1, ..., A_k : V \to V$ be linear operators on V. If $ImA_i \subset KerA_{i+1}$ for $1 \le i \le k-1$, then we call $\mathbb{D} = (A_1, ..., A_k)$ a* ***chain complex****, and we write*

$$V \xrightarrow{A_1} \dots \xrightarrow{A_i} V \xrightarrow{A_{i+1}} \dots \xrightarrow{A_k} V$$

If $\mathbb{D}_1, ...\mathbb{D}_l$ are chain complexes, we call $\mathbb{E} = \cup_{i=1}^l \mathbb{D}_i$ a ***compound chain complex*** *and we write*

$$
\begin{array}{ccccccccc}
V & \xrightarrow{A_1} & ... & \xrightarrow{A_i} & V & \xrightarrow{A_{i+1}} & ... & \xrightarrow{A_k} & V \\
\\
V & \xrightarrow{B_1} & ... & \xrightarrow{B_i} & V & \xrightarrow{B_{i+1}} & ... & \xrightarrow{B_k} & V \\
 & & & & \vdots & & & & \\
V & \xrightarrow{T_1} & ... & \xrightarrow{T_i} & V & \xrightarrow{T_{i+1}} & ... & \xrightarrow{T_k} & V
\end{array}
$$

We will study linearly dependent sets of algebraic curvature tensors constructed from a set of operators which satisfy a chain complex. Studying the operator from which an algebraic curvature tensor was built can give us information about the tensor itself, by the following result.

Theorem 1. *Let V be a real, finite-dimensional vector space, and let $A : V \to V$ be an operator on V. Then,*

1. *If $RkA \leq 1$, then $R_A = 0$*
2. *If $RkA \geq 2$, then $KerR_A = KerA$*

Thus, a chain complex structure also gives us information about the algebraic curvature tensors constructed from the operators. Depending on the assumptions involved, we may be able to put bounds on the dimension of the vector space as well.

For a given compound chain complex with a linear dependence equation, We wish to combine the information derived from a the complex with our knowledge about tensor behavior. To that end, we introduce an operation which allows for some convenient results.

Definition 1. *Let A, B be operators on a real, finite-dimensional vector space V, and let R_A be the canonical algebraic curvature tensor built from A. Then, we define* ***pre-composition by B****, denoted B^*R_A, by the following*

$$
\begin{aligned}
B^*R_A(x,y,z,w) &= R_A(Bx,By,Bz,Bw) = R(ABx,ABy,Bz,Bw) \\
&= R(B^*ABx,B^*ABy,z,w) = R_{B*AB}(x,y,z,w)
\end{aligned}
$$

It follows readily from this construction that for symmetric or anti-symmetric operators A and B, if BA = 0, then $A^*R_B = B^*R_A = 0$. In working with chain complexes and associated linear dependence equations, it is often the case that a tensor will vanish under precomposition by a certain operator. We make use of this result to reduce the number of terms in our linear dependence equation.

Since a chain complex with a single base vector space is similar in construction to a directed graph, we also use some basic techniques from graph theory to help us classify chain complexes. For a compound chain complex on a vector space V, we wish to associate each instance of V with a distinct vertex and associate the operators on V to maps between vertices that preserve the component complexes. To allow for a unique association between a compound chain complex and a directed graph, we need notions of a source and a sink.

Definition 1. *Let G be a finite directed graph. We call a vertex $v \in G$ a* ***source*** *if v receives no edges in G. We call v a* ***sink*** *if v sends no edges in G.*

Since it is possible that in a compound chain complex, there are many different "sources" (in the way that different component complexes may begin with different operators) and different "sinks" (in the way that different compononet complexes may end with different operators), in our complex-to-graph injection, we map each "source" space to the same vertex and each "sink" space to the same vertex. This gives a unique graphic representation for each compound chain complex, and restricts the kinds of graphs we need to consider for our purposes.

2 Previous Work

The association of a chain complex to a linear dependence equation for algebraic curvature tensors was previously studied in citeElise. Notable results are reproduced here without proof.

Lemma 1. *If A, B are symmetric or antisymmetric operators with $ImB \subset KerA$ or $ImA \subset KerB$, then*

$$B * R_A = R_{B^*AB} = R_{\pm BAB} = R_{BAB} = 0$$

This is a corollary to THEOREM3, and the connecting bridge between linear dependence of algebraic curvature tensors and compound chain complexes.

