Linear Dependence in Sets of Three Canonical Algebraic Curvature Tensors

Susan Ye

Abstract

This paper will examine sets of three canonical algebraic curvature tensors for linear dependence and extend several previous results to provide a means of determining when any three given algebraic curvature tensors can be linearly dependent when one is defined by a positive definite inner product. We will examine the ranks of our linear operators in a new approach of studying linear dependence. We will then use the results to conclude that in every nontrivial case, linear dependence of canonical algebraic curvature tensors implies that all operators can be simultaneously diagonalized.

1 Introduction and Definitions

Let V be a real valued vector space with finite dimension n, with a positive definite inner product φ. We will start with the following definitions:

Definition 1.1. Let R be a 4-multilinear form such that $R : V \times V \times V \times V \to \mathbb{R}$. R is called an *algebraic curvature tensor* when it satisfies the following:

$$\begin{aligned}
&R(x, y, z, w) = -R(y, x, z, w)\\
&R(x, y, z, w) = R(z, w, x, y)\\
&R(x, y, z, w) + R(x, z, w, y) + R(x, w, y, z) = 0
\end{aligned}$$

for all $x, y, z, w \in V$. The last equality is known as the *Bianchi Identity*. Let $\mathcal{A}(V)$ denote the space of all algebraic curvature tensors on V [4].

Definition 1.2. Let φ be an *inner product* on V such that $\varphi : V \times V \to \mathbb{R}$. We have that $\varphi \in S^2(V)$, in other words φ is a *symmetric bilinear form*, when it satisfies the following properties:

$$\begin{aligned}
&\varphi(u + v, w) = \varphi(u, w) + \varphi(v, w)\\
&\varphi(cu, v) = c\varphi(u, v)\\
&\varphi(u, v) = \varphi(v, u)
\end{aligned}$$

for all $u, v, w \in V$ and $c \in \mathbb{R}$.

Definition 1.3. An inner product φ is called *positive definite* if $\varphi(u,u) \geq 0$ for all $u \in V$ and only equals 0 when $u = 0$.

Definition 1.4. A basis $\{e_1, \ldots, e_n\}$ on V is an *orthonormal basis* if $e_1, \ldots, e_n$ are mutually orthogonal and $|e_i| = 1$.

Definition 1.5. If $A : V \to V$ is a linear map, define R_A as

$$R_A(x,y,z,w) = \varphi(Ax,w)\varphi(Ay,z) - \varphi(Ax,z)\varphi(Ay,w).$$

Now we can define an *algebraic curvature tensor* R_φ such that for all $x,y,z,w \in V$,

$$R_\varphi(x,y,z,w) = \varphi(x,w)\varphi(y,z) - \varphi(x,z)\varphi(y,w).$$

The following lemma follows from the definition and properties of algebraic curvature tensors:

Lemma 1.1. *Let $R_\varphi \in A(V)$. Then $R_{-\varphi} = R_\varphi$, and $cR_\varphi(x,y,z,w) = sign(c)R_{\sqrt{|c|}\varphi}$, where $sign(c)$ denotes the signature of c.*

Proof.

$$\begin{aligned}
R_{-\varphi} &= (-\varphi(x,w))(-\varphi(y,z)) - (-\varphi(x,z))(-\varphi(y,w)) \\
&= \varphi(x,w)\varphi(y,z) - \varphi(x,z)\varphi(y,w) \\
&= R_\varphi \\
cR_\varphi(x,y,z,w) &= c\varphi(x,w)\varphi(y,z) - c\varphi(x,z)\varphi(y,w) \\
&= sign(c)\varphi(\sqrt{|c|}x,w)\varphi(\sqrt{|c|}y,z) - sign(c)\varphi(\sqrt{|c|}x,z)\varphi(\sqrt{|c|}y,w) \\
&= sign(c)R_{\sqrt{|c|}\varphi}.
\end{aligned}$$

□

Definition 1.6. Let the *kernel of* R, denoted $ker(R)$, be defined as

$$ker(R) = \{v \in V | R(v,x,y,z) = 0 \text{ for all } x,y,z \in V\}.$$

One important property of the kernel is that it is not biased towards the first argument of (x,y,z,w) [3]; more specifically, the kernel is defined as $ker(R) = \{v \in V | R(v,x,y,z) = R(x,v,y,z) = R(x,y,v,z) = R(x,y,z,v) = 0\}$.

Lemma 1.2. *Let A be a symmetric bilinear form and $R_A \in A(V)$. Then Rank R_A = Rank A [4].*

In this paper we will also use the operators ψ and τ, where $\psi, \tau \in S^2(V)$. Since φ is positive definite, ψ can be represented by a unique linear transformation $[\psi_{ij}] : V \to V$ on the orthonormal basis $\{e_1, \ldots, e_n\}$. $\psi(e_i, e_j) = \psi_{ij}$, the (i,j) entry of the matrix denoted by $[\psi_{ij}]$. Since ψ is symmetric, $[\psi_{ij}]$ is also symmetric, and hence $R_\psi \in A(V)$ [2]. Also, $\psi(u,v) = \varphi([\psi_{ij}]u, v)$, where $\{e_1, \ldots, e_n\}$ is a basis for V. The same reasoning holds for τ.

Definition 1.7. The *spectrum of A* is the set of eigenvalues of $[A_{ij}]$, not necessarily distinct, and is denoted $Spec(A)$. We will use $|Spec(A)|$ to denote the number of distinct eigenvalues of A.

Definition 1.8. Two matrices A and B are *simultaneously diagonalizable* if and only if there exists a basis composed of eigenvectors of both A and B.

We will discuss the motivations behind our study as well as several past contributions Section 2.

2 Preliminaries

2.1 Linear Dependence

Now we will provide some background regarding the study of linear dependence of algebraic curvature tensors, including several previous results that will be used for further proofs.

Previous work on the linear dependence of $\{R_\varphi, R_\psi, R_\tau\}$, with φ being positive definite, has focused on cases in vector spaces of dimension at least 3; in vector spaces of dimension less than 3, the algebraic curvature tensors reduce to the zero tensor or the trivial case, and the study of linear dependence is much less interesting, so we will also require that dim $V \geq 3$.

First, we have the following theorem in the case where we have linear dependence of only two algebraic curvature tensors.

