Cusp Density in Nested Octahedral Links

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Abstract

This paper studies link invariants of generalized fully augmented links. We look specifically at nested octahedral links. We outline the process of augmenting and nesting a link. We geometrically realize the link complement as hyperbolic polyhedra using a cell decomposition given by Purcell. Finally, we prove cusp density is equal to $3/V_8$ for nested octahedral links.

1 Introduction

A knot is a closed, non self-intersecting curve embedded in 3-space that cannot be untangled to produce a simple loop. In this paper we will focus mainly on links. A link is a collection of knots with mutual entanglements. There exist three different classes of knots: satellite, torus and hyperbolic knots. We will be interested in hyperbolic knots and link; those whose complement can be geometrically realized as regular, ideal, hyperbolic polyhedra with faces identified. In order to distinguish between these links we use invariants. In this paper we will explore link invariants on generalized fully augmented links.

2 Hyperbolic Geometry

Hyperbolic geometry is a non-Euclidean geometry that can be described by it's constant negative curvature. We will use the Upper Half Space model and Poincare Ball model. The upper half space model, $\mathbb{H}^3$, is defined as the set $\{(x, y, z)|z > 0\}$ with ideal points $z = 0$ considered to be at infinity, together with a hyperbolic metric. In this space planes are vertical Euclidean planes and hemispheres lying on ideal points. Lines are vertical rays and half circles with ideal endpoints perpendicular to the z-axis, both which exist in hyperbolic planes.

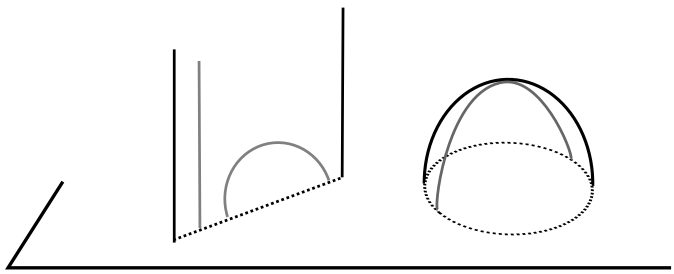


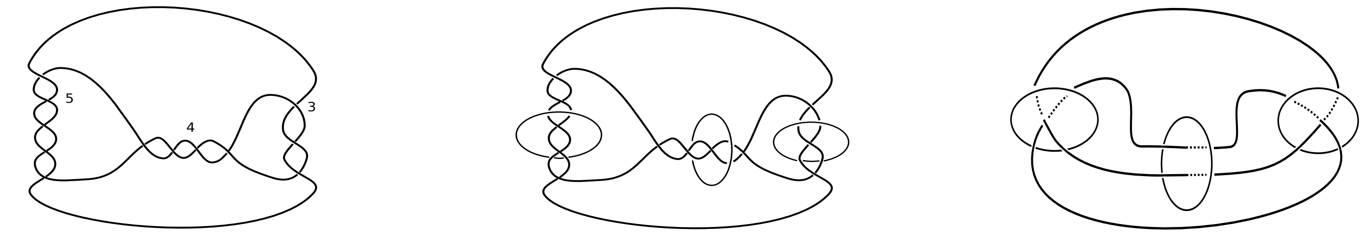
Figure 1: Upper Half Space Model

3 Augmenting links

In this section we define the classes of links to be investigated.

3.1 Fully Augmented Links

To study hyperbolic links we will use fully augmented links, introduced by Adams [2], as they have much simpler hyperbolic geometry. The process of augmenting a link, as described by Purcell, [5] is to introduce a simple closed loop around a twist region in a link and reduce the crossings modulo 2, that is, to remove all full twists. We define a *twist region* to be a maximal string of bigons arranged end to end. A *fully augmented link*, F.A.L., is one in which all twist regions are augmented. See Figure 2. Note that the knotting strands are embedded in the plane of the page and the closed loops are perpendicular to that plane. We call these closed loops *crossing circles*, and the region bounded by these circles *crossing disks.*



(a) Link with three twist regions. (b) Introduce trivial components. (c) Reduce crossings *mod*2.