Theorem 1. *Let A, B, C, D be operators on V with $RkA, RkB, RkC, RkD \geq 4$. Suppose $R_A + \alpha_1 R_B + \alpha_2 R_C + \alpha_3 R_D = 0$ and the operators satisfy the chain complex*

$$V \xrightarrow{A} V \xrightarrow{B} V \xrightarrow{C} V \xrightarrow{D} V \xrightarrow{A} V$$

Then, we have

1. *R_A and R_C have the same build, and $A^3C = \pm ACAC = AC^3$*
2. *R_B and R_D have the same build, and $B^D = \pm BDBD = BD^3$*

Theorem 1. *Let $A, B_1, ..., B_k$ be operators on vector space V such that $0 = R_A + \sum \alpha_i R_{B_i}$. Suppose the operators fit one of the following compound chain complexes:*

$$V \xrightarrow{A} V \xrightarrow{B_i} V$$

$$\begin{array}{ccccc} & B_i & & A & \\ V & \longrightarrow & V & \longrightarrow & V \end{array}$$

For symmetric or antisymmetric A, $R_A = 0$. For antisymmetric A, if the sequence is exact for some B_i, then B_i is invertible.

These results were achieved through precomposition by the operator A and use of LEMMA4.1. Following are some other previous results that we will reference.

Lemma 1. *CITEELISE If $A = \pm A^*$ is an operator on V and $p, k \in \mathbb{N}$, then*

$$RkA = p \iff RkA^k = p$$

The backwards direction was proven in CITEELISE. To show the forward direction, first assume A is symmetric. Diagonalize A, so that A^k is diagonalized as well. Then RkA is the number of nonzero diagonal entries of A and RkA^k is the number of nonzero diagonal entries of A^k. So if $A = [a_{ij}]$ and $A^3 = [\hat{a}_{ij}]$, then for all i,

$$\hat{a}_{ii} = a_{ii}^3$$

So, $\hat{a}_{ii} = a_{ii}^3 = 0 \iff a_{ii} = 0$. Therefore, the number of nonzero diagonal entries of A^3 is exactly the same as the number of nonzero diagonal entries of A.

Lemma 1. *CITEGILKEY If A, B are operators on V with $R_A = R_B$, then $A = \pm B$ if*

1. *A, B are symmetric and $RkA \geq 3$*
2. *A, B are antisymmetric*

Lemma 1. *CITEDIAZANDDUNN CITETREADWAY If A, B are operators on V with $RkA \geq 3$,*

1. *if A, B are symmetric, then $R_A^S \neq -R_B^S$*
2. *if A, B are antisymmetric, then $R_A^\Lambda \neq -R_B^\Lambda$*

Lemma 1. *CITETREADWAY CITELOVELL Let A be an antisymmetric operator on V with $RkA \geq 4$. If B is a symmetric operator on V, then*

$$R_A^\Lambda \neq \pm R_B^S$$

For convenience, we combine the previous three lemmas for the following result:

Corollary 1. *Suppose A, B are symmetric or antisymmetric operators on V with $RkA \geq 3$ and we have*

$$R_A = \pm R_B$$

Then, R_A and R_B have the same build, $R_A = R_B$, and

$$A = \pm B$$

3 Motivation

The primary problem I have studied follows. Let A, B, C, D be symmetric or antisymmetric operators on V with $RkA, RkB, RkC, RkD \geq 3$. Suppose these operators satisfy the compound chain complex

$$\begin{array}{ccccccc} & A & & C & & D & \\ V & \longrightarrow & V & \longrightarrow & V & \longrightarrow & V \end{array}$$

$$\begin{array}{ccccc} & B & & D & \\ V & \longrightarrow & V & \longrightarrow & V \end{array}$$

Together with this diagram, we associate the linear dependence relationship $R_A + \alpha_1 R_B + \alpha_2 R_C + \alpha_3 R_D = 0$. Examination begins by precomposing this equation with each of A, B, C, D in order to achieve four new linear dependence equations:

$$\begin{aligned}
&R_{A^3} + \alpha_1 R_{ABA} + \alpha_3 R_{ADA} = 0 \\
&R_{BAB} \alpha_1 R_{B^3} + \alpha_2 R_{BCB} = 0 \\
&\alpha_2 R_{CBC} + \alpha_3 R_{C^3} - = 0 \\
&R_{DAD} + \alpha_3 R_{D^3} = 0
\end{aligned}$$

We omit tensors which are identically 0 by LEMMA4.1. Note that each of our new equations has fewer terms than our original equation. Intuitively, we have traded working with simple matrices and many tensors for working with more complicated matrices and fewer tensors. This is the convenience of precomposition, and we are especially interested in cases where we are left with two-term equations, such as

$$\begin{aligned}
R_{C^3} &= -\alpha_1 \alpha_2 R_{CBC} \\
R_{D^3} &= -\alpha_3 R_{DAD}
\end{aligned}$$

By LEMMA7 and LEMMA10, we have that $RkC^3, RkD^3 \geq 3$, and thus

1. R_{D^3} and R_{DAD} have the same build and $\alpha_3 = -1$
2. R_{C^3} and R_{CBC} have the same build and $\alpha_1 = -\alpha_2$

With a result we will develop later, it can be shown further that R_A, R_D must have the same build and R_B, R_C must have the same build. Additionally, we note the following about the complex

1. If A is invertible, then C = 0.
2. If B is invertible, then D = 0.
3. If C is invertible, then A = D = 0, and $R_C = \pm R_B$.

