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Constant Vector Curvature in 3-Dimensional Lorentzian Space with Diagonalized Ricci Tensor

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Abstract

This research revealed that Lorentzian, 3-Dimensional model spaces, whose curvature tensor's associated Ricci operator is diagnolizable, will have constant vector curvature (we say M has $cvc(\epsilon)$ for some $\epsilon \in \mathbb{R}$) under some cicumstances. In these circumstances, we know the value of ϵ, and ϵ is unique. In the circumstances where the model space does not have $cvc(\epsilon)$, we know which vectors in the model space prevent it from having $cvc(\epsilon)$. These vectors form a subspace tangent to the light cone.

1 Introduction & Background

Every n-dimensional vector space has a means of measuring the distance of and the curvature along paths. Differential Geometry is a branch of mathematics that generalizes what we would think of as the *metric* to something called the **inner product** and what we would think of as a curvature function to an **algebraic curvature tensor** and then studies the properties of both of these functions, amongst other things. Every n-dimensional vector space can be talked about with respect to a basis, an inner product and an algebraic curvature tensor. Intuitively, this is so that we know what sort of paths we can make in a given space. We call such a space a **model space**.

Constant vector curvature is a relatively newly discovered property on model spaces. In three dimensions, studying this property is made easier due to the fact that every curvature tensor, a function that takes in four vectors and ouputs a real number, can be uniquely represented by a more simple function, the Ricci Tensor. The Ricci tensor only requires two inputs. There is no such unique representation in higher dimensions.

Furthermore, the associated Ricci Operator can take one of four Jordan-Normal forms. This research considers only curvature tensors whose Ricci operator is diagonalized (one of the Jordan Normal forms).

Definition 1.1. A **model space** $M = (V, < \cdot, \cdot >, R)$ is comprised of a vector space $V = span\{e_1, ..., e_n\}$; an inner product $< \cdot, \cdot >$ which is a symmetric, bi-linear form; and an algebraic curvature tensor R. Working with a diagonalized Ricci operator reduces the number of non-zero curvature entries.

Definition 1.2. Let V be a real, finite-dimensional vector space. Let $R : V \times V \times V \times V \to \mathbb{R}$ be a multilinear function. R is an **algebraic curvature tensor** (or ACT) if it satisfies

1. $R(x, y, z, w) = -R(y, x, z, w)$
2. $R(x, y, z, w) = R(z, w, x, y)$
3. $R(x, y, z, w) + R(z, x, y, w) + R(y, z, x, w) = 0$

for all $x, y, z, w \in V$.

Definition 1.3. Let $M = (V, < \cdot, \cdot >, R)$ be a model space and $< \cdot, \cdot >$ non-degenerate. A 3-dimensional model space is **Lorentzian** if we can find an orthonormal basis $\{e_1, e_2, e_3\}$ for V s.t.

$$< \cdot, \cdot > = \begin{array}{c c} & \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} \\ \begin{array}{c} e_1 \\ e_2 \\ e_3 \end{array} & \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

We would then call e_1 a **time-like** vector and e_2 and e_3 **space-like**. Furthermore, all basis vectors e_i s.t.$< e_i, e_i >= 0$ are reffered to as **light-like**, though none of the basis vectors are light-like in Lorentzian space.

Definition 1.4. Let V be a real, finite-dimensional vector space and $v, w \in V$. Suppose $\pi = span\{v, w\}$ is non-degenerate. Then the **sectional curvature** $\kappa(\pi)$ is defined

$$\kappa(\pi) = \frac{R(v, w, w, v)}{< v, v >< w, w > - < v, w >^2}$$

Definition 1.5. A model space $M = (V, <, \cdot >, R)$ has **constant sectional curvature** ϵ, denoted $csc(\epsilon)$, if $\kappa(span\ v, w\}) = \epsilon$ for all $v, w \in M$ and $span\{v, w\}$ is non-degenerate.

Definition 1.6. A model space $M = (V, <, \cdot >, R)$ has **constant vector curvature** ϵ, denoted $cvc(\epsilon)$, if for every $v \in V$ exists a $w \in V$ s.t. $\kappa(span\{v, w\}) = \epsilon$ and $span\{v, w\}$ is non-degenerate.

Definition 1.7. Let $M = (V, < \cdot, \cdot >, R)$ be an n-dimensional model space with $\{e_1, ..., e_n\}$ an orthonormal basis for V. Let ρ be a symmetric bi-linear form with respect to $< \cdot, \cdot >$ (i.e. $\rho(x, y) =< \Phi x, y >$ where Φ is referred to as the Ricci Operator). The the **Ricci Tensor** ρ is defined

$$\rho(x, y) = \sum_{i=1}^{n} < e_i, e_i > R(x, e_i, e_i, y)$$

An $n \times n$ matrix in **Jordan-Normal Form** is a matrix representation of a linear transformation on some basis. Generally, the matrix has eigenvalues on the diagonal, $1s$ or $0s$ on the diagonal above the main diagonal and $0s$ everywhere else. The number of $1s$ depends on the number of unique eigenvalues. For an explicit definition see [5].