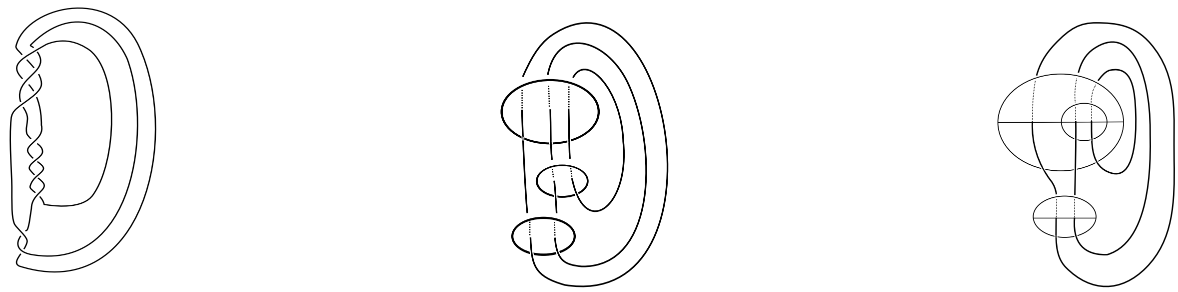
Figure 2: Augmenting a link

3.2 Generalized Fully Augmented Links

Purcell extends this idea of a fully augmented link to *generalized fully augmented links* [4] in which she allows for more than two strands in a given twist region. With the added condition that additional strands can only added within the two 2-strand twist positions, no twist, or a half twist. Augmenting this types of link will result in some crossing disks with more than two punctures by the knotting strands. See Figure 3b.

3.3 Nested Links

In order to use the cell decomposition given by Purcell, we want each crossing disk to have exactly two punctures. To rectify disks with more than two punctures, in some cases, we can nest disks inside of others. *Nested links* are created by sliding 2-puncture crossing disks until they lie in the same plane as disk with more punctures. See Figure 3.



(a) Link with >2 knotting strands in a twist region. (b) Generalized Fully Augmented Link. (c) Nested Link.

Figure 3: Augmenting and nesting a generalized fully augmented link

4 Polyhedral Decomposition

In this section we discuss the process of putting a hyperbolic structure on a *link complement*, $S^3 \backslash L$. We are interested in hyperbolically realizing the link complement as right angled, ideal polyhedra. We will use a decomposition described by Purcell [5] that gives two identical ideal polyhedra. Purcell also proves in [5] that "faces of the poyhedra can be checkerboard colored, with shaded faces all triangles corresponding to 2-punctured disks, and white faces corresponding to components of the projection plane". In this decomposition 3-cells are S^3 cut along the plane of projection into S^3_+ and S^3_-. Two-cells are the plane of projection and crossing disks. Finally, 1-cells are intersections of 2-cells, namely where crossing disks intersect the plane of projection.

We obtain the polyhedra by first cutting along the 2-cells, which cuts S^3 in half and gives two copies of each crossing disk within S^3. Recall, that topologically expanding and shrinking a component are homeomorphic, thus we collapse crossing circles and knotting strands to points. In the case of fully augmented links and nested links we will obtain a packing of triangles. We can consider three vertices in a triangle to correspond to a hyperbolic plane, or hemisphere, and raise the triangular face onto the plane. Raising all faces to the hyperbolic hemispheres they lie on will give a hyperbolic ideal polyhedron. Recall that slicing along the projection plane gives two copies of the process defined above. Thus one of polyhedra will occur in S^3_+ which will we call P_+, similarly the other in S^3_-, P_-. See figure 4.