4. If D is invertible, then B = C = 0, and $R_A = \pm R_D$.
5. We can rewrite our equation to achieve $R_A = -\alpha_1 R_B + \alpha_1 R_C + R_D$. Thus, $KerA = KerR_A = Ker(-\alpha_1 R_B + \alpha_1 R_C + R_D) \subset KerR_B \cap KerR_C \cap KerR_D = KerB \cap KerC \cap KerD$. We could rearrange terms to show something similar for $KerB, KerC, KerD$.

From this examination, some natural questions arise:

1. How much information about operators does a chain complex encode?
2. For a linear dependence equation with a chain complex, does reducing the number of terms by precomposition yield more information about the operators?
3. For a given number of operators, how many chain complexes are possible?

In this paper, I will begin to answer all of these questions. My hope is to lay a strong foundation for the study of chain complexes with linear dependence equations of algebraic curvature tensors, and to motivate deeper study in the field.

4 Compound Chain Complexes and Their Operators

We want to study compound chain complexes in general, especially with respect to operators. Given a chain complex, we want to decide what kinds of operators could satisfy. We take a kernel-based approach, so that an operator splits the vector space V into two parts: the image of A and the kernel of A. The following result justifies this perspective.

Lemma 1. *Let $A = \pm A^*$ be an operator on V. Then,*

$$ImA \cap KerA = \{0\}$$

Proof by contradition. Suppose we have a nonzero $v \in ImA \cap KerA$. Let $\mathcal{B} = \{v, e_1, ..., e_{m-1}\}$ be a basis for ImA. Then,

$$\{Av, Ae_1, ..., Ae_{m-1}\}$$

is a spanning set for ImA^2. But $Av = 0$ by assumption, so $\mathcal{B}' = \{Ae_1, ..., Ae_{m-1}\}$ is also a spanning set for imA^2. Thus,

$$RkA^2 = dim(ImA^2) < dim(ImA) = RkA$$

But this is a contradiction to LEMMA7, so no such v exists.

Theorem 1. *Let $\mathbb{D} = (A_1, ..., A_k)$ be a chain complex relating k operators on an n-dimensional vector space V. Suppose that $A_i = \pm A_i^*$ for $0 \leq i \leq k$. Then, we may assume without loss of generality that*

$$U = \cap_{i=0}^{k} kerA_i = \{0\}$$

.

Proof. Suppose there exists $v \in U$ such that $v \neq 0$. Define

$$\begin{aligned}
&\bar{V} = V/U \\
&\pi : V \to \bar{V} \text{ given by } \pi(v) = v + U \text{ for all } v \in V \\
&\pi^* R = \bar{R} \text{ so that } \pi^* R(x, y, z, w) = \bar{R}(x + U, y + U, z + U, w + U) \\
&\bar{A}_i : \bar{V} \to \bar{V} \text{ such that } \pi(A_i v) = \bar{A}_i(v + U) \text{ for all } v \in V
\end{aligned}$$

We need to verify that the above is well-defined and simplifies our chain complex so that $\bar{U} = \bigcap_{i=0}^{k} Ker\bar{A}_i = \{0\}$.

By THEOREM3.01 that for $R_A \neq 0$, $KerR_A = KerA$, and so $U \subset KerR_A$. Let $x_1, x_2 \in V$ such that $x_1 + U = x_2 + U$. Then,

$$\begin{aligned}
&x_1 - x_2 \in U \\
\implies &R_A(x_1 - x_2, y, z, w) = 0 \\
\implies &R_A(x_1, y, z, w) = R_A(x_2, y, z, w) \\
\implies &\bar{R_A}(x_1 + U, y + U, z + U, w + U) = \bar{R_A}(x_2 + U, y + U, z + U, w + U)
\end{aligned}$$

So, $\bar{R}$ is well-defined for R on V. Also, we have

$$\begin{aligned}
&x_1 - x_2 \in U \subset kerA_i \\
\implies &A_i(x_1 - x_2) = 0 \\
\implies &A_i(x_1) = A_i(x_2) \\
\implies &\pi(A_i x_1) = \pi(A_i x_2) \\
\implies &\bar{A}_i(x_1 + U) = \bar{A}_i(x_2 + U)
\end{aligned}$$

for all $0 \leq i \leq k$. So $\bar{A}_i$ is well-defined.