Theorem 2.1. [2]. Suppose Rank $\varphi \geq 3$. The set $\{R_\varphi, R_\psi\}$ is linearly dependent if and only if $R_\psi \neq 0$ and $\psi = c\varphi$ for some $c \in \mathbb{R}$.

With three algebraic curvature tensors, linear dependence is satisfied when $c_1R_\varphi + c_2R_\psi + c_3R_\tau = 0$. We will let φ represent the positive definite symmetric bilinear form. From this equation, we can divide the problem into three cases:

1. If just one of $c_1, c_2, c_3 \neq 0$, then we have that R_ψ, or R_τ equals the zero tensor. We will exclude this trivial case. Also, note that $R_\varphi \neq 0$ since φ is positive definite, and c_2, c_3 cannot simultaneously equal 0, otherwise $c_1R_\varphi = 0$ which is a contradiction.

2. If exactly two of $c_1, c_2, c_3 \neq 0$, then $R_{\varphi_1} = \pm R_{\varphi_3}$, and since we are working in dimensions of at least 3, Rank $\varphi_1 \geq 3$ so Theorem 2.1 holds and $\varphi_2 = \pm\varphi_1$. We will investigate each specific case of this situation in the following sections.

3. If all three of $c_1, c_2, c_3 \neq 0$, then we can rearrange our equation into $R_\varphi + \frac{c_2}{c_1}R_\psi = -\frac{c_3}{c_1}R_\tau$. From the properties of the algebraic curvature tensors, we know that we can write $R_\varphi + \epsilon R_\psi = \delta R_\tau$, where $\epsilon, \delta \in \{\pm 1\}$.

We will exclude the first case and look at the latter two in different rank settings in Sections 3 and 4.

2.2 Simultaneous Diagonalizability

Since φ is positive definite, there exists an orthonormal basis on V which can simultaneously diagonalize $[\varphi_{ij}]$ and $[\psi_{ij}]$ with respect to φ, so that we have $[\varphi_{ij}]$ as the identity matrix while $[\psi_{ij}]$ has only its eigenvalues on the diagonal and zeroes elsewhere. The following theorem clarifies when $[\psi_{ij}]$ and $[\tau_{ij}]$ can both be simultaneously diagonalized with respect to φ under the conditions of an existing dependence relationship:

Theorem 2.2. [2]. Suppose φ is positive definite, Rank $\tau = n$, and Rank $\psi \geq 3$. If $\{R_\varphi, R_\psi, R_\tau\}$ is linearly dependent, then $[\psi_{ij}]$ and $[\tau_{ij}]$ are simultaneously orthogonally diagonalizable with respect to φ .

We will look at the ranks of ψ and τ to determine when and under what conditions the set $\{R_\varphi, R_\psi, R_\tau\}$ can be linearly dependent, that is when $c_1 R_\varphi + c_2 R_\psi + c_3 R_\tau = 0$, where $c_1, c_2, c_3 \in \mathbb{R}$.

However, we will first state two other results which have helped immensely with determining what restrictions exist on the possibilities for the ranks of our algebraic curvature tensors.

Theorem 2.3. [5]. Let V be a vector space with $R = R_\varphi \pm R_\psi$, an algebraic curvature tensor, on V. If φ is positive definite, and dim $V = n \geq 3$, then dim $ker(R_\varphi \pm R_\psi) = 0, 1$, or n.

Theorem 2.3 gives us three possible values for the dimension of the kernel of an algebraic curvature tensor that is known to be a linear combination of two others, one of which is positive definite.

The Rank-Nullity Theorem tells us that, given vector spaces V and W and a linear transformation $V \to W$, the dimension of V is the sum of the rank and nullity of the linear transformation. Since the nullity of a linear transformation is simply the dimension of its kernel, we know that if the dimension of $ker(R_\varphi \pm R_\psi) = 0, 1$, or n, then

$$\begin{aligned} \text{dim } ker(R_\tau) &= \text{dim } ker(R_\varphi \pm R_\psi) \\ &= 0,\ 1,\ \text{ or } n, \end{aligned}$$

and when all coefficients are nonzero, $R_\varphi + \epsilon R_\psi = \delta R_\tau$, we can conclude that the rank of τ must equal n, $n-1$, or 0. In the case that Rank $\tau = 0$, $R_\tau(x, y, z, w) = 0$ for all $x, y, z, w \in V$, and we would simply have the zero tensor. We will exclude this case.

The following corollary illustrates the case where dim $ker(R_\varphi \pm R_\psi) = 1$ and determines the relationship that must occur in the eigenvalues of ψ.

Corollary 2.1. [5]. If φ is positive definite, dim $V \geq 3$, and dim $ker(R_\varphi \pm R_\psi) = 1$, then $Spec(\psi) = \{\mp\frac{1}{\lambda}, \lambda, \dots\}$.

Now we can state the main result that will be proved in Sections 3 and 4:

Main Result. If φ is positive definite, dim $V \geq 3$, and neither one of R_ψ, R_τ is the zero tensor, then we have the following:

1. If $\{R_\varphi, R_\psi, R_\tau\}$ is a linearly dependent set of canonical algebraic curvature tensors, then ψ and τ are simultaneously diagonalizable with respect to φ. In addition, the eigenvalues of ψ and τ must satisfy specific relationships which are determined by the ranks of the operators.

2. Conversely, if ψ and τ are simultaneously diagonalizable with respect to φ, and in each separate case the corresponding eigenvalue relationships are satisfied, then $\{R_\varphi, R_\psi, R_\tau\}$ is linearly dependent.

We will go case by case to determine when and under what conditions linear dependence is possible, as well as what the specific eigenvalue relationships are in each case, by changing the rank requirements of our operators ψ and τ.