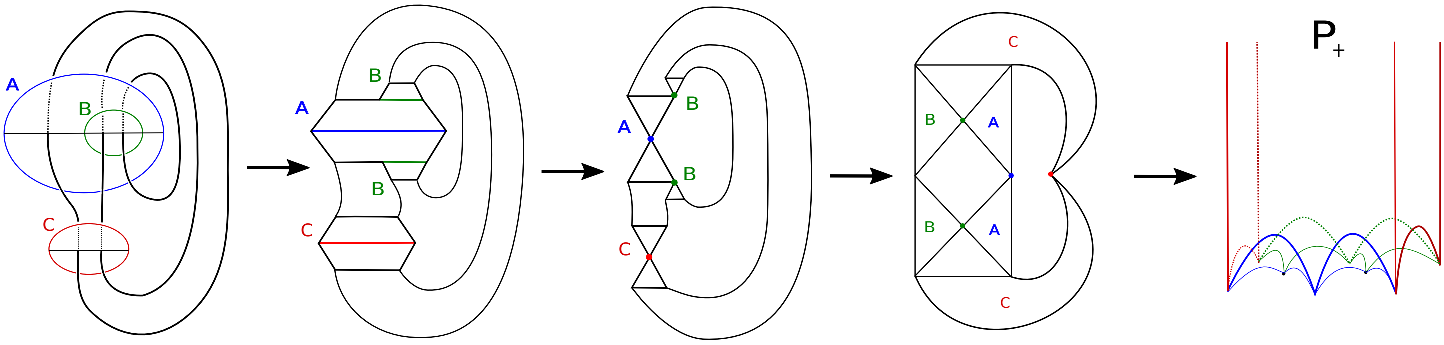


Figure 4: Cell decomposition into regular, ideal polyhedron

4.1 Octahedral Links

An octahedral link is one whose complement can be geometrically realized as some number of regular, ideal hyperbolic octahedron. Different links may have the same octahedral decomposition with different gluing instructions. For example, the nested link in Figure 4 lifts to an 11-sided polyhedron which can be constructed by gluing 2 regular, ideal octahedron along a face. Since these are in P_+ there are 2 identical octahedron in P_-, so we call this nested link a 4 octahedral link.

5 Link Invariants

A single link may take more than one form, or be embeded in a number of ways. A characteristic that is independent of the way the link is presented is an *invariant*. Due to Mostow-Prasad Rigidity, any geometric invariant of a hyperbolic link is a link invariant. In this section we discuss several different invariants of hyperbolic links, in particular cusp density.

5.1 Volume

Volume is a natural invariant for hyperbolic knots and links. This is the volume of the link complement realized as hyperbolic polyhedra, $Vol(L)$. If the link is octahedral, made up of n octahedra, its volume is nV_8; where V_8 is the volume of a regular ideal octahedron.

5.2 Cusp Volume

A *cusp* is a tubular neighborhood around a component missing only the part in which it intersects it. The volume of just one component is the *cusp volume* and is calculated at a *maximal cusp*. A maximal cusp occurs when the cusp of a link is expanded until it first touches itself. To calculate the maximal cusp, we lift the cusp to the hyperbolic polyhedron and tessellate $\mathbb{H}^3$ with copies of the polyhedron to obtain the universal cover. As a result the horoball corresponding to the cusp is mapped to infinitely many other horoballs. Next, expand all horoballs corresponding to the cusp equivariantly and stop the first time any two horoballs touch at a point of tangency. This configuration is the maximal cusp. At this point we compute the volume of the cusp.

To compute the volume of a given horoball we use a Möbius Transformation to map the *center* of the horoball to the point at infinity. In this context the center of the horoball is the ideal point which touches the $z = 0$ plane in hyperbolic space, thus it occurs at infinity. Mapping in this way will transform the horoball to some horizontal plane at infinity. By keeping track of the image of the other points in the octahedron we can obtain a length and width for a fundamental region of this plane and compute its area. It is well known that the hyperbolic volume above a plane is half its area.

The *maximal cusp volume* is the largest total volume of all cusps in which the cusps are tangent to each other. The maximal cusp volume, $mcv(L)$ may be obtained by maximizing some component first, and then maximally expanding the remaining cusps. [Adams, private communication].