We now need to show that the new operators satisfy the original chain complex. Since the information encoded by a chain complex is a containment of the images of some operators in the kernels of others, it is sufficient to show that $Im\bar{A}_i \subset Ker\bar{A}_j$ for $ImA_i \subset KerA_j$. Fix such i, j and let $v + U \in im\bar{A}_i$ so that $\bar{A}_i(u + U) = v + U$. Then,

$$\begin{aligned}
&v - A_i u \in U \subset KerA_j \\
\implies &A_j(v - A_i u) = 0 \\
\implies &A_j v = A_j A_i u = 0 \text{ since } ImA_i \subset KerA_j \\
\implies &\bar{A}_j(v + U) = \pi(A_j v) = \pi(0) = 0 + U
\end{aligned}$$

Thus, our new operators satisfy the same chain complex on $\bar{V}$ that our original operators satisfied on V. We also have that if $v + U \in \bigcap_{i=0}^{k} Ker\bar{A}_i$, then

$$\begin{aligned} & v + U \in \bigcap_{i=0}^{k} Ker\pi(A_i) \\ \Longrightarrow\ & \pi(A_i v) = 0 + U \text{ for } i \leq k \\ \Longrightarrow\ & A_i v \in U \subset KerA_i \text{ for } i \leq k \\ \Longrightarrow\ & A_i v \in KerA_i \cap ImA_i = \{0\} \text{ for } i \leq k \end{aligned}$$

by LEMMA12. Finally, to guarantee that $Rk\bar{A}_i = RkA_i$, let $v = A_i u$ be nonzero for some $i \leq k$. Since $ImA \cap U \subset ImA \subset KerA = \{0\}$ by LEMMA12, $v \notin U$. Thus, $v + U = \pi(v) = \pi(A_i u) = \bar{A}_i(u + U)$, and so $v + U \in Im(\bar{A}_i)$.

This result is very convenient, because it allows us to assume for every chain complex that the intersection of the kernels of all operators is trivial. In other words, $dimV$ can be assumed to be only large enough to meet our assumptions, and no larger.

Putting restrictions on $dimV$ is generally a tedious process in which one must consider the interrelated kernels of all operators in the complex. However, assuming that the kernels have a trivial intersection gives us some leverage when considering how the kernels interact. The following result is especially nice in cases when $dimV \leq 5$.

Theorem 1. *Let $A_1, ..., A_k$ be operators on a vector space V such that $\{R_{A1}, ..., R_{Ak}\}$ is a properly linearly dependent set. Suppose that $\bigcap_{i=1}^{k} KerR_{Ai} = 0$, and let v be a vector such that v is in the kernels of k-1 of the A_i's. Then, $v = 0$.*

Proof. Since $\{R_{A_1}, ..., R_{A_k}\}$ is a properly linearly dependent set, $R_{A_i} \neq 0 \implies RkA_i \geq 2$ for all i by LEMMA3.01. Thus, $KerR_{A_i} = KerA_i$ and so $\bigcap_{i=1}^{k} KerA_i = \{0\}$. Furthermore, we can choose $\alpha_1, ..., \alpha_k$ all nonzero such that

$$\sum_{i=1}^{k} \alpha_i R_{Ai} = 0$$

Now let v be a vector such that v is in the kernels of k-1 of the A_i's. Fix j such that $v \in KerA_i$ for $i \neq j$. We can now rewrite the linear dependence equation

$$\begin{aligned}
-\alpha_j R_{Aj} &= \sum_{i=1}^{j-1} \alpha_i R_{Ai} + \sum_{i=j+1}^{k} \alpha_i R_{Ai} \\
\implies R_{Aj} &= -\frac{1}{\alpha_j}(\sum_{i=1}^{j-1} \alpha_i R_{Ai} + \sum_{i=j+1}^{k} \alpha_i R_{Ai}) \\
\implies R_{Aj}(v,y,z,w) &= -\frac{1}{\alpha_j}(\sum_{i=1}^{j-1} \alpha_i R_{Ai}(v,y,z,w) + \sum_{i=j+1}^{k} \alpha_i R_{Ai}(v,y,z,w)) \\
&= -\frac{1}{\alpha_j}(\sum_{i=1}^{j-1} 0 + \sum_{i=j+1}^{k} 0) \\
&= 0
\end{aligned}$$

for all $y, z, w \in V$. Thus, $v \in KerR_{Aj} \implies v \in KerA_j$. But by choice of v, we now have $v \in \bigcap_{i=1}^{k} KerR_{Ai} = \{0\}$ by assumption. So $v = 0$.

5 Precomposed Linear Dependence Equations

The process of precomposition by a symmetric or antisymmetric operator is key to our study of chain complexes and the linear dependence of canonical algebraic curvature tensors. The equations that result from precomposition are often simpler at the tensor level, but the operators from which the tensors are built are often much more complicated. However, because we can often reduce our equations to 2 or 3 nontrivial terms, we can begin to apply previous knowledge to derive matrix equations.