3 Vector Spaces of Dimension $n = 3$

The theorems presented in this section determine exactly when a set of three algebraic curvature tensors can be linearly dependent when in dimension 3. By altering the restrictions on the ranks of our symmetric bilinear forms ψ and τ, we have three settings:

3.1 Rank $\psi =$ Rank $\tau = 3$

We start with $c_1 R_\varphi + c_2 R_\psi + c_3 R_\tau = 0$ and exclude the case where two or more of the coefficients equal 0. If $c_3 = 0$, then by [4] we have

$$R_\varphi = \tfrac{c_3}{c_1} R_\tau = \pm R_{c\tau} \text{ for } c = \sqrt{|\tfrac{c_3}{c_1}|}.$$

Since Rank $\varphi = 3$, then $\pm c\tau = \varphi$. Now, if we diagonalize ψ with respect to φ on the orthonormal baisis $\{e_1, e_2, e_3\}$, we obtain

$$[\varphi_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, [\psi_{ij}] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}, \text{ and } [\tau_{ij}] = \pm \begin{bmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{bmatrix} \text{ for } c \neq 0.$$

In this specific case where $c_2 = 0$, we obtain the requirements $Spec(\psi) = \{\lambda_1, \lambda_2, \lambda_3\}$ and $Spec(\tau) = \{\pm c, \pm c, \pm c\}$. A similar situation occurs if $c_2 = 0$; we will obtain that $Spec(\psi) = \{\pm d, \pm d, \pm d\}$ for some $d \in \mathbb{R}$, and $Spec(\tau) = \{\eta_1, \eta_2, \eta_3\}$. We will omit the $c_1 = 0$ case because we are using the positive definite R_φ to diagonalize the other operators.

Conversely, suppose that ψ and τ have eigenvalues that satisfy the above requirements. Then there exists a basis on which ψ, τ are simultaneously diagonalized with respect to φ. Then we have

$$\begin{aligned} c^2 R_\varphi - R_\tau &= c^2 R_\varphi - R_{\pm c\varphi} \\ &= c^2 R_\varphi - c^2 R_\varphi \\ &= 0, \end{aligned}$$

so linear dependence holds and our Main Result from Section 1 is satisfied.

If all three of $c_1, c_2, c_3 \neq 0$, then we have the following theorem:

Theorem 3.1. [2]. Let φ be a positive definite symmetric bilinear form on a real vector space V of dimension 3. Suppose $Spec(\tau) = \{\eta_1, \eta_2, \eta_3\}$. Set

$$\eta(i, j, k) = (-\epsilon)\sqrt{\tfrac{(1-\delta\eta_i\eta_j)(1-\delta\eta_i\eta_k)}{(-\epsilon)(1-\delta\eta_j\eta_k)}}.$$

If Rank $\psi =$ Rank $\tau = 3$, and $Spec(\psi) = \{\lambda_1, \lambda_2, \lambda_3\}$, where

$$\lambda_1 = (-\epsilon)\eta(1,2,3), \lambda_2 = (-\epsilon)\eta(2,3,1), \lambda_3 = \eta(3,1,2),$$

then ψ and $-\psi$ are the only solutions to the equation $R_\varphi + \epsilon R_\psi = \delta R_\tau$.

3.2 Rank $\psi = 2$, Rank $\tau = 3$

When one of ψ or τ is not full rank, the following theorem states under what conditions linear dependence can occur. Obviously, the same result holds if we switch the role of ψ and τ, that is if Rank $\tau = 2$, Rank $\psi = 3$.

Suppose $\{R_\varphi, R_\psi, R_\tau\}$ is linearly dependent and there exist $c_1, c_2, c_3 \in \mathbb{R}$, not all equal to 0, such that $c_1 R_\varphi + c_2 R_\psi + c_3 R_\tau = 0$.

If c_1 or $c_3 = 0$, then the equation can be rearranged to the form $R_\alpha = \pm R_\beta$, but Rank $\alpha \neq$ Rank β and both are at least 2, which is a contradiction.

If $c_2 = 0$, then by [4] we have

$$R_\varphi = \tfrac{c_3}{c_1} R_\tau = \pm R_{c\tau} \text{ for } c = \sqrt{|\tfrac{c_3}{c_1}|}.$$

Since Rank $\varphi = 3$, then $\pm c\tau = \varphi$. Now, if we diagonalize ψ with respect to φ on the orthonormal baisis $\{e_1, e_2, e_3\}$, we obtain

$$[\varphi_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, [\psi_{ij}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}, \text{ and } [\tau_{ij}] = \pm \begin{bmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{bmatrix} \text{ for } c \neq 0.$$

In this specific case where $c_2 = 0$, we obtain the requirements $Spec(\psi) = \{0, \lambda_2, \lambda_3\}$ and $Spec(\tau) = \{\pm c, \pm c, \pm c\}$ for $\lambda_2\lambda_3 \neq 0$.

Conversely, suppose that ψ and τ have eigenvalues that satisfy the above requirements. Then there exists a basis on which ψ, τ are simultaneously diagonalized with respect to φ. Then we have

$$\begin{aligned} c^2 R_\varphi - R_\tau &= c^2 R_\varphi - R_{\pm c\varphi} \\ &= c^2 R_\varphi - c^2 R_\varphi \\ &= 0, \end{aligned}$$

so linear dependence holds and our Main Result from Section 1 is satisfied.

If all $c_1, c_2, c_3 \neq 0$, then our linear dependence equation becomes $R_\varphi + \epsilon R_\psi = \delta R_\tau$, and we have the following theorem which gives the eigenvalue relationships for this setting:

Theorem 3.2. Let V be a vector space with dim $V = 3$, φ is positive definite, Rank $\psi = 2$, and Rank $\tau = 3$. Then there exists a solution to $R_\varphi + \epsilon R_\psi = \delta R_\tau$ if and only if the following conditions are satisfied:

I. τ and ψ are simultaneously diagonalizable with respect to φ

II. $\delta = 1, \epsilon = \pm 1$

III. $Spec(\tau) = \{\pm\frac{1}{\eta}, \eta, \eta\}$, and $Spec(\psi) = \{0, \lambda_2, \lambda_3\}$ where $\lambda_2\lambda_3 = \epsilon(\eta^2 - 1)$.

Proof. Assume φ is positive definite, $R_\varphi + \epsilon R_\psi = \delta R_\tau$, Rank $\psi = 2$, and Rank $\tau = 3$. Let $\{e_1, e_2, e_3\}$ be an orthonormal basis on V.

