Curvature Homogeneous Types in Generalized Dunn Manifolds

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Abstract

Throughout the study of curvature homogeneous Riemannian and pseudo-Riemannian manifolds, many homogeneity conditions have been introduced imposing different restrictions on the model space of a manifold. In this paper we investigate the curvature characteristics of generalized Dunn manifolds under a variety of these homogeneous model spaces including the weak, homothety, variable, and the "regular" curvature homogeneity conditions. The Dunn manifolds have already been shown to provide a rich family of curvature homogeneous examples in the higher-signature, pseudo-Riemannian setting and by creating the generalizing form we can further explore other homoegeneity types.

1 Introduction

Let (M, g) denote a pseudo-Riemannian manifold of signature (p, q) equipped with a Levi-Civita connection, ∇, and let TM and T^*M be the tangent bundle on M and its dual. Similarly, let g_p, T_pM and T_p^*M denote the metric, tangent bundle, and dual evaluated at a point $p \in M$.

Definition 1.1. If $X, Y, Z, W \in T_pM$ are vector fields, then we can define the Riemannian curvature operator, $\mathcal{R} \in TM \otimes (T^*M)^3$, as

$$\mathcal{R}(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Additionally, we can define the Riemannian curvature tensor, $R \in \otimes^4(T^*M)$ using the metric, g, as

$$R(X,Y,Z,W) = g(\mathcal{R}(X,Y)Z, W).$$

We also define $\nabla^i R \in \otimes^{4+i} T^*M$ for $i \in \mathbb{N}$ as the ith covariant derivative of R and following previous notation, R_p denote the curvature tensor evaluated at a point $p \in M$.

The curvature operator and tensor wholly determine the curvature of a manifold and thus is the primary objects of study when looking at curvature homogeneity.

When Singer pioneered the study of curvature homogeneity in 1960 [8], he introduced the notion of a curvature homogeneous manifold in terms of the Riemannian curvature tensor:

Definition 1.2. A manifold, (M, g), is *k-curvature homogeneous* if for any points, $p, q \in M$ and $i = 0, 1, ..., k$ there exists a linear isomorphism $\Phi_{pq} : T_pM \to T_qM$ such that:

$$\Phi_{pq}^* g_q = g_p \text{ and } \Phi_{pq}^* \nabla^i R_q = \nabla^i R_p$$

If M is k-curvature homogeneous, we say that M is CH_k, and if it is also the case that M is CH_k but not CH_{k+1}, then M is *properly* CH_k.

The manifolds central to this paper, the Dunn family of manifolds [5] , $\mathcal{M}_f$, were created to allow greater flexibility in signature, primarily motivated by the rigid signature restrictions of previously existing example families in the higher signature setting [1].

Definition 1.3 ($\mathcal{M}_f$ **Manifolds).** Let $M := \mathbb{R}^{3k+2}$ be a Euclidean space wtih a coordinate basis $\{u_0, ..., u_k, v_0, ..., v_k, s_1, ..., s_k\}$ and $F := \{f_1(u_1), ..., f_k(u_k)\}$ be a collection of smooth functions such that $f_i(u_i) + 1 \neq 0$ for all u_i. Then we can define the metric, g_F, on the coordinate frame as:

$$g_F(\partial_{u_0}, \partial_{u_i}) = 2f_i(u_i)s_i, \qquad g_F(\partial_{u_i}, \partial_{u_i}) = -2u_0 s_i,$$
$$g_F(\partial_{u_i}, \partial_{v_j}) = \delta_{ij}, \qquad g_F(\partial_{s_i}, \partial_{s_i}) = \epsilon_i \ .$$

If $M_f := (\mathbb{R}^{3k+2}, g_F)$, then M_f is a pseudo-Riemannian manifold with signature $(k + 1 + a, k + 1 + b)$; where $a + b = k$ is a choice of signs for ϵ_i.

The initial examination of these manifolds was primarily concerned with curvature homogeneity in the "regular" sense, introduced by Singer, showing that:

Theorem 1.4. Let $(M, g) = \mathcal{M}_f$ for a fixed family of functions, F, such that $f_i'(u_i) + 1 \neq 0$ for all i, then:

- **(a)** $\mathcal{M}_f$ is CH_0
- **(b)** $\mathcal{M}_f$ is locally indecomposable at every point.
- **(c)** If $f_i''(u_i) \neq 0$ for any $i = 0, ..., k$ then $\mathcal{M}_f$ is not CH_2
- **(d)** $\mathcal{M}_f$ is generalized plane wave and thus Ricci Flat and VSI

We will build off these manifolds, generalizing them by introducing a second collection of functions, giving more flexibility in choosing the defining functions.