The following result justifies our use of precomposition to form new canonical algebraic curvature tensors.

Lemma 1. *Let $A = \alpha A^*$ and $B = \beta B^*$ for some $\alpha, \beta = 1, -1$. Then, we have*

$$(ABA)^* = \beta ABA$$

Proof. By properties of adjoints, we have

$$(ABA)^* = A^* B^* A^* = (\alpha A)(\beta B)(\alpha A) = \alpha^2 \beta ABA = \beta ABA$$

since $\alpha^2 = 1$.

As a corollary to the above, precomposing a canonical algebraic curvature tensor by a symmetric or antisymmetric operator produces another canonical algebraic curvature tensor of the same build. This allows us to apply previous results to tensors that have undergone precomposition.

Theorem 1. *Let A,B be operators on a vector space V such that $A = A^*$, $B = \pm B^*$ and $3 \le RkB \le RkA$. Let $A = [a_{ij}]$ and $B = [b_{ij}]$. Suppose that*

$$\begin{aligned} R_{A^3} &= \pm R_{ABA} \\ R_{B^3} &= \pm R_{BAB} \end{aligned}$$

We choose a basis for V such that A is diagonal. Then, we have that for distinct $i, j \le dimV$,

$$\begin{aligned} a_{ii} &= a_{jj}, \text{ or} \\ b_{ij} &= b_{ji} = 0 \end{aligned}$$

Proof. By COROLLARY11, we have that $B = B^*$, $R_{A^3} = R_{ABA}$ and $R_{B^3} = R_{BAB}$. Also,

$$\begin{aligned} &\beta A^3 = ABA \text{ for some } \beta = 1, -1 \\ &\eta B^3 = BAB \text{ for some } \eta = 1, -1 \\ \implies &\beta A^3 B = ABAB = \eta AB^3 \\ \implies &A^3 B = \gamma AB^3 \text{ for } \gamma = \beta\eta \end{aligned}$$

Thus with diagonalized A, we have

$$A = \begin{bmatrix} \lambda_1 & 0 & \dots & \dots & & 0 \\ 0 & \ddots & & & 0 & \\ & & \lambda_h & & & \\ \vdots & & & 0 & & \vdots \\ & 0 & & & \ddots & \\ 0 & & \dots & \dots & & 0 \end{bmatrix}$$

$$\implies A^3 = \begin{bmatrix} \lambda_1^3 & 0 & \dots & \dots & & 0 \\ 0 & \ddots & & & 0 & \\ & & \lambda_h^3 & & & \\ \vdots & & & 0 & & \vdots \\ & 0 & & & \ddots & \\ 0 & & \dots & \dots & & 0 \end{bmatrix}$$

where $h = RkA$. Let $B^3 = [\tilde{b_{ij}}]$. Thus, from the above relation, we have

$$A^3B = \begin{bmatrix} \lambda_1^3 b_{11} & \lambda_1^3 b_{12} & \dots & \lambda_1^3 b_{1h} & 0 & \dots & 0 \\ \lambda_2^3 b_{21} & \lambda_2^3 b_{22} & & \vdots & & & \\ \vdots & & \ddots & & & & \\ \lambda_h^3 b_{h1} & \dots & & \lambda_h^3 b_{hh} & & & \vdots \\ 0 & & & & 0 & & \\ \vdots & & & & & \ddots & \\ 0 & & & \dots & & & 0 \end{bmatrix} = [\lambda_i^3 b_{ij}]$$

$$= \gamma AB^3 = \gamma \begin{bmatrix} \lambda_1 \tilde{b_{11}} & \lambda_1 \tilde{b_{12}} & \dots & \lambda_1 \tilde{b_{1h}} & 0 & \dots & 0 \\ \lambda_2 \tilde{b_{21}} & \lambda_2 \tilde{b_{22}} & & \vdots & & & \\ \vdots & & \ddots & & & & \\ \lambda_h \tilde{b_{h1}} & \dots & & \lambda_h \tilde{b_{hh}} & & & \vdots \\ 0 & & & & 0 & & \\ \vdots & & & & & \ddots & \\ 0 & & & \dots & & & 0 \end{bmatrix} = \gamma[\lambda_i \tilde{b_{ij}}]$$

Since B, B^3 are symmetric matrices, we have that $b_{ij} = b_{ji}$ and $\tilde{b_{ij}} = \tilde{b_{ji}}$ for all i, j. Thus, for distinct i, j we have

$$\begin{aligned} &\lambda_i^3 b_{ij} = \gamma \lambda_i \tilde{b_{ij}} \text{ and } \lambda_j^3 b_{ji} = \gamma \lambda_j \tilde{b_{ji}} \\ \Longrightarrow\ &\lambda_i^2 b_{ij} = \gamma \tilde{b_{ij}} = \gamma \tilde{b_{ji}} = \lambda_j^2 b_{ji} \text{ since } \lambda_k \neq 0 \text{ for all } k \leq m \\ \Longrightarrow\ &(\lambda_i^2 - \lambda_j^2) b_{ij} = 0 \\ \Longrightarrow\ &\lambda_i = \pm \lambda_j \text{ or } b_{ij} = b_{ji} = 0 \text{ for distinct } i, j \end{aligned}$$