We can rearrange our equation to $\delta R_\tau - R_\varphi = \epsilon R_\psi$. Since Rank $\psi = n - 1 =$ Rank R_ψ by Lemma 1.3, Theorem 2.3 and its corollary can be applied to diagonalize τ with respect to φ on our orthonormal basis:

$$[\tau_{ij}] = \begin{bmatrix} \pm\frac{1}{\eta} & 0 & 0 \\ 0 & \eta & 0 \\ 0 & 0 & \eta \end{bmatrix}, \text{ while } [\psi_{ij}] = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}.$$

If $\eta = \pm 1$, then $\tau = \pm I = \pm\varphi$, and

$$\begin{aligned} R_\varphi + \epsilon R_\psi &= \delta R_\tau = \delta R_{\pm\varphi} = \delta R_\varphi \\ \Rightarrow \epsilon R_\psi &= (\delta - 1) R_\varphi \end{aligned}$$

which is a contradiction because Rank $R_\psi = 2 \neq$ Rank R_φ. So we know that $\eta \neq \pm 1$.

If $e \neq 0$, we can make calculations easier by changing the orthonormal basis for the η eigenspace, $E_\eta = span\{e_2, e_3\}$, to a different orthonormal basis $\{f_1, f_2, f_3\}$, of the form

$$\begin{aligned} f_1 &= e_1 \\ f_2 &= \cos\theta e_2 + \sin\theta e_3 \\ f_3 &= -\sin\theta e_2 + \cos\theta e_3 \end{aligned}$$

Now, $E_\eta = span\{f_2, f_3\}$.

It can be easily checked that $\{f_1, f_2, f_3\}$ is still a basis for V and preserves φ and τ, that is,

$$[\varphi_{ij}]_f = [\varphi_{ij}]_e \text{ and } [\tau_{ij}]_f = [\tau_{ij}]_e.$$

With respect to this basis,

$$\begin{aligned}
0 = ([\psi_{ij}]_f)_{23} &= \psi(f_2, f_3) = \psi_{23} \\
&= \psi(\cos\theta e_2 + \sin\theta e_3, -\sin\theta e_2 + \cos\theta e_3) \\
&= -\cos\theta\sin\theta\psi_{22} + \cos^2\theta\psi_{23} - \sin^2\theta\psi_{32} + \cos\theta\sin\theta\psi_{33} \\
&= \cos\theta\sin\theta(f-d) + (\cos^2\theta - \sin^2\theta)(e) \\
&= \sin 2\theta(\frac{f-d}{2}) + \cos 2\theta(e)
\end{aligned}$$

when $-\sin 2\theta(\frac{f-d}{2}) = \cos 2\theta(e)$, that is when $(\frac{d-f}{2e}) = \cot 2\theta$, and there exists some θ which satisfies this condition. So we know that there exists some θ for our change of basis that makes $\psi_{23} = \psi_{32} = 0$. So we now have a basis which leaves φ and τ diagonal, but

$$[\psi_{ij}]_f = \begin{bmatrix} g & h & i \\ h & j & 0 \\ i & 0 & k \end{bmatrix}.$$

From the linear dependence hypothesis we have the following equalities:

Equation	(x, y, z, w)	$R_\varphi + \epsilon R_\psi = \delta R_\tau$
(1)	(e_1, e_2, e_2, e_1)	$1 + \epsilon(gj - h^2) = \delta$
(2)	(e_1, e_3, e_3, e_1)	$1 + \epsilon(gk - i^2) = \delta$
(3)	(e_2, e_3, e_3, e_2)	$1 + \epsilon(jk) = \delta\eta^2$
(4)	(e_1, e_2, e_3, e_1)	$\epsilon(-ih) = 0$
(5)	(e_2, e_3, e_1, e_2)	$\epsilon(ji) = 0$
(6)	(e_3, e_1, e_2, e_3)	$\epsilon(kh) = 0.$

Equations (1) and (2) give us that $gj - h^2 = gk - i^2$, and Equation (3) implies that $jk = \epsilon(\delta\eta^2 - 1) \neq 0$ since $\eta \neq \pm 1$. So $j, k \neq 0$ which means that $h = i = 0$ due to equations $(4) - (6)$. Then, looking back at Equations (1) and (2), either $g = 0$ or $j = k$.

However, since all the off-diagonal terms are 0 and Rank $\psi = 2$, exactly one of g, j, or k must equal 0. The only option which makes this true is $g = 0$.

Since ψ is diagonalized, we conclude that its eigenvalues must be $g = \lambda_1$, $j = \lambda_2$, and $k = \lambda_3$, so

$$[\psi_{ij}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \text{ and } \lambda_2\lambda_3 = \pm(\eta^2 - 1)$$

when there exists a solution to $R_\varphi + \epsilon R_\psi = \delta R_\tau$. Furthermore, (1) and (2) now reduce to $1 + \epsilon(0) = \delta$, so $\delta = 1$.

Conversely, if

$$[\varphi_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, [\psi_{ij}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}, \text{ and } [\tau_{ij}] = \begin{bmatrix} \pm\frac{1}{\eta} & 0 & 0 \\ 0 & \eta & 0 \\ 0 & 0 & \eta \end{bmatrix},$$

that is, τ and ψ are simultaneously diagonalizable with respect to φ on some basis $\{e_1, e_2, e_3\}$ so that $Spec(\tau) = \{\pm\frac{1}{\eta}, \eta, \eta\}$, $Spec(\lambda) = \{0, \lambda_1, \lambda_2\}$, where $\lambda_1\lambda_2 = \epsilon(\eta^2 - 1)$, and $\delta = 1$, then we can check that there exists a solution to the equation $R_\varphi + \epsilon R_\psi = \delta R_\tau$ for all $x, y, z, w \in V$ and $\epsilon = \pm 1$ by checking that Equations $(1) - (6)$ hold true. $\square$

Note that this agrees with Theorem 3.1, where we replace $\{\eta_1, \eta_2, \eta_3\}$ with $\{\pm\frac{1}{\eta}, \eta, \eta\}$. We will show with an example that every λ_i is indeed determined by $Spec(\tau)$.

Example 3.2.1.

To demonstrate an application of Theorem 3.2, we will simply choose the eigenvalues for τ and construct $[\psi_{ij}]$ and $[\tau_{ij}]$ by Conditions I and III and set $\delta = 1$ to satisfy Condition II. We will show that our constructed algebraic curvature tensors satisfy the equation $R_\varphi + \epsilon R_\psi = \delta R_\tau$ for all $x, y, z, w \in V$.