Definition 1.5 ($\mathcal{M}_\delta$ **Manifolds).** Define M as in Definition 1.3 and let

$$\left.\begin{array}{l} H := \{h_i(u_i) \\ F := \{f_i(u_i) \end{array}\right| \quad i = 1, ..., k \left.\vphantom{\begin{array}{l} H \\ F \end{array}}\right\}$$

be two independent collections of smooth functions. Then define the metric, g_δ, as having the following non-zero entries:

$$\begin{aligned} g_\delta(\partial_{u_0}, \partial_{u_i}) &= 2f_i(u_i)s_i, & g_\delta(\partial_{u_i}, \partial_{u_i}) &= -2u_0 s_i, \\ g_\delta(\partial_{u_0}, \partial_{s_i}) &= h_i(u_i), & g_\delta(\partial_{s_i}, \partial_{s_i}) &= \epsilon_i, \\ g_\delta(\partial_{u_0}, \partial_{v_0}) &= 1, & g_\delta(\partial_{u_i}, \partial_{v_i}) &= 1. \end{aligned}$$

If $M_\delta := (M, g_\delta)$, then M_δ is pseudo- Riemannian with the same choice of signature as M_f in Definition 1.3.

Remark. This "generalized" family is so called because it allows much more flexibility in choosing the defining sets of functions F and H. Instead of imposing the restrictions for curvature homogeneity directly on the $f_i(u_i)$ for each i, the restriction is instead imposed on a new function

$$\delta_i = \frac{\epsilon_i}{2}[h_i'(u_i) - 2f_i(u_i)].$$

This means that for any chosen collection, F, we have another potentially infinite number of choices for H that still exhibit the same curvature characteristics.

It is obvious that the original Dunn manifolds described in Definition 1.3 are a subset of $\mathcal{M}_\delta$, but it is still unclear if the two families are distinctly different as they might still be isomorphic.

2 Preliminiaries

Since Springer's introduction of curvature homogeneity, many authors have conducted work on classifying curvature homogeneous manifolds but this also prompted the introduction of new homogeneity conditions.

The work of Kowalski and Vanžurová [6, 7] proposed a generalization of curvature homogeneity which preserves the curvature operator instead of the curvature tensor, called curvature homogeneity of type (1,3). This property was later renamed in [4], whose notation we will use.

Definition 2.1. A manifold, (M, g) is *variable homothety k-curvature homogeneous*, or VCH$_k$, if for every $i = 0, ..., k$ there exists a homothety (an isometry followed by a dilation), $h_i : T_pM \to T_qM$ for every $p, q \in M$ such that $h_i^* \nabla^i \mathcal{R}_q = \nabla^i \mathcal{R}_p$.

This prompted the study of a special case of this property, by García-Río et. al. [3] where the curvature operator is preserved at all levels by a single homothety.

Definition 2.2. A manifold, (M, g) is *homothety k-curvature homogeneous*, or HCH$_k$, if there exists a homothety, $h : T_pM \to T_qM$ for every $p, q \in M$ such that $h^* \nabla^i \mathcal{R}_q = \nabla^i \mathcal{R}_p$ for all $0 \leq i \leq k$.

Central to the study of curvature homogeneous manifolds are model spaces which provide the primary tool with which to differentiate sub-classes in a family of manifolds. We will be adopting the notation used in [2].

Definition 2.3. Let V and V^* be a finite-dimensional real vector space and its dual and $\langle\cdot,\cdot\rangle$ be a symmetric, nondegenerate, bilinear form.
An element $A^0 = A_{abcd} \in \otimes^4 V^*$ is said to be an *algebraic curvature tensor* if for all $a, b, c, d, e \in V$ it satisfies:

$$\begin{gathered} A_{abcd} = A_{cdab} \\ A_{abcd} = -A_{bacd} = -A_{abdc} \\ A_{abcd} + A_{adbc} + A_{acdb} = 0 \end{gathered}$$

Similarly, $A^1 = A_{abcd;e} \in \otimes^5 V^*$ is an *algebraic covariant derivative curvature tensor* if it satisfies:

$$\begin{gathered} A_{abcd;e} = A_{cdab;e} \\ A_{abcd;e} = -A_{bacd;e} = -A_{abdc;e} \\ A_{abcd;e} + A_{adbc;e} + A_{acbd;e} = 0 \\ A_{abcd;e} + A_{abec;d} + A_{abde;c} = 0 \end{gathered}$$

These structures are clearly analoguous to R and ∇R and thus we can extend them naturally to define $A^i \in \otimes^{4+i} V^*$ as the *ith algebraic covariant derivative curvature tensor* following the symmetries of $\nabla^i R$.