This result is very nearly sufficient to show commutativity for symmetric A, B given our assumptions. For diagonalized A, we have $AB = [a_{ii}bij]$ and $BA = [bija_{jj}]$. Thus, in order for A, B to commute, we must have

$$a_{ii}bij = a_{jj}bij$$

for all i, j. This is true for our matrices except in the case where $a_{ii} = -a_{jj}$. Thus, we can force commutativity if we assume that A has only positive (or only negative) eigenvalues.

We wish to extend this result to antisymmetric A, B. This is nontrivial, since antisymmetric matrices cannot be diagonalized. They can, however, be block-diagonalized with 2-by-2 blocks down the diagonal with zeros elsewhere. Each of these blocks must be a scalar multiple of

$$M = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

We'll now formalize a process by which we use M to "diagonalize" an antisymmetric operator. First, we must show that a block partitioning on an antisymmetric matrix preserves the antisymmetric properties of the matrix.

Lemma 1. *Suppose B is a $2n$-by-$2n$ antisymmetric matrix over V. Suppose we express $B = [b_{ij}]$ as a block matrix, so that*

$$B = \begin{bmatrix} B_{11} & B_{12} & \dots & B_{1,n} \\ B_{21} & B_{22} & & B_{2,n} \\ \vdots & & \ddots & \vdots \\ B_{n,1} & B_{n,2} & \dots & B_{n,n} \end{bmatrix}$$

where

$$B_{ij} = \begin{bmatrix} b_{2i-1,2j-1} & b_{2i-1,2j} \\ b_{2i,2j-1} & b_{2i,2j} \end{bmatrix}$$

Then, for all i, j, we have

$$B_{ij} = -(B_{ji})^T$$

Proof. Let $1 \leq i, j \leq n$. Then,

$$\begin{aligned} b_{2i-1,2j-1} &= -b_{2j-1,2i-1} \\ b_{2i,2j-1} &= -b_{2j-1,2i} \\ b_{2i-1,2j} &= -b_{2j,2i-1} \\ b_{2i,2j} &= -b_{2j,2i} \end{aligned}$$

Thus, we have

$$\begin{aligned} B_{ij} &= \begin{bmatrix} b_{2i-1,2-1} & b_{2i,2j-1} \\ b_{2i-1,2j} & b_{2i,2j} \end{bmatrix} \\ &= \begin{bmatrix} b_{2i-1,2j-1} & b_{2i-1,2j} \\ b_{2i,2j-1} & b_{2i,2j} \end{bmatrix}^T \\ &= -\begin{bmatrix} b_{2j-1,2i-1} & b_{2j-1,2i} \\ b_{2j,2i-1} & b_{2j,2i} \end{bmatrix}^T = -B_{ji}^T \end{aligned}$$

This directly shows that an antisymmetric operator on an even-dimensional V retains antisymmetry under block-partitioning. If $n = dimV$ is odd instead, then we can augment the matrix representation of B by one row and one column, and fill all the new entries with zeroes. This new (n+1)-by-(n+1)matrix is antisymmetric, so we can apply LEMMA19 to it.

We now extend the result from THEOREM18 to antisymmetric operators.

Theorem 1. *Let $A = [a_{ij}], B = [b_{ij}]$ be n x n antisymmetric matrices on a real, finite-dimensional vector space V. Let $RkA \geq RkB \geq 4$. Suppose A, B satisfy the following relationship:*

$$A^3 B = \alpha A B^3$$

for some $\alpha \in \{1, -1\}$. Choose a basis for V so that A is block-diagonalized:

$$A = \begin{bmatrix} A_{11} & 0 & \dots & 0 \\ 0 & \ddots & 0 & \vdots \\ \vdots & 0 & A_{h,h} & 0 \\ 0 & \dots & 0 & 0 \end{bmatrix}$$

where $2h = RkA$ and

$$A_{ii} = \begin{bmatrix} 0 & \lambda_i \\ -\lambda_i & 0 \end{bmatrix} = \lambda_i M$$

where

$$M = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Then, for distinct indices i, j, we have

$$\lambda_i = \pm\lambda_j, \text{ or}$$
$$b_{ij} = b_{ji} = 0$$

Proof. We partition B as a block matrix, so that $B = [B_{ij}]$, where

$$B_{ij} = \begin{bmatrix} b_{2i-1,2j-1} & b_{2i-1,2j} \\ b_{2i,2j-1} & b_{2i,2j} \end{bmatrix}$$

and similarly $B^3 = [\tilde{B_{ij}}]$. Note that since $RkB \leq RkA$, $b_{ij} = 0$ for $i \geq 2h$ or $j \geq 2h$. Thus, $B_{ij} = [0]$ for $i \geq h$ or $j \geq h$.