For example, any 3×3 matrix will take one of the following J-N Forms:

3 real eigenvalues

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

2 real eigenvalues

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_2 \end{pmatrix}$$

1 real eigenvalue

$$\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$$

1 complex eigenvalue

$$\begin{pmatrix} a & b & 0 \\ -b & a & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}$$

$\lambda_1 = a + bi$, $\lambda_2 \in \mathbb{R}$

Theorem 1.1. *For a 3-dimensional model space, a diagonalized Ricci Operator implies that R_{1221}, R_{1331}, and R_{2332} are the only possible non-zero curvature entries [needs citation].*

Remark. For a Lorentzian model space $M = (V, < \cdot, \cdot >, R)$, consider $v, w \in V$ s.t. $v = xe_1 + ye_2 + ze_3$ and $w = a_1 + be_2 + ce_3$. Then

$$\kappa(span\{v, w\}) = \frac{\alpha(xb - ay)^2 + \beta(xc - az)^2 + \gamma(yc - zb)^2}{-(xb - ay)^2 - (xc - az)^2 + (yc - bz)^2}$$

Lemma 1.2. *If $\{e_1, e_2, e_3\}$ is a basis for V with $R(e_1, e_2, e_2, e_1) = \alpha$ and $R(e_1, e_3, e_3, e_1) = \beta$ where e_1 is time-like and e_2, e_3 are space-like, then $\exists$ another basis for V, $\{f_1, f_2, f_3\}$, defined as $e_1 = f_1$, $e_2 = f_3$, $e_3 = f_2$, then $R(e_1, e_2, e_2, e_1) = \beta$ and $R(e_1, e_3, e_3, e_1) = \alpha$.*

Proof. This is pretty straight forward to see. Note, f_1 is still space-like and f_2, f_3 time-like. □

Thompson showed in [1] that all 3-dimensional model spaces in the Reimannian setting have $cvc(\epsilon)$ for some $\epsilon \in \mathbb{R}$. In [2], Peng showed that when the Ricci operator is diagonalized,

$$\alpha > \beta \geq \gamma \geq 0 \implies M \text{ has } cvc(-\alpha)$$
$$\beta > \alpha \geq \gamma \geq 0 \implies M \text{ has } cvc(-\beta)$$

This research started by considering the proofs in [2] and found a generalization for those results. The result is we now know when M has $cvc(\epsilon)$ for some $\epsilon \in \mathbb{R}$ and when it doesn't. Furthermore, this is the first instance of a 3-dimensional model space that does not have $cvc(\epsilon)$ for some $\epsilon \in \mathbb{R}$.

2 RESULTS

2.1 WHAT CAN ϵ BE?

Theorem 2.1. *Let $M = (V, <\cdot,\cdot>, R)$ be a 3-dimensional Lorentzian model space and $\{e_1, e_2, e_3\}$ an orthonormal basis for V with respect to $<\cdot,\cdot>$. Let $R_{1221} = \alpha, R_{1331} = \beta, R_{2332} = \gamma$ and, without less of generality, say $\alpha > \beta$. Then If M has $cvc(\epsilon)$ for some $\epsilon \in \mathbb{R}$ and*

1. *If $-\alpha < -\beta \leq \gamma$, then $\epsilon = -\alpha$.*
2. *If $\gamma \leq -\alpha, -\beta$, then $\epsilon = -\beta$.*
3. *If $-\alpha < \gamma < -\beta$, then $\epsilon \in \{-\alpha, \gamma, -\beta\}$.*
4. *If $\alpha = \beta$, then $\epsilon = -\alpha = -\beta$.*

Proof. Suppose $-\alpha < -\beta \leq \gamma$.

From Peng's paper (theom 2.2):
Either $\epsilon \geq -\alpha$ and $\epsilon \leq -\beta$ or $\epsilon \leq -\alpha$ and $\epsilon \geq -\beta$, but $\alpha > \beta$ so $\epsilon \in \{-\alpha, -\beta\}$.
Also, $\epsilon \geq max\{-\alpha, \gamma\}$ or $\epsilon \leq min\{-\alpha, \gamma\}$. But $\gamma > -\alpha$, so $\epsilon \in (-\infty, -\alpha] \cup [\gamma, \infty)$.
Finally, $\epsilon \geq max\{-\beta, \gamma\}$ or $\epsilon \leq min\{-\beta, \gamma\}$. But $\gamma > -\beta$, so $\epsilon \in (-\infty, -\beta] \cup [\gamma, \infty)$. So $\{-\alpha, -\beta\} \cap (-\infty, -\alpha] \cup [\gamma, \infty) \cap (-\infty, -\beta] \cup [\gamma, \infty) = -\alpha$. So $\epsilon = -\alpha$.