Let $\{e_1, e_2, e_3\}$ be an orthonormal basis for V, and $\eta = 2$. Choose $Spec(\tau) = \{\frac{1}{2}, 2, 2\}$, so $\lambda_1\lambda_2 = \epsilon(3)$. Let us choose $\lambda_2 = 1, \lambda_3 = 3$ by setting $\epsilon = 1$. Note that we have chosen the signs to be positive in this example so that our calculations will be easier.

Now we have the following matrices:

$$[\psi_{ij}]_e = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}, [\tau_{ij}]_e = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Even on a basis where $[\psi_{ij}]$ is not diagonalized, Theorem 3.2 guarantees that linear dependence will hold. To show this, we will use the change of basis

$$\begin{aligned} f_1 &= e_1 \\ f_2 &= \cos\theta e_2 + \sin\theta e_3 \\ f_3 &= -\sin\theta e_2 + \cos\theta e_3. \end{aligned}$$

We can verify that $\{f_1, f_2, f_3\}$ are mutually orthogonal and then calculate the entries of $[\psi_{ij}]_f$, $[\tau_{ij}]_f$, and $[\varphi_{ij}]_f$. Any entries that are zero on the $\{e_1, e_2, e_3\}$ basis will still be zero on $\{f_1, f_2, f_3\}$ since those are the kernel elements, so we can omit those calculations.

For φ_{ij},

$$\begin{aligned}
\varphi(f_1, f_1) &= \varphi(e_1, e_1) \\
&= 1 \\
\varphi(f_2, f_2) &= \varphi(\cos\theta e_2 + \sin\theta e_3, \cos\theta e_2 + \sin\theta e_3) \\
&= \cos^2\theta\varphi_{22} + \cos\theta\sin\theta\varphi_{23} + \sin\theta\cos\theta\varphi_{32} + \sin^2\theta\varphi_{33} \\
&= 1 \\
\varphi(f_3, f_3) &= \varphi(-\sin\theta e_2 + \cos\theta e_3, -\sin\theta e_2 + \cos\theta e_3) \\
&= \sin^2\theta\varphi_{22} - \cos\theta\sin\theta\varphi_{23} - \sin\theta\cos\theta\varphi_{32} + \cos^2\theta\varphi_{33} \\
&= 1.
\end{aligned}$$

For ψ_{ij},

$$\begin{aligned}
\psi(f_2, f_2) &= \psi(\cos\theta e_2 + \sin\theta e_3, \cos\theta e_2 + \sin\theta e_3) \\
&= \cos^2\theta\psi_{22} + \cos\theta\sin\theta\psi_{23} + \sin\theta\cos\theta\psi_{32} + \sin^2\theta\psi_{33} \\
&= 1 + 2\sin^2 2\theta \\
\psi(f_2, f_3) &= \psi(\cos\theta e_2 + \sin\theta e_3, -\sin\theta e_2 + \cos\theta e_3) \\
&= -\cos\theta\sin\theta\psi_{22} + \cos^2\theta\psi_{23} - \sin^2\theta\psi_{32} + \cos\theta\sin\theta\psi_{33} \\
&= \sin 2\theta \\
\psi(f_3, f_2) &= \psi(f_2, f_3) \\
&= \sin 2\theta \\
\psi(f_3, f_3) &= \psi(-\sin\theta e_2 + \cos\theta e_3, -\sin\theta e_2 + \cos\theta e_3) \\
&= \sin^2\theta\psi_{22} - \cos\theta\sin\theta\psi_{23} - \sin\theta\cos\theta\psi_{32} + \cos^2\theta\psi_{33} \\
&= 1 + 2\cos^2 2\theta.
\end{aligned}$$

For τ_{ij},

$$\begin{aligned}
\tau(f_1, f_1) &= \tau(e_1, e_1) \\
&= \frac{1}{2} \\
\tau(f_2, f_2) &= \tau(\cos\theta e_2 + \sin\theta e_3, \cos\theta e_2 + \sin\theta e_3) \\
&= \cos^2\theta\tau_{22} + \cos\theta\sin\theta\tau_{23} + \sin\theta\cos\theta\tau_{32} + \sin^2\theta\tau_{33} \\
&= 2 \\
\tau(f_3, f_3) &= \tau(-\sin\theta e_2 + \cos\theta e_3, -\sin\theta e_2 + \cos\theta e_3) \\
&= \sin^2\theta\tau_{22} - \cos\theta\sin\theta\tau_{23} - \sin\theta\cos\theta\tau_{32} + \cos^2\theta\tau_{33} \\
&= 2.
\end{aligned}$$

Now we have

$$[\varphi_{ij}]_f = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, [\psi_{ij}]_f = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1+2\sin^2 2\theta & \sin 2\theta \\ 0 & \sin 2\theta & 1+2\cos^2 2\theta \end{bmatrix}, \text{ and } [\tau_{ij}]_f = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

$[\psi_{ij}]_f$ and $[\tau_{ij}]_f$ are the matrices we will use to demonstrate the theorem. By [2], we know that τ can be diagonalized with respect to φ. And by construction, we know that there exists the basis $\{e_1, e_2, e_3\}$ where ψ and τ are indeed simultaneously diagonalizable with respect to φ.

Now we must show that $R_\varphi + \epsilon R_\psi = R_\tau$ for all $x, y, z, w \in V$. By the properties of algebraic curvature tensors, we know that checking that all the permutations of the basis vectors e_1, e_2, e_3 satisfy the equation is sufficient. Then we have

Equation	(x, y, z, w)	$R_\varphi + \epsilon R_\psi = R_\tau$
(1)	(e_1, e_2, e_2, e_1)	$1 + \epsilon(0) = 1$
(2)	(e_1, e_3, e_3, e_1)	$1 + \epsilon(0) = 1$
(3)	(e_2, e_3, e_3, e_2)	$1 + \epsilon(3) = 4$
(4)	(e_1, e_2, e_3, e_1)	$0 = 0$
(5)	(e_2, e_3, e_1, e_2)	$0 = 0$
(6)	(e_3, e_1, e_2, e_3)	$0 = 0.$

All of these equalities will hold when $\epsilon = 1$. Therefore, $\{R_\varphi, R_\psi, R_\tau\}$ is indeed properly linearly dependent when we construct ψ and τ according to the conditions in Theorem 3.2.