By hypothesis, we have

$$A^3 B = \alpha A B^3$$

for some $\alpha \in \{1, -1\}$. So,

$$A^3 B = \begin{bmatrix} A_{11}^3 B_{11} & \dots & A_{11}^3 B_{1,h} & 0 & \dots \\ \vdots & \ddots & & & \\ A_{h,h}^3 B_{h,1} & \dots & A_{h,h}^3 B_{h,h} & 0 & \dots \\ 0 & & & 0 & \\ \vdots & & & & \ddots \end{bmatrix} = [A_{ii}^3 B_{ij}]$$

$$= \alpha AB^3 = \alpha \begin{bmatrix} A_{11}\tilde{B_{11}} & \dots & A_{11}\tilde{B_{1,h}} & 0 & \dots \\ \vdots & \ddots & & & \\ A_{h,h}\tilde{B_{h,1}} & \dots & A_{h,h}\tilde{B_{h,h}} & 0 & \dots \\ 0 & & & 0 & \\ \vdots & & & & \ddots \end{bmatrix} = \alpha[A_{ii}\tilde{B_{ij}}]$$

So, for distinct indices i, j we have

$$\begin{aligned} & A_{ii}^3 B_{ij} = \alpha A_{ii}\tilde{B_{ij}} \text{ and } A_{jj}^3 B_{ji} = \alpha A_{jj}\tilde{B_{ji}} \\ \implies & \lambda_i^3 M^3 B_{ij} = \alpha\lambda_i M\tilde{B_{ij}} \text{ and } \lambda_j^3 M^3 B_{ji} = \alpha\lambda_j M\tilde{B_{ji}} \\ \implies & \lambda_i^2 M^2 B_{ij} = \alpha\tilde{B_{ij}} \text{ and } \lambda_j^2 M^2 B_{ji} = \alpha\tilde{B_{ji}} \end{aligned}$$

since $M^4 = I$ and $\lambda_k \neq 0$ for $k \leq h$. But by LEMM19, we have

$$\tilde{B_{ij}} = -\tilde{B_{ji}}^T$$

and so

$$\begin{aligned} & \lambda_i^2 M^2 B_{ij} = \alpha\tilde{B_{ij}} = -\alpha\tilde{B_{ji}}^T = -\lambda_j^2 (M^2 B_{ji})^T \\ \implies & -\lambda_i^2 B_{ij} = (-\lambda_j^2)(-B_{ji})^T && \text{since } M^2 = -I \\ \implies & -\lambda_i^2 B_{ij} = -\lambda_j^2 B_{ij} \end{aligned}$$

by LEMMA19. So, we have

$$B_{ij} = 0 \text{ or } \lambda_i = \pm\lambda_j$$

6 Questions

7 Projects for Further Study

1. For $n \geq 5$, classify directed graphs of n edges/operators that can be associated with some chain complex. Impose a linear dependence on the operators in the chain complex, and don't consider graphs which force any operator to be 0 (these are not useful for our purposes). For valid graphs on $n \geq 4$ edges/operators, form the hierarchy of graphs from least restrictive to most restrictive. For example, on 4 edges/operators, there are 14 valid graphs, some of which are stricter than others. Which ones are the least strict? Which are the most strict? If we start with a graph which is not very strict and impose more containment relationships on the images and kernels of the operators, what other graphs can we derive? What are the possible *restriction paths* we could take from a least-strict graph to a most-strict graph? Answer these questions for $4 \leq n \leq 10$ edges/operators.

2. For a set of CACT's which is known to be linearly dependent, consider what other conditions are *necessary* for us to conclude that a chain complex structure must exist on the underlying operators. This is intended to fit Elise's and my work with chain complexes into the greater discussion about linearly dependent sets of ACT's, since satisfying a chain complex structure seems to be a strong condition. Other problems along this vein are: For an ACT R, how does chain complex analysis interact with our knowledge about $\nu(R), \eta(R)$, and $\mu(R)$? For a given vector space V, can we use our knowledge about $A(V)$ to restrict the kinds of possible chain complexes on V? Some of my work is very similar to the work that other REU students have done, and I think there are ways to combine my methods and theirs to acquire a fuller *toolbox* for analyzing ACT's.