The tuple, $(V, \langle\cdot,\cdot\rangle, A^0, A^1, ..., A^k)$ is a *curvature k-model* and we say a manifold (M, g) can be *k-modeled* by a given model space if there exist isomorphisms such that at every point, $p \in M$,

$$(T_pM, g_p, R_p, ..., \nabla^k R_p) \cong (V, \langle\cdot,\cdot\rangle, A^0, ..., A^k).$$

In other words, there exists an isometry $\Gamma : T_pM \to V$ such that for $x, y \in T_pM$:

$$\langle \Gamma^* x, \Gamma^* y\rangle = g_p(x, y) \qquad \text{and} \qquad \Gamma^* A^i = \nabla^i R \qquad \text{for} \qquad 0 \le i \le k.$$

In [2], Dunn and McDonald reformulated and summarized the homogeneity conditions in terms of the model spaces. Here we will also introduce the concept of weak curvature homogeneity:

Theorem 2.4. Let $\mathcal{M} = (M, g)$ be a smooth pseudo-Riemannian manifold, then

(a) $\mathcal{M}$ is weak k-curvature homogeneous, or WCH_k, if and only if it can be k-modeled by
$$\mathfrak{W}_k = (V, A^0, ..., A^k)$$

(b) $\mathcal{M}$ is CH_k if and only if can be k-modeled by
$$\mathfrak{M}_k = (V, \langle\cdot,\cdot\rangle, A^0, ..., A^k)$$

(c) $\mathcal{M}$ is VCH_k if and only if there exist smooth, positive functions, φ_i, for every $0 \le i \le k$ such that $\mathcal{M}$ can be k-modeled by
$$\mathfrak{V}_k = (V, \langle\cdot,\cdot\rangle, \varphi_0 A^0, ..., \varphi_k A^k)$$

(d) $\mathcal{M}$ is HCH_k if and only if there exists a smooth, positive function, λ, such that $\mathcal{M}$ can be k-modeled by
$$\mathfrak{H}_k = (V, \langle\cdot,\cdot\rangle, \lambda A^0, \lambda^{\frac{3}{2}} A^1, ..., \lambda^{\frac{1}{2}(k+2)} A^k)$$
.

In the rest of this paper, we will establish the following curvature properties of the generalized Dunn manifolds, $\mathcal{M}_\delta$:

Theorem 2.5. Let $(M,g) = M_\delta \in \mathcal{M}_\delta$ as in Definition 1.5 for fixed collections of smooth functions, F and H, and $\delta_i = \frac{\epsilon_i}{2}[h_i'(u_i) - 2f_i(u_i)]$ then:

- **(a)** $\mathcal{M}_\delta$ is never flat for any collections F and H.
- **(b)** M_δ is CH_0.
- **(c)** CH_0 and WCH_1 imply CH_1.
- **(d)** M_δ is locally symmetric if $\delta_i = 0$ or $\delta_i' = \frac{1}{2}$
- **(e)** If δ_i is constant, then M_δ is CH_1.

Considering variable curvature homogeneity we find:

Theorem 2.6. Consider M_δ as in Theorem 2.5, then M_δ is VCH_1 if it is CH_0, not locally symmetric, and one of the following is true:

- **(1)** $\delta_i' = 1$ and $(\epsilon_i\delta_i) > 0$.
- **(2)** $\delta_i' \neq 1$, $\delta_i'' = 0$, and $(\epsilon_i\delta_i) > 0$.
- **(3)** δ_i is a non-linear solution to the differential equation $\delta_i\delta_1'' = 2\epsilon_i(\delta_i' - 1)(2\delta_i' - 1)$ and $(\delta_i''\frac{\delta_i'-1}{|\delta_i'-1|}) > 0$.
- **(4)** For some positive function, φ_1, δ_i is a non-linear solution to the differential equation $2\delta_i\epsilon_i(\delta_i' - 1)(2\delta_i' - 1) - \delta_i^2\delta_i''\epsilon_i = \pm\varphi_1\delta_i''(\delta_i' - 1)$ with $\delta_i'' > 0$.

Finally, we find the following results when examining the homothety curvature homogeneous examples in the manifold family:

Theorem 2.7. Consider a manifold M_δ as in Theorem 2.5, then M_δ is HCH_1 if one of the following is true:

- **(1)** $\delta_i' = 1$.
- **(2)** δ_i is linear but $\delta_i' \neq 1$.
- **(3)** δ_i is a non-linear solution to the differential equation:

$$\delta_i\delta_1'' = 2\epsilon_i(\delta_i' - 1)(2\delta_i' - 1)$$

- **(4)** For some fixed, positive function φ_1, δ_i is a non-linear solution and to the differential equation:

$$2\delta_i\epsilon_i(\delta_i' - 1)(2\delta_i' - 1) - \delta_i^2\delta_i''\epsilon_i = \pm\varphi_1\delta_i''(\delta_i' - 1)$$

3 Geometry of $\mathcal{M}_\delta$

We begin our discussion of curvature by computing the Riemannian curvature tensor and its first and second covariant derivatives.

Lemma 3.1. Let $\{\partial_{u_0}, ..., \partial_{u_k}, \partial_{v_0}, ..., \partial_{v_k}, \partial_{s_1}, ..., \partial_{s_k}\}$ be the coordinate basis of TM and for convenience, adopt the notation $h_i = h_i(u_i)$ and $f_i = f_i(u_i)$ and define $\delta_i := \frac{\epsilon_i}{2}[h_i' - 2f_i]$.