Suppose $\gamma \leq -\alpha, -\beta$.
The intervals are the same as above, but given this relationship between $\gamma, -\alpha, -\beta$, we get: $\{-\alpha, -\beta\} \cap (-\infty, -\alpha] \cup [\gamma, \infty) \cap (-\infty, -\beta] \cup [\gamma, \infty) = -\beta$. So $\epsilon = -\beta$.

Suppose $-\alpha < \gamma < -\beta$.
Then $\{-\alpha, -\beta\} \cap (-\infty, -\alpha] \cup [\gamma, \infty) \cap (-\infty, -\beta] \cup [\gamma, \infty) = \{-\alpha, \gamma, -\beta\}$. So $\epsilon \in \{-\alpha, \gamma, -\beta\}$.

Suppose $\alpha = \beta$.
Then it is clear to see that since $\epsilon \in [-\alpha, -\beta]$ and $\alpha = \beta$, $\epsilon = -\alpha = -\beta$.

□

2.2 WHEN DOES M HAVE $cvc(\epsilon)$?

Theorem 2.2. *Let $M = (V, <\cdot,\cdot>, R)$ be a 3-dimensional Lorentzian model space and $\{e_1, e_2, e_3\}$ an orthonormal basis for V with respect to $<\cdot,\cdot>$. Let $R_{1221} = \alpha, R_{1331} = \beta, R_{2332} = \gamma$ and, without less of generality, say $\alpha > \beta$.*

1. *If $\gamma > -\alpha, -\beta$ then M has $cvc(-\alpha)$.*

2. *If* $\gamma < -\alpha, -\beta$ *then* M *has* $cvc(-\beta)$.

3. *If* $-\alpha \leq \gamma < -\beta$ *or* $-\alpha < \gamma \leq -\beta$ *then* M *does not have* $cvc(\epsilon)$ *for any* $\epsilon \in \mathbb{R}$.

4. *If* $-\alpha = -\beta \neq \gamma$ *then* M *has* $cvc(-\alpha = -\beta)$.

5. *If* $-\alpha = -\beta = \gamma$ *then* M *has* $csc(\gamma = -\alpha = -\beta)$.

Proof. (1 of 5)
See Peng's proof for Theorem 2.3 in [2]. It turns out that her proof for this theorem also proves (1) of *this* theorem. However, what Peng did not explicitly consider in [2] was whether $span\{v, w\}$ will ever be degenerate for some $v \in V$. But $\kappa(span\{v, w\})$ is undefined when the denominator is zero:

$$-x^2 - y^2\frac{\gamma+\alpha}{\alpha-\beta} + 2xy\sqrt{\frac{\gamma+\alpha}{\alpha-\beta} - \frac{\gamma+\beta}{\alpha-\beta}} = 0$$

This happens when either (a) $x, y = 0$ and $\gamma = -\beta$ OR (b) $x = 0$ and $\gamma = -\alpha = -\beta$. By the hypotheses of the theorem, neither (a) or (b) can happen. Thus the proof holds.

□

Proof. (2 of 5)
Case 1: Suppose $v = xe_1 + ze_3$ where only one of x, z or neither are zero, and let $w \in V$ be any vector s.t. $span\{v, w\} = span\{e_1, e_3\}$. Then

$$\begin{aligned}\kappa(span\{e_1, e_3\}) &= \frac{R_{1331}}{<e_1, e_1><e_3, e_3> - <e_1, e_3>^2} \\ &= \frac{\beta}{(-1)(1) - (0)} \\ &= -\beta\end{aligned}$$