5.3 Cusp Density

Cusp density is the ratio of *maximal cusp volume* to *volume* of the link.

$$cd(L) = \frac{mcv(L)}{Vol(L)}.$$

Theorem 1. *Let L be an octahedral, nested link made up n octahedra, then* $cd(S^3\backslash L) = \frac{3}{V_8}$.

Proof. Since L is octahedral, it can be obtained by gluing n regular, ideal octahedra together. We construct a horoball packing in a single octahedron in P_+. In this octahedron, O, expand horoballs centered at each vertex. Expand them until they are tangent at midpoints of edges. Direct calculation shows the total volume of cusps in O is 3 (see case I for proof). Consider $n/2$ of these octahedron in both P_+ and P_-. We glue these octahedra together by gluing octahedron in P_+ to each other along shaded faces and octahedron in P_+ to octahedron in P_- along unshaded faces. We obtain a collection of maximal cusps in $S^3\backslash L$ whose volume is $3n$. Thus $cd(s^3\backslash L) \geq \frac{3n}{nV_8}$.

Proposition 1. *A single octahedron in an octahedral nested link construction contributes at most mod 3 to the cusp volume of a maximal cusp.*

Recall that we can maximize cusps one at a time to obtain the maximal cusp. So we prove this proposition in cases based on which component we expand first.

Proof. Case I: Adjacent Horoballs

In this case the component we maximize first lifts to two adjacent horoballs in a single octahedron. Maximally expand these horoballs, and then maximize the horoballs centered at the remaining vertices in the octahedron. In the maximal cusp the four corner horoballs and the plane are tangent at exactly 1 point pairwise. Let the octahedron have length and width 2. We notice adjacent horoballs touch when their diameter is 2 (see Figure 5), thus the plane has height 2. We consider the portion of the plane within the octahedron. This square has hyperbolic area is $\frac{2 \cdot 2}{2^2} = 1$. Recall volume is 1/2 area, thus the volume$=\frac{1}{2}$.

If we consider this octahedron in the Poincare ball model, we see that the octahedron is completely symmetric. Thus all horoballs have volume 1/2, and so the maximal cusp volume is 3.

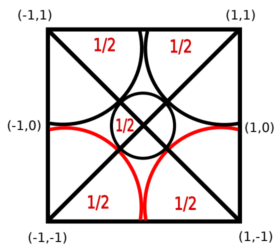


Figure 5: Shadow obtained by maximizing adjacent horoballs and then remaining horoballs. **Note: Vol(plane at ∞)=1/2.

Case II: Opposite Horoballs

We now consider the case in which the component we maximize first lifts to two opposite horoballs in a single octahedron. Again, maximally expand these horoballs first, and maximize the remaining horoballs centered at vertices of the octahedron.

We let the octahedron have length and width 2 and notice opposite horoballs are tangent with diameter $\sqrt{2}$, see Figure 6. If we rotate the octahedron so that one of the opposite horoballs is a horizontal plane and the other is a horoball centered at the opposite vertex, the plane has height $\sqrt{2}$. The portion of the plane within the the octahedron is a square, and thus has hyperbolic area $\frac{2 \cdot 2}{(\sqrt{2})^2} = 2$, and volume 1. By symmetry we see that we may rotate the octahedron as to switch the plane and the opposite horoball. Thus the horoball centered at the opposite vertex also has volume 1.

We now consider maximizing the remaining horoballs centered around the waist of the octahedron. We expand these horoballs until they touch the previous horoballs, which occurs at diameter $\sqrt{2}$. In order to calculate the volume of one of these horoballs we use a Möbius transformation to send one horoball to the horizonal plane at infinity. To do this we use inversion. Since inversion is an isometry, the new configuration is isometric to the original. We call the center of a single horoball $(0,0,0)$ and invert through the hemisphere defined by $(-1,0,0)$,$(\frac{-\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0)$ and $(0,1,0)$. Under this transformation we consider the image of the points defining the length and width of the octahedron and two points on the horoball. Inverting over this hemisphere corresponds to the transformation $\vec{v} \to \frac{1}{2||\vec{v}||^2}\vec{v}$.