(a) The nonzero coordinate covariant derivatives are

$$\begin{aligned}
&\nabla_{\partial_{u_0}}\partial_{u_i} = \nabla_{\partial_{u_i}}\partial_{u_0} = \delta_i\partial_{s_i} - h_i\delta_i\epsilon_i\partial_{v_0} - s_i\partial_{v_i},\\
&\nabla_{\partial_{u_0}}\partial_{s_i} = \nabla_{\partial_{s_i}}\partial_{u_0} = -\delta_i\epsilon_i\partial_{v_i},\\
&\nabla_{\partial_{u_i}}\partial_{s_i} = \nabla_{\partial_{s_i}}\partial_{u_i} = \delta_i\epsilon_i\partial_{v_0} - u_0\partial_{v_i},\\
&\nabla_{\partial_{u_i}}\partial_{u_i} = \epsilon_i u_0\partial_{s_i} + (2f_i's_i - h_i\epsilon_i u_i - s_i)\partial_{v_0}.
\end{aligned}$$

(b) Up to symmetry, the non-zero curvature tensor entries are

$$\begin{aligned}
&R_0(i) = R(\partial_{u_0}, \partial_{u_i}, \partial_{u_0}, \partial_{u_i}) = \delta_i^2\epsilon_i,\\
&R_s(i) = R(\partial_{u_0}, \partial_{u_i}, \partial_{u_i}.\partial_{s_i}) = \delta_i' - 1.
\end{aligned}$$

(c) Up to symmetry, the non-zero first covariant derivative curvature tensor entries are

$$\begin{aligned}
&\nabla R_0(i) = \nabla R(\partial_{u_0}, \partial_{u_i}, \partial_{u_0}, \partial_{u_i}; \partial_{u_i}) = 2\delta_i\epsilon_i(2\delta_i' - 1),\\
&\nabla R_s(i) = \nabla R(\partial_{u_0}, \partial_{u_i}, \partial_{u_i}, \partial_{s_i}; \partial_{u_i}) = \delta_i''.
\end{aligned}$$

Proof. Lemma 3.1 follows from direct calculation of the Christoffel symbols, curvature tensor, and curvature operators as defined in Section 1, and will be omitted for brevity. □

For our discussion of curvature homogeneity, it will be necessary to find a normalizing basis for our manifolds, $\mathcal{M}_\delta$.

Definition 3.2. Define a change of basis such that for each $1 \le i \le k$,

$$u_0 := b_{00}\partial_{u_0} + b_{0v_0}\partial_{v_0} + \textstyle\sum_{j=1}^k b_{0s_j}\partial_{s_j}, \qquad v_0 := b_{v_{00}}\partial_{v_0},$$

$$u_i := b_{ii}\partial_{u_i} + b_{is_i}\partial_{s_i} + b_{iv_i}\partial_{v_i}, \qquad v_i := b_{v_{0i}}\partial_{v_i},$$

$$s_i := \epsilon_i b_{s_{ii}}\partial_{s_i} + b_{sv_0}\partial_{v_0} + \textstyle\sum_{j=1}^k \partial_{v_j}$$

Lemma 3.3. For convienience of notation, assume all formualtions are given for a fixed $1 \le i \le k$ such that $R_0 = R_0(i)$ and $R_s = R_s(i)$. Then under the change of basis described in Definition 3.2,

(a) the non-zero curvature tensor entries are:

$$\begin{aligned}
&\bar{R}_0 = R(u_0, u_i, u_0, u_i) = b_{ii}^2 b_{00}[b_{00}R_0 - 2b_{0s_i}R_s],\\
&\bar{R}_s = R(u_0, u_i, u_i, s_i) = b_{ii}^2 b_{00} b_{s_{ii}}\epsilon_i R_s.
\end{aligned}$$

(b) Similarly, the non-zero covariant derivative curvature tensor entries are:

$$\nabla\bar{R}_0 = \nabla R(u_0, u_i, u_0, u_i; u_i) = b_{ii}^3 b_{00}[b_{00}\nabla R_0 - 2b_{0s_i}\nabla R_s],$$
$$\nabla\bar{R}_s = \nabla R(u_0, u_i, u_i, s_i; u_i) = b_{ii}^3 b_{00} b_{s_{ii}}\epsilon_i \nabla R_s.$$

4 Curvature 0-Models

This section is dedicated to examining the curvature characteristics of $\mathcal{M}_\delta$ at the 0-model level where we will establish assertions (a) and (b) of Theorem 2.5.

Lemma 4.1. $\mathcal{M}_\delta = (\mathbb{R}^{3k+2}, g_\delta)$ cannot be zero-modeled by $(V, \langle\cdot,\cdot\rangle, 0)$.