Equation (3) makes it evident that, despite that ψ_{ij} is not diagonal on the $\{f_1, f_2, f_3\}$ basis, linear dependence still holds. Essentially, as long as there exists some basis where ψ and τ are simultaneously diagonalized, then linear dependence can hold, as long as the other conditions are also met. The calculation follows:

$$\begin{aligned}
R_\varphi(e_2, e_3, e_3, e_2) &= (1)(1) - 0 = 1 \\
R_\psi(e_2, e_3, e_3, e_2) &= (1 + 2\sin^2 2\theta)(1 + 2\cos^2 2\theta) - \sin^2 2\theta \\
&= 1 + 2(\sin^2 2\theta + \cos^2 2\theta) + 4\sin^2\theta\cos^2\theta - 4\sin^2\theta cos^2\theta \\
&= 3 \\
R_\tau(e_2, e_3, e_3, e_2) &= (2)(2) - 0 = 4.
\end{aligned}$$

As long as there exists a basis composed of the eigenvectors of ψ and τ such that they are simultaneously diagonalized, and the eigenvalue relationships hold, then $R_\varphi + \epsilon R_\psi = \delta R_\tau$ holds.

We have shown that linear dependence occurs if and only if simultaneous diagonlizability of ψ and τ is possible, and if certain case-specific eignenvalue relationships hold.

3.3 Rank ψ = Rank $\tau = 2$

Theorem 3.3. Let V be a vector space with dim $V = 3$, Rank ψ = Rank $\tau = 2$. Then there is no solution to the equation $R_\varphi + \epsilon R_\psi = \delta R_\tau$.

Proof. We will start by assuming that $R_\varphi + \epsilon R_\psi = \delta R_\tau$.

Let $\{e_1, e_2, e_3\}$ be an orthonormal basis for V so that we can diagonalize ψ with respect to the positive definite φ. Then we have

$$[\varphi_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, [\psi_{ij}] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ and } [\tau_{ij}] = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}.$$

By plugging basis vectors into the equation, we get the following equalities:

Equation	(x, y, z, w)	$R_\varphi + \epsilon R_\psi = \delta R_\tau$
(1)	(e_1, e_2, e_2, e_1)	$1 + \epsilon\lambda_1\lambda_2 = \delta(ad - b^2)$
(2)	(e_1, e_3, e_3, e_1)	$1 = \delta(af - c^2)$
(3)	(e_2, e_3, e_3, e_2)	$1 = \delta(df - e^2)$
(4)	(e_1, e_2, e_3, e_1)	$0 = \delta(ae - cb)$
(5)	(e_2, e_3, e_1, e_2)	$0 = \delta(dc - eb)$
(6)	(e_3, e_1, e_2, e_3)	$0 = \delta(fb - ec)$.

We can also calculate the determinant of $[\tau_{ij}]$:

$$\begin{aligned} det([\tau_{ij}]) &= a(df - e^2) + b(ce - bf) + c(be - cd) \\ &= a(\delta) + b(0) + c(0) \\ &= \pm a. \end{aligned}$$

Since Rank $\tau < n$, the determinant must be equal to 0. When we plug $a = 0$ into the equations, we have that $c^2 = 1$ and $\delta = 1$ from (2), and so $b = 0$ from (4). Now $d = 0$ from (5), so $e^2 = 1$ from (3). But then (6) cannot be true because we will have $0 = \delta(\pm 1)$.

We need $a \neq 0$ for the equations to hold, but then our determinant would be nonzero, and $[\tau_{ij}]$ would be invertible, but that contradicts our assumption that Rank $\tau = 2$. Therefore, linear dependence cannot occur if both Rank $\tau = 2$ and Rank $\psi = 2$ in a vector space of dimension 3. □

4 Vector Spaces of Dimension $n \geq 4$

This section will determine exactly when three algebraic curvature tensors can be linearly dependent in any dimension ≥ 4.

4.1 Rank $\psi = 2$, Rank $\tau = n$

Theorem 4.1. [1]. Let V be a vector space with dim $V = 4$. The equation $R_\varphi = \epsilon R_\psi + \delta R_\tau$ has no solution when φ is positive definite, Rank $\psi = 2$, and Rank $\tau =$ dim V.

The implication of this theorem is that there may exist a minimum lower bound for the ranks of our operators in order for linear dependence to occur. In Section 4.3 we will work towards determining what that lower bound may be.

4.2 Rank $\psi \geq 3$, Rank $\tau = n$

Theorem 4.2. [2]. Suppose dim $V \geq 4$, φ is positive definite, Rank $\tau = n$, and Rank $\psi \geq 3$. The set $\{R_\varphi, R_\psi, R_\tau\}$ is linearly dependent if and only if one of the following is true:

I. $|Spec(\tau)| = |Spec(\psi)| = 1$.

II. $Spec(\psi) = \{\lambda_1, \lambda_2, \lambda_2, \dots\}$ and $Spec(\tau) = \{\eta_1, \eta_2, \eta_2, \dots\}$, with

$$\eta_1 \neq \eta_2, (\lambda_2)^2 = \epsilon(\delta(\eta_2)^2 - 1), \text{ and } \lambda_1 = \frac{\epsilon}{\lambda_2}(\delta\eta_1\eta_2 - 1).$$

The above theorems address what happens when τ has full rank. Our choice of τ and ψ does not matter; we could just as easily allow ψ to be full rank and alter the rank of τ.

In addition, note that condition II of Theorem 4.2 only gives a linear dependence relationship, not *proper linear dependence*, since we would have that $\eta_i = a\lambda_i$ for some $a \in \mathbb{R}$, so $[\tau_{ij}]$ and $[\psi_{ij}]$ are just multiples of the identity matrix I. However, the question remains of whether or not the set $\{R_\varphi, R_\psi, R_\tau\}$ exhibits proper linear dependence when both ψ and τ have less than full rank.

Also, Theorem 4.1 suggests that there might be a lower bound on the ranks of our algebraic curvature tensors in order for linear dependence to occur. We will seek to find exactly what this lower bound must be.

The theorem that follows answers both these questions and generalizes for all cases where at least one of the tensors has less than full rank in vector spaces of dimension 4 or more. However, if both ψ and τ have full rank, then Theorem 4.2 would still apply.

4.3 $3 \leq$ Rank $\tau < n$

As in the dimension 3 case, we will now try to lower the ranks of our operators to see if linear dependence can still hold. We will start by setting Rank τ strictly less than n but at least 3 and determine what requirements we must put on Rank ψ and the eigenvalue relationships in order for linear dependence to occur.