Boundary points:

∞	$\to$	$(0,0,0)$
$(-2,0,0,)$	$\to$	$(\frac{-1}{4},0,0)$
$(-2,-2,0)$	$\to$	$(\frac{-1}{8},0,0)$
$(0,2,0)$	$\to$	$(0,\frac{1}{4},0)$

Horoball points

$(0,0,0)$	$\to$	∞
$(0,0,\sqrt{2})$	$\to$	$(0,0,\frac{\sqrt{2}}{4})$

Thus the fundamental region of the plane within the octahedron has length and width $\frac{1}{4}$ at height $\frac{\sqrt{2}}{4}$.

$$\text{Thus hyperbolic area} = \frac{\frac{1}{4}\frac{1}{4}}{(\frac{\sqrt{2}}{4})^2} = \frac{1}{2} \to \text{volume} = \frac{1}{4}.$$

Again by symmetry we notice that all horoballs on the waist of the octahedron have volume 1/4 thus the maximal cusp volume for this collection of horoballs is 3.

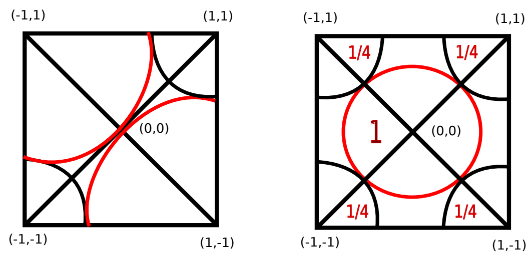


Figure 6: Shadow obtained by maximizing adjacent horoballs and then remaining horoballs. **Note: Vol(plane at ∞)=1.

Case III: Single Horoball

In this case we consider a component that corresponds to one horoball in a single octahedron. Map this point to infinity, and so it lifts to a horizontal plane in the octahedron. This point lies in P_+, so we consider its reflection in P_-. We obtain P_- by reflecting P_+ through the hyperbolic plane defined by the verticies of the face that is adjacent to a vertical shaded face. The point at infinity is mapped to the point directly below the shaded face.

We now expand the horoball in P_- and the plane in P_+ equivariantly until they touch. This occurs when the diameter of the horoball and the height of the plane are 1. Thus the hyperbolic area of the plane is $\frac{2 \cdot 2}{1^2} = 4 \rightarrow$ volume=2.

We then maximally expand the horoballs centered at the remaining 5 vertices. By using inversion on the horoball opposite to the plane, we map this horoball to the plane at infinity. We compute this volume to be 1/2. We use the same technique for a horoball centered at a vertex adjacent to the plane and find its volume to be 1/8. By symmetry the remaining horoballs adjacent to the plane also have volume 1/8. Thus the maximal cusp volume is 3.

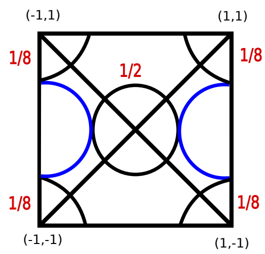


Figure 7: Shadow obtained by maximizing horoballs shown in P_+ and P_- and then remaining horoballs.**Note: Vol(plane at ∞)=2.

□

We return to the proof of the theorem and examine the horoball expansion in an arbitrary n octahedral link.

Suppose $C_1, ..., C_j$ are cusps in the order we expand them. By Proposition 1, expanding C_1 gives one of the cases above. Now consider expanding C_i. We will either obtain one of the cases above, or the cusp will touch another horoball before it can be maximally expanded, in which case the maximal volume of that octahedron would be less than 3.

In either case the maximal cusp volume in a single octahedron is at most 3. Therefore the maximal cusp volume in an n octahedral link is at most $3n$. In the first step of the proof we have presented such a link with maximal cusp volume equal to $3n$. Thus $cd(S^3 \backslash L) = \frac{3n}{nV_8} = \frac{3}{V_8}$. □

References

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