Proof. Even under a change of basis, $\bar{R}$ is identically zero only if both R_0 and $R_s = 0$. Suppose $R_0 = 0$, then $\delta_i = 0$ for all i. If $R_s = 0$, then $\delta_i' = 1$ which would imply that $\delta_i = u_i + c$ for some constant, c. This of course is a contradiction, so R_0 and R_s cannot be simultaneously zero. $\square$

Remark. Lemma 4.1 is equivalent to showing that no members of $\mathcal{M}_\delta$ are isomorphic to flat, Euclidean space. This also proves assertion (a) of Theorem 2.5.

To establish the CH_0 condition, we can use assertion (a) of Theorem 2.4 to show that through a change of basis $\mathcal{M}_\delta$ can be made to have constant curvature tensor entries.

Definition 4.2. Let $\mathfrak{M}_{0_n}$ be the family of constant curvature 0-model spaces, such that $\mathfrak{M}_{0_n} = (\mathbb{R}^{3k+2}, \langle\cdot,\cdot\rangle, R_n)$ and the inner product and curvature tensors have only the following non-zero entries:

$$\langle u_0, v_0\rangle = 1, \qquad R_1 := \{\bar{R}_0 = 0, \bar{R}_s = 1\}$$
$$\langle u_i, v_i\rangle = 1 \qquad R_2 := \{\bar{R}_0 = 1, \bar{R}_s = 0\}$$
$$\langle s_i, s_i\rangle = 1 \qquad R_3 := \{\bar{R}_0 = 1, \bar{R}_s = 1\}$$

Notice that $\mathfrak{M}_{0_n} \subset \mathfrak{M}_0$, thus if $\mathcal{M}_\delta \cong \mathfrak{M}_{0_n}$ then it is CH_0.

Lemma 4.3. If $M_\delta \in \mathcal{M}_\delta$, then M_δ can be zero-modeled by $\mathfrak{M}_{0_n}$ for some n.

Proof. Since we want a normalizing basis such that we make $g_\delta \cong \langle\cdot,\cdot\rangle$, we establish the following relations from Definition 3.2:

$$b_{sv_0} = \frac{-b_{s_{ii}}(b_{00}h_i + b_{0s_i})}{b_{00}} \qquad b_{iv_i} = 2u_0 s_i - \frac{b_i s^2}{2b_{ii}}$$
$$b_{0v_0} = \frac{1}{2b_{00}}\sum_{j=1}^{k} b_{0s_i}^2 + 2h_i b_{00} b_{0s_i} \qquad b_{sv_i} = \frac{b_{is}b_{s_{ii}}}{b_{ii}}$$
$$b_{is} = \frac{-4b_{ii}f_i s_i}{b_{0s_i} + b_{00}} \qquad b_{v_{ii}} = \frac{1}{b_{ii}}$$
$$b_{v_{00}} = \frac{1}{b_{00}}$$

If each coefficient is set in terms of $b_{00}, b_{s_{ii}}, b_{ii},$ and b_{0s_i} as above, one can verify through direct calculation that the metric will only have the following non-zero entries:

$$g_\delta(u_0, v_0) = 1, \quad g_\delta(u_i, v_i) = 1, \quad g_\delta(s_i, s_i) = 1.$$

From Lemma 3.3 we can use the remaining, undefined coefficients to fit R into our model space.

We will start by dividing the manifolds $\mathcal{M}_\delta$ into three cases.

First suppose that $R_s \neq 0$ and $R_0 \neq -2R_s$, and let:

$$b_{0s_i} = \frac{R_0}{2R_s}, \;\; b_{00} = 1, \;\; b_{ii} = \frac{1}{\sqrt{b_{s_{ii}}\epsilon_i R_s}}, \;\; b_{s_{ii}} = \epsilon_i \text{sign}(R_s).$$

This will leave $\bar{R}_0 = 0$ and $\bar{R}_s = 1$, then this branch of $\mathcal{M}_\delta \cong \mathfrak{M}_{0_1}$.

Now, suppose instead the $R_s = 0$ and $R_0 = -2R_s$. These correspond to those δ_i which satisfy the Riccati equation $\delta_i' + \frac{1}{2}\delta_i^2\epsilon_i - 1 = 0$. In this case let $b_{s_{ii}}$ and b_{ii} be defined as in the previous case but instead let b_{00} and b_{0s_i} be such that

$$b_{00} + b_{0s_i} = -\frac{1}{2}b_{s_{ii}}\epsilon_i.$$

With this basis, $\bar{R}_0 = 1$ and $\bar{R}_s = 1$ and clearly, $\mathcal{M}_\delta \cong \mathfrak{M}_{0_3}$.