3. Find classes of solutions for common relationships found in chain complexes, such as
$$R_{A^3} + \alpha R_{ABA} + \delta R_{ACA} = 0$$
for A, B, C symmetric or anti-symmetric and for $\alpha, \delta \in \{1, -1\}$. Examine these equations as matrix polynomials and also as systems of equations, probably with very many unknowns. This seems like a very tedious problem that may involve heavy use of CAS, but getting information about the the properties or entries of these matrices could help us characterize the solutions to a given chain complex. Partitioning this problem into cases based on rank assumptions seems to be the most logical way to progress.

4. Reexamine the work Elise and I have done with the new assumption that all sequences are exact. This has the effect of making every containment assumption an equality assumption instead. How does this change the number of valid graphs on 4 or more operators/edges? Are the solutions to matrix equations more readily derived? Are the linear dependence equations easier to work with? Many cases should become trivial, and nontrivial examples seem like they should simplify immensely. With exact sequences, is it possible to make concrete statements about sets of higher numbers of tensors?

5. Use homology theory to dissect the kernels of operators in chain complexes, and determine if there is a clear connection between homologies and operators. For sequences that are necessarily not exact, there is potential variance in the size of the operator's image. Just as I was able to draw many conclusions by examining the kernels of the operators, it might be possible to derive new conclusions by studying the homologies of the operators. Additionally, it should be noted that my assumption that the intersection of the kernels of the operators is trivial is equivalent to assuming that the intersection of the homologies of the operators is trivial in the chain complex.

6. The convenient (and restrictive) property of a chain complex is that if A immediately precedes B in complex, BA = 0. This is certainly sufficient to show that

$R_{ABA} = 0 = R_{BAB}$, but it is not *necessary.* A less restrictive (and so more plausible) condition is to require that $RkABA \leq 1 \leq RkBAB$, which *is* necessary to show $R_{ABA} = 0 = R_{BAB}$. For a given A, define the **flattening** of A, denoted F(A), to be the set of matrices such that $RkABA \leq 1 \geq RkBAB$. For a given A, what kind of set is F(A)? Is it ever a group? An abelian group? Which elements $B \in F(A)$ satisfy $ImA \subset KerB$? Furthermore, when we use this new condition to form *pseudo-chain-complexes*, how does this change the properties of the operators? Are the implications similar or different from the results on normal chain complexes? How does this new assumption affect the matrix equations and graphcial classifications?

7. Consider more abstract graph-theoretic properties of the directed graphs that are derived from chain complexes? If the graphs of two chain complexes are duals of each other, is there any connection to the complexes themselves? Is there a canonical/logical flow on the edges of a directed graph which corresponds somehow to the operators on the underlying chain complex? Also, it seems that the number of valid graphs which contain no directed cycles is fairly low for every number of operators. Why is this? Is there any way to "replace" a directed cycle in a graph with a non-cycle construct that maintains the linear dependence of the operators? Consider the polarization formula and matrix splitting, as well as Elise's identity and other identities for tensors and matrices.

8. For symmetric matrix A and anti-symmetric matrices C, D find general forms or classifications for the products C^3D, CD^3, C^3A, CA^3, A^3C, AC^3, $ABAB$, $ACAC$, $CACA$, $CDCD$. These operators appear frequently in working with precomposed ACT's, but their properties are tedious to determine in general.

9. Reexamine my work by restating my results in terms of the images and ranks of operators rather than kernels and nullities. How does this change the underlying assumptions about our operators? Does this make any of my results more intuitive? Less intuitive?

10. Given operators A, B on V, find relations for $ImAB, RkAB, KerAB, NlAB$ in terms of $ImA, RkA, KerA, NlA, ImB, RkB, KerB,$ and NlB.

11. Determine whether the work Elise and I have done holds for cases in which some the ranks of the operators are exactly 2.I often avoided studying cases in which the ranks of the operators were exactly 2, as there were few previous results that applied to them. Do my results still hold? Do Elise's?

12. Let M be defined as follows.
$$M = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$
This matrix is notable because it allows us to express an antisymmetric block-diagonal matrix as a diagonal block matrix. Additionally, M has several properties

that are very similar to the imaginary number i: $M^2 = -I$ for example, and $M = -M^3$. I have a hunch that by studying M alongside antisymmetric matrices, there may be a way to express antisymmetric matrices in a more convenient way. By my proof of RESULTHERE, we know that there exists a function ϕ which associates a block-diagonal antisymmetric matrix with a unique diagonal matrix. Ideally, we would be able to expand $Dom(\phi)$ to all antisymmetric matrices and $Im(\phi)$ to all symmetric matrices so that ϕ is a bijection. If such a ϕ is found, many previous results (including mine) could be simplified to just the symmetric case, and we would have a new tool with which to study canoncial algebraic curvature tensors.

8 Acknowledgments

I would like to thank the National Science Foundation and California State University, San Bernardino for funding my work. Also a big thank-you to Drs. Corey Dunn and Rollie Trapp for their insight, guidance, and dedication to their students.