Suppose $\{R_\varphi, R_\psi, R_\tau\}$ is linearly dependent and there exist $c_1, c_2, c_3 \in \mathbb{R}$, not all equal to 0, such that $c_1 R_\varphi + c_2 R_\psi + c_3 R_\tau = 0$.

If two of $c_1, c_2, c_3 = 0$, then one of our curvature tensors must be the zero tensor. We will exclude this case once again.

If $c_1 = 0$, then the equation can be rearranged to be of the form $R_\psi = \pm R_{c\tau}$. Rank $\psi =$ Rank τ, and $\psi = \pm c\tau$. But then $R_\varphi = \pm(1 \pm c)R_\tau$, and the ranks are unequal. No dependence can occur.

If $c_2 = 0$, then we rearrange the equation into $R_\varphi = -\frac{c_3}{c_1} R_\tau$, but φ must have full rank since it is positive definite, and τ must have less than full rank by our hypothesis, so this is a contradiction.

If $c_3 = 0$, then by [4] we have

$$R_\varphi = -\tfrac{c_2}{c_1} R_\psi = \pm R_{c\tau} \text{ for } c = \sqrt{|\tfrac{c_3}{c_1}|}.$$

Then since Rank $\psi \geq 3$, we can conclude that $\pm c\psi = \varphi$, and diagonalizing τ with respect to φ on the orthonormal basis $\{e_1, e_2, e_3\}$ will automatically diagonalize ψ as well, and we obtain

$$[\varphi_{ij}] = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & 1 \end{bmatrix}, [\psi_{ij}] = \pm \begin{bmatrix} c & 0 & \dots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & c \end{bmatrix}, \text{ and } [\tau_{ij}] = \begin{bmatrix} \eta_1 & 0 & \dots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & \eta_n \end{bmatrix}.$$

In this case, the eigenvalue requirements are that $Spec(\psi) = \{\pm c, \dots, \pm c\}$ and $Spec(\tau) = \{\eta_1, \dots, \eta_n\}$.

Conversely, when ψ and τ ahve eigenvalues that satisfy the above requirements such that $[\varphi_{ij}], [\psi_{ij}]$, and $[\tau_{ij}]$ have the representations shown above on some orthonormal basis, then linear dependence will hold.

Now if we assume that all $c_1, c_2, c_3 \neq 0$, then we can once again rearrange the equation for linear dependence into $R_\varphi + \epsilon R_\psi = \delta R_\tau$, and the following theorem applies.

Theorem 4.3. Let V be a vector space with dim $V = n \geq 4$, φ is positive definite, and $3 \leq$ Rank $\tau < n$. Then there exists a solution to $R_\varphi + \epsilon R_\psi = \delta R_\tau$ if and only if the following conditions are satisfied:

I. Rank $\psi = n$.

II. ψ and τ are simultaneously diagonalizable with respect to φ.

III. $Spec(\psi) = \{\pm\frac{1}{\lambda}, \lambda, \lambda, \dots\}$ and $Spec(\tau) = \{0, \eta, \eta, \dots\}$,
where $\lambda^2 = 1 - \delta\eta^2$.

Proof. Assume that $R_\varphi + \epsilon R_\psi = \delta R_\tau$, ψ has eigenvalues $\{\lambda_1, \dots, \lambda_n\}$, and τ has eigenvalues $\{\eta_1, \dots, \eta_n\}$.

From Theorem 2.3 and our assumption that $3 \leq \text{Rank } \tau < n$, we know that the rank of τ must be $n-1$. This tells us that one eigenvalue of τ must be 0, which we will call η_1.

Since φ is positive definite, there exists an orthonormal basis $\{e_1, \dots, e_n\}$ which we can use to diagonalize ψ. Since Rank $R_\tau = n-1 = \text{Rank } (R_\varphi \pm R_\psi)$, we can use Corollary 2.1, to determine that $spec(\psi) = \{\pm\frac{1}{\lambda}, \lambda, \dots\}$. Since $\lambda \neq 0$, ψ must be invertible and have rank n.

Now since Rank $\psi = n$, and Rank $\tau \geq 3$, and we can rearrange our original equation such that $R_\varphi - \delta R_\tau = -\epsilon R_\psi$, we can use Theorem 2.2 to conclude that τ can also be simultaneously diagonalized with respect to φ, and $\eta_1 = 0$.

Then we have

$$[\psi_{ij}] = \begin{bmatrix} \pm\frac{1}{\lambda} & 0 & \dots & 0 \\ 0 & \lambda & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda \end{bmatrix} \text{ and } [\tau_{ij}] = \begin{bmatrix} \eta_1 & 0 & \dots & 0 \\ 0 & \eta_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \eta_n \end{bmatrix}$$

on the orthonormal basis $\{e_1, \dots, e_n\}$.

Since dim $V \geq 4$, there exist at least three distinct indices $j, k, l \neq 1$, and from the equation $R_\varphi + \epsilon R_\psi = \delta R_\tau$, we have

$$\begin{aligned} &(e_1, e_j, e_j, e_1) \text{ gives } 1 + \epsilon = 0, \\ &(e_j, e_k, e_k, e_j) \text{ gives } 1 + \epsilon\lambda^2 = \delta\eta_j\eta_k, \text{ and} \\ &(e_j, e_l, e_l, e_j) \text{ gives } 1 + \epsilon\lambda^2 = \delta\eta_j\eta_l, \end{aligned}$$

which can be subtracted to obtain

$$\delta\eta_j\eta_k = \delta\eta_j\eta_l.$$

Hence, $\eta_k = \eta_l$ for all $k \neq l$, and $\epsilon = -1$.