Lastly, consider the case in which $R_s = 0$. From assertion (a) of Theorem 2.5, we know that R_0 cannot also be zero. These manifolds correspond to those linear $\delta_i' = 1$. To show that this case is also CH_0, let us adjust our basis by setting

$$b_{00} = 1, \;\; b_{0s_i} = 0, \;\; b_{ii} = \frac{1}{\sqrt{b_{s_{ii}}R_0}}, \;\; b_{s_{ii}} = \text{sign}(R_0).$$

Plugging these in will show that $\bar{R}_0 = 1$ and $\bar{R}_s = 0$ showing that $\mathcal{M}_\delta \cong \mathfrak{M}_{0_2}$, showing the last case.

This also proves assertion (b) of Theorem 2.5. $\square$

Remark. Although it makes little difference for the purposes of this paper, with the isometry invarients outlined by Dunn in [1] it can be shown that $\mathfrak{M}_{0_1}$ and $\mathfrak{M}_{0_3}$ are equivalent model spaces.

5 Curvature 1-Models

This section will provide the main examination into the curvature characteristics of the $\mathcal{M}_\delta$ family of manifolds which at the 1-model level begin to exhibit a more diverse series of examples under different curvature homogeneity conditions.

The nature of the homogeneity types in the Dunn manifold reveals a nested pattern, with more restrictive conditions such as regular curvature homogeneity implying all the more general conditions as well. Because of this, it is the most natural to begin with the least restrictive, WCH_1 and moving inward.

Proof of Theorem 2.5 (c). Consider a manifold $M_\delta \in \mathcal{M}_\delta$ such that it is CH_0. Then by Lemma 4.3, there exists a change of basis, outlined in Definition 3.2, such that the curvature tensor is constant. If the manifold is also WCH_1, then exist a set of coefficients, $\{b_{00}, b_{ii}, b_{s_{ii}}, b_{0s_i}\}$ such that the first curvature tensor is also constant. Using the relations for the rest of the coefficients found in the proof of Lemma 4.3 the same set of coefficients can be used to normalize the metric , makin the manifold also CH_1. $\square$

Definition 5.1. Let (M, g) be a pseudo-Riemannian manifold, then the following are equivalent:

1. M is locally symmetric

2. ∇R is identically zero.

3. There exist isomorphisms such that $(TM, g, R, \nabla R) \cong (V, \langle\cdot,\cdot\rangle, A_0, 0)$

Proof of Theorem 2.5 (d). Assume $\delta_i = 0$ or $\delta_i' = \frac{1}{2}$ then by direct calculation it can be seen that both ∇R_s and ∇R_0 are identically zero, thus M_δ is locally symmetric. $\square$

Similar to the method used to prove the CH_0 condition, we must define seperate model spaces for different branches of manifolds.

Definition 5.2. Let $\mathfrak{V}_{1_{n,m}}$ be the family of variable homothety 1-models such that $\mathfrak{V}_{1_{n,m}} = (\mathbb{R}^{3k+2}, \langle\cdot,\cdot,\rangle, \varphi_1 R_n, \varphi_2 \nabla R_m)$ for positive functions φ_1 and φ_2 and with the inner product, curvature tensor and its covariant derivative having only the following non-zero entries:

$$\begin{array}{lll} \langle u_0, v_0\rangle = 1 & & \nabla R_1 = \{\nabla \bar{R}_0 = 0, \nabla \bar{R}_s = 1\} \\ \langle u_i, v_i\rangle = 1 & R_1 = \{\bar{R}_s = 0, \bar{R}_0 = 1\} & \nabla R_2 = \{\nabla \bar{R}_0 = 1, \nabla \bar{R}_s = 0\} \\ \langle s_i, s_i\rangle = 1 & R_2 = \{\bar{R}_s = 1, \bar{R}_0 = 0\} & \nabla R_3 = \{\nabla \bar{R}_0 = 1, \nabla \bar{R}_s = 1\} \end{array}$$

To prove the variable homothety curvature homogeneity condition, we will show that any manifold, M_δ that is CH_0 but not locally symmetric can be 1-modeled by some $\mathfrak{V}_{1_{n,m}}$.

Remark. It may be the case that many of these model spaces are equivalent and a more elegant proof exists, however without a well-defined notion of a homothety curvature invariant to differenciate model spaces from point to point, there is no way of proving whether or not we need to consider all these model spaces.

Proof of Theorem 2.6. First consider those manifolds with $\delta_i' = 1$. Then since by assumption they are CH_0, using the notation of Definition 3.2 and Lemma 3.3 we have:

$$\begin{array}{ll} \bar{R}_s = 0 & \bar{\nabla} R_s = 0 \\ \bar{R}_0 = b_{ii}^2 b_{00}^2 R_0 & \bar{\nabla} R_0 = b_{ii}^3 b_{00}^2 \nabla R_0 \end{array}$$

By setting, the coefficients similar to as they were in proof of Lemma 4.3 (b) with:

$$b_{s_{ii}} = \ \mathrm{sign}(R_0), \ \ b_{ii} = \frac{1}{\sqrt{b_{s_{ii}}R_0}}, \ \ b_{0s_i} = 0, \ \ b_{00}^2 = \varphi_1$$

where φ_1 some positive, real-valued function, the non-zero curvature tensor entries become:

$$\bar{R}_0 = \varphi_1 \ \text{ and } \ \nabla\bar{R}_0 = \varphi_1\frac{\nabla R_0}{(bs_{ii}R_0)^{3/2}}.$$

If we define $\varphi_2 = \nabla\bar{R}_0$ as above, φ_2 will also be positive if we assume $(\epsilon_i\delta_i) > 0$, showing VCH$_1$ under $\mathfrak{V}_{1_{1,2}}$.

For the rest of the cases we will be assuming that $\delta_i' \neq 1$, and since all manifolds in this family are CH$_0$, $\bar{R}_0 = 0$ and b_{00} remains our only free variable. Because of this we will always be setting $\varphi_1 = b_{00}$.

First let us consider the case of a linear δ_i with $\delta_i'' = 0$ for all i but not locally symmetric. With this condition, both ∇R_s and $\nabla\bar{R}_s$ are zero and we have:

$$\bar{R}_s = \varphi_i \ \text{ and } \ \nabla\bar{R}_0 = \varphi_1^2\frac{\nabla R_0}{(b_{s_{ii}}\epsilon_i R_s)^{3/2}}$$

Here we set $\varphi_2 = \nabla\bar{R}_0$ which will be positive only if $(\epsilon_i\delta_i) > 0$, and the manifold is VCH$_1$ under $\mathfrak{V}_{1_{2,2}}$.

Next consider the case in which $\nabla\bar{R}_0$ is zero, in other words $R_0\nabla R_s = R_s\nabla R_0$ or δ_i is a non-linear solution to the second order differential equation $\delta_i\delta_1'' = 2\epsilon_i(\delta_i - 1)(2\delta_i - 1)$. In this case, we have:

$$\bar{R}_s = \varphi_1 \ \text{ and } \nabla\bar{R}_s = \varphi_1\frac{b_{s_{ii}}\epsilon_i\nabla R_s}{(b_{s_{ii}}\epsilon_i R_s)^{3/2}}$$

Letting $\varphi_2 = \nabla\bar{R}_s$ will give us the second function we need for VCH$_1$ under $\mathfrak{V}_{1_{2,1}}$ which will be positive if $(\delta_i''\frac{\delta_i'-1}{|\delta_i'-1|}) > 0$.

Lastly, we consider in which neither $\nabla\bar{R}_0$ or $\nabla\bar{R}_s$ are zero. Unlike the other two cases which encompassed an infinite family of model spaces for any choice of positive functions φ_1, this case has the restriction that $\nabla\bar{R}_0 = \nabla\bar{R}_s$ and thus if $\varphi_1 = b_{00}$, we get the relations:

$$R_s\nabla R_0 - R_0\nabla R_s = \varphi_1(b_{s_{ii}}\epsilon_i R_s)\nabla R_s$$

or by substituting back to make it in terms of δ_i,

$$2\delta_i\epsilon_i(\delta_i' - 1)(2\delta_i' - 1) - \delta_i^2\delta_i''\epsilon_i = \pm\varphi_1\delta_i''(\delta_i' - 1).$$

If we assume that φ_1 is a positive function such that there exists a non-linear δ_i that satisfies the above differential equation, then we can simply let

$$\varphi_2 = \varphi_1(b_{s_{ii}}\epsilon_i R_s)\nabla R_s = R_s\nabla R_0 - R_0\nabla R_s$$

which will be positive if $\delta_i'' > 0$.

Of course which positive functions will allow a non-llinear solution to exist, if any, is unclear; however, if there does exist one, we have shown that if $\delta_i'' > 0$, we can define a positive φ_2 that will be sufficient to be 1-modeled by $\mathfrak{V}_{2_{2,3}}$. $\square$

Now we will look into the HCH_1 condition by use of its connection to the variable homothety curvature condition.

Definition 5.3. Let $\mathfrak{H}1_{n,m}$ be the family of homothety 1-models such that for a positive real-valued function, λ, $\mathfrak{H}1_{n,m} = (\mathbb{R}^{3k+2}, \langle\cdot,\cdot\rangle, \lambda R_n, \lambda^{3/2}\nabla R_m)$ with the inner product and curvature tensor entries defined as in Definition 5.2.

Lemma 5.4. If a manifold $M_\delta \in \mathcal{M}_\delta$, is VCH_k then it is HCH_k.

Proof. This family of manifold poses a special property in regards to VCH_1 in that the two positive functions, φ_1 and φ_2, are already related. Because of this, by imposing the usual relations for homothety 1-homogeneity namely, $\lambda = \varphi_1$ and $\lambda^{3/2} = \varphi_2$, we find λ as the square of a ratio of the two φ's.