By solving for λ, we find that

$$\begin{aligned} 1 - \lambda^2 &= \delta\eta^2 \\ \Rightarrow \lambda^2 &= 1 - \delta\eta^2 \end{aligned}$$

and we can determine that the diagonalized forms of ψ and τ are in fact

$$[\psi_{ij}] = \begin{bmatrix} \pm\frac{1}{\lambda} & 0 & \dots & 0 \\ 0 & \lambda & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda \end{bmatrix} \text{ and } [\tau_{ij}] = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & \eta & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \eta \end{bmatrix}.$$

Conversely, when the conditions I–III hold true, then the equation $R_\varphi + \epsilon R_\psi = R_\tau$ holds for all $x, y, z, w \in V$, so the set $\{R_\varphi, R_\psi, R_\tau\}$ is linearly dependent. $\square$

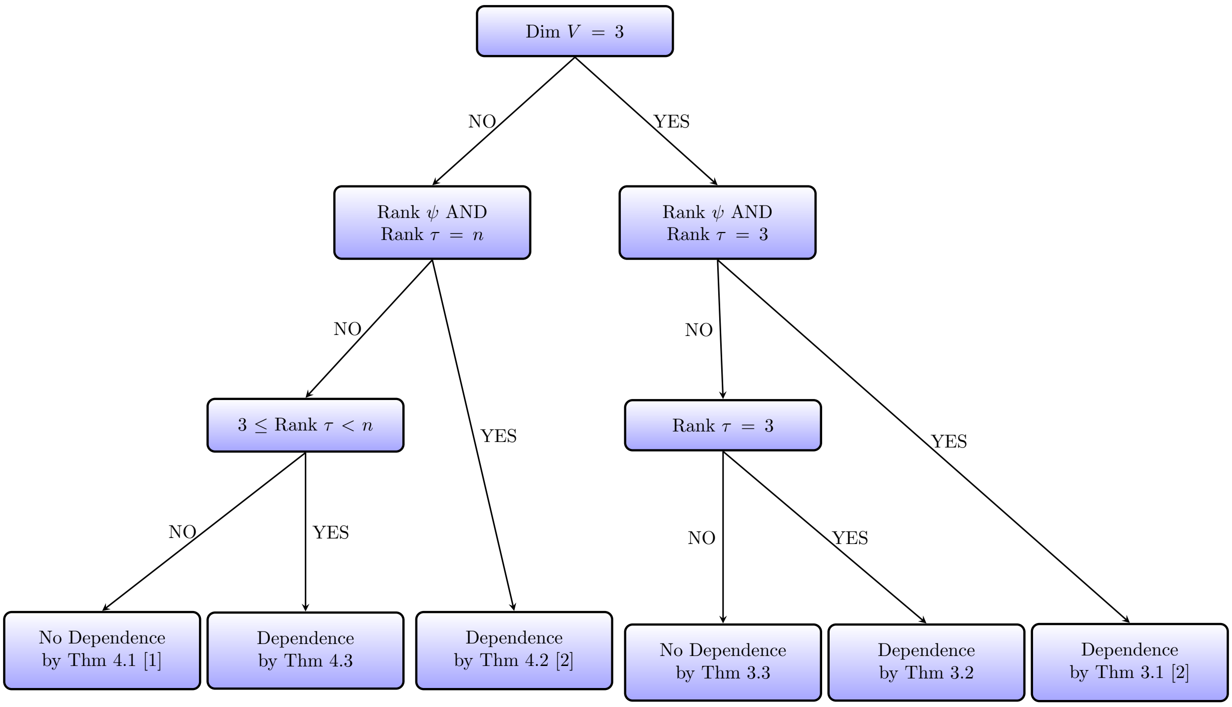
An interesting observation is that, given a set of three algebraic curvature tensors $\{R_\varphi, R_\psi, R_\tau\}$ in a vector space V of dimension $n \geq 3$, where one is determined by a positive definite operator, the tensors can only be linearly dependent if one operator has full rank and the other has rank no less than $n-1$.

Corollary 4.1. When dim $V = n \geq 3$ and φ is a positive definite symmetric bilinear form, $R_\varphi + \epsilon R_\psi = \delta R_\tau$ has a solution only when one of ψ or τ has rank n and the other has rank no less than $n-1$.

Theorem 4.3 combined with Section 3 gives us stronger conditions for linear dependence when one tensor has less than full rank by providing a stronger lower bound for the rank requirement.

5 Summary of Results

As a visual aid, this dichotomous key helps determine when and exactly what conditions exist under the linear dependence of $\{R_\varphi, R_\psi, R_\tau\}$.



In Section 2, we stated a result from [2] regarding simultaneous diagonalization of three algebraic curvature tensors. The theorem helped us with the study of linear dependence. Now we have shown that linear dependence of the set $\{R_\varphi, R_\psi, R_\tau\}$ always occurs if and only if ψ and τ are simultaneously diagonalizable with respect to positive definite φ, and specific eigenvalue relationships hold.

This completes all cases of linear dependence of three algebraic curvature tensors in vector spaces of dimension at least 3, where at least one tensor is defined by a positive definite inner product.

6 Future Questions

1. All of the results in this paper use the condition that φ is positive definite, which we have depended upon in conjunction with the results of [2] and [5] to simultaneously diagonalize one of either ψ or τ. How would the linear dependence relationship change if we altered this hypothesis and allowed φ to be positive semi-definite or nondegenerate? With the weaker condition of a semi-definite or nondegenerate operator, do the conditions for linear dependence also become weaker, or is linear dependence less common?
2. What are the conditions for four or more algebraic curvature tensors to be linearly dependent? A reasonable hypothesis after the results of this paper is that one must be positive definite so that the other three can be simultaneously diagonalized.
3. What happens when we choose a combination of symmetric, anti-symmetric, and skew-symmetric bilinear forms?

7 Acknowledgments

I would like to thank Dr. Corey Dunn for his guidance and insight as well as Dr. Rolland Trapp for his support and encouragement during this program. This research program was supported by Cal State University, San Bernardino and the National Science Foundation grant DMS-1461286.

References

[1] A. Acevedo. *Linearly Dependent Sets of Algebraic Curvature Tensors.* CSUSB REU, 2014.

[2] A. Diaz. *The Linear Independence of Sets of Two and Three Canonical Algebraic Curvature Tensors.* CSUSB REU, 2009.

[3] C. Dunn, C. Franks, J. Palmer. *On the Structure Group of a Decomposable Model Space.* http://arxiv.org/pdf/1108.2224.pdf, 2013. Accessed 12 August 2015.

[4] P. Gilkey. *The Geometry of Curvature Homogeneous Pseudo-Riemannian Manifolds.* Imperial College Press, 2007. ISBN: 978-1-86094-785-8.

[5] D. Strieby. *The decomposability of full and weak model spaces.* CSUSB REU, 2009.