For the $\mathfrak{V}1_{1,2}$ case, if $\delta_i' = 1$ for all i, we can make it $\mathfrak{H}1_{1,2}$ if we set

$$\lambda = \frac{(\nabla R_0)^2}{(bs_{ii}R_0)^3}.$$

Next, consider the case in which $\delta_i' \neq 1$ and $\delta_i'' = 0$, the $\mathfrak{V}1_{2,2}$ case. If we set

$$\lambda = \frac{(b_{s_{ii}}\epsilon_i R_s)^3}{(\nabla R_0)^2}$$

we find it is HCH_1 under $\mathfrak{H}1_{2,2}$ as well. The $\mathfrak{V}1_{2,1}$ case, with a δ_i which is a nonlinear solution to $\delta_i\delta_1'' = 2\epsilon_i(\delta_i' - 1)(2\delta_i' - 1)$ can be shown to be HCH_1 with $\mathfrak{H}1_{2,1}$ by setting

$$\lambda = \frac{(\nabla R_s)^2}{(b_{s_{ii}}\epsilon_i R_s)^3} = \frac{(R_s)^2(\nabla R_0)^2}{(R_0)^2(b_{s_{ii}}\epsilon_i R_s)^3}.$$

The final case, in which δ_i is a nonlinear solution to monsterous differential equation described in the previous proof, then setting

$$\lambda = (R_s)^2(\nabla R_s)^2$$

will work to show HCH_1 under $\mathfrak{H}1_{2,3}$. □

Remark. Lemma 5.4 shows that any manifold considered in Theorem 2.6, those which are VCH_1 are also HCH_1, but in so doing, we eliminated the sign restrictions. This effectively loosened the restrictions on possible δ_i's or choices for ϵ_i's. However it is unclear whether the set of homothety curvature 1-homogeneous manifold is larger than the set of variable homothety curvature 1-homogeneous ones, since there exists potentially infinitely many choices for positive functions, φ_1 that will work.

The following section is dedicated to finding those manifolds which are CH_1, however because it involves solving very non-linear differential equations, it will not be comprehensive. In fact, we will only be able to consider the case of a constant δ_i.

Proof of Theorem 2.5 (d). Assume that δ_i is a constant. The δ_i' and δ_i'' are zero, so consider the $\mathfrak{H}_{1_{2,2}}$ model space. CH_1 is equivalent to being HCH_1 under the same model space with a constant λ. Thus, set $\lambda = a$ for some positive constant, a.

Then, we get:

$$a4\delta_i^2(2\delta_i' - 1)^2 = |\delta_i' - 1|^3.$$

Since $\delta_i' = 0$, we get,

$$\delta_i = \frac{1}{2}\sqrt{a} = A$$

which is a positive constant. Equivalently, we can find when $\nabla \bar{R}_0$ is identically constant, meaning

$$2\delta_i\epsilon_i(2\delta_i' - 1) = b|\delta_i' - 1|^{3/2}$$

Solving for δ_i we find:

$$\delta_i = -2\epsilon_i b = B$$

Where B can be any constant. □

Remark. The rationale behind showing the existance of CH_1 manifolds from constant δ_i's using both the homothety curvature 1-solutions as well as directly from the basis is to show that although intuitively it would seem that CH_1 is just a special case of HCH_1, there is actually quite the loss of information and solving for constant homothety 1-curvature homogeneous manifolds will infact not give all of the curvature 1- homogeneous manifolds.

6 Open Questions

1. All proofs in this paper were done on a case by case basis in regards to the model spaces because we lack a well-defined notion of of a homothety isometry invariant. Such an object would serve to differentiate model spaces from point to point, and it'd be easier to definitively say if two manifolds are isomorphic to each other. The author conjectured that such an invariant would be a generalization of Singer invariants, necessarily also containing the solutions to regular curvature homogeneous manifolds within the family however the nature of the homothety solutions only revealing an very restrictive set of CH_1 solutions, it seems unlikely.

2. The formulation of $\mathcal{M}_\delta$ seems to be a generalization of the original Dunn manifolds as defined in [1] however it is unknown if the two families are still isomorphic to each other. Also, the structure group of the $\mathcal{M}_\delta$ family is mostly a mystery at this point. More insight would possibly allow the creation of a new isometry invariant.

3. Exploration into higher order model spaces and exploring the different curvature conditions on those would aid in potentially defining a homothety invariant but also in finding a Singer-type number for homothety homogeneity, that is, a positive integer $k_{p,q}$ such that if a manifold is

homothety curvature $k_{p,q}$-homogeneous, then it it necessarily homothety homogeneous.

4. The nested relationship between the curvatue homogeneity conditions exhibited by $\mathcal{M}_\delta$ brings up the question of if it is possible to create manifolds such that certain conditions are satisfied but others are not. For example, VCH but not HCH, or HCH but not CH, or vice versa?

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