Case 2: Suppose $\tilde{v} \in V$ s.t. $\tilde{v} = \tilde{x}e_1 + \tilde{y}e_2 + \tilde{z}e_3$ and $\tilde{y} \neq 0$. Scale $\tilde{v}$ by $\frac{1}{\tilde{y}}$and call it $v = xe_1 + e_2 + ze_3$. (Note: for any $w \in V$, $span\{\tilde{v}, w\} = span\{v, w\}$.)
Consider $w = \sqrt{\frac{\gamma+\beta}{\beta-\alpha}}e_1 + e_3$.
Then

$$\begin{aligned}\kappa(span\{v, w\}) &= \frac{R(v, w, w, v)}{<v, v><w.w> - <v, w>^2} \\ &= \frac{\beta(x^2 + z^2\frac{\gamma+\beta}{\beta-\alpha} - 2xz\sqrt{\frac{\beta+\gamma}{\beta-\alpha}} + \frac{\alpha+\gamma}{\beta-\alpha})}{(-1)((x^2 + z^2\frac{\gamma+\beta}{\beta-\alpha} - 2xz\sqrt{\frac{\beta+\gamma}{\beta-\alpha}} + \frac{\alpha+\gamma}{\beta-\alpha})} \\ &= -\beta.\end{aligned}$$

Furthermore, $span\{v, w\}$ will never be degenerate for any $v \in V$. Note that $\kappa(span\{v, w\})$ is undefined when the denominator is zero:

$$-x^2 + z^2\frac{\gamma+\beta}{\beta-\alpha} + 2xy\sqrt{\frac{\gamma+\beta}{\beta-\alpha} - \frac{\gamma+\alpha}{\beta-\alpha}} = 0$$

This happens when either (a) $x, z = 0$ and $\gamma = -\alpha$ or (b) $x = 0$ and $\gamma = -\alpha = -\beta$. By the hypotheses of the theorem, neither (a) or (b) are true.

$\square$

Proof. (3 of 5)

First, show: for $-\alpha < \gamma < -\beta$, M does not have $cvc(\gamma)$.

Consider the vectors $v = e_1 + ye_2 \pm (\sqrt{1 - \frac{\alpha+\gamma}{\beta+\gamma}} + y\sqrt{\frac{\alpha+\gamma}{\beta+\gamma}})e_3$.

By way of contradiction, suppose $\exists w = ae_1 + be_2 + ce_3$ s.t. $\kappa(span\{v, w\}) = \gamma$.

Then

$$\kappa(span\{v,w\}) = \frac{\alpha(b-ay)^2 + \beta(c-az)^2 + \gamma(yc-bz)^2}{-(b-ay)^2 - (c-az)^2 + (yc-bz)^2} = \gamma$$
$$\implies \frac{\alpha+\gamma}{-(\beta+\gamma)} = \frac{(c-az)^2}{(b-ay)^2} \implies \pm\sqrt{\frac{\alpha+\gamma}{-(\beta+\gamma)}} = \frac{c-az}{b-ay}.$$

Case 1: $a = 0$

Then $\frac{\alpha+\gamma}{-(\beta+\gamma)}b^2 = c^2 \implies c = \pm\sqrt{\frac{\alpha+\gamma}{-(\beta+\gamma)}}b$. Consider the denominator of $\kappa(span\{v, w\})$:

$$\begin{aligned}-(b-ay)^2 - (c-az)^2 + (yc-bz)^2 &= -b^2 - c^2 + (yc-bz)^2 \\ &= -b^2 + \frac{\alpha+\gamma}{\beta+\gamma}b^2 + [y(\pm\sqrt{\frac{\alpha+\gamma}{-(\beta+\gamma)}}) - zb]^2 \\ &= b^2(\frac{\alpha+\gamma}{\beta+\gamma} - 1 + [\pm y\sqrt{1 + \frac{\alpha+\gamma}{-(\beta+\gamma)}} - (\sqrt{\frac{\alpha+\gamma}{-(\beta+\gamma)}} \pm y\sqrt{\frac{\alpha+\gamma}{-(\beta+\gamma)}})]^2) \\ &= b^2(\frac{\alpha+\gamma}{\beta+\gamma} - 1 + 1 - \frac{\alpha+\gamma}{\beta+\gamma}) = 0.\end{aligned}$$

Thus $\kappa(span\{v, w\})$ is undefined for all $w \in V$ with $a = 0$.

Case 2: $a \neq 0$

Without loss of generality, say $a = 1$.

Then $\frac{\alpha+\gamma}{-(\beta+\gamma)}(b-y)^2 = (c-z)^2 \implies c = \pm\sqrt{\frac{\alpha+\gamma}{-(\beta+\gamma)}}(b-y) + z$.

Consider the denominator of $\kappa(span\{v, w\})$:

$$\begin{aligned}-(b-y)^2 - (c-z)^2 + (yc-bz)^2 &= -(b-y)^2 + \frac{\alpha+\gamma}{\beta+\gamma}(b-y)^2 + [y(\pm\sqrt{\frac{\alpha+\gamma}{-(\beta+\gamma)}}(b-y) + z) - zb]^2 \\ &= (b-y)^2(\frac{\alpha+\gamma}{\beta+\gamma} - 1 + [\pm y\sqrt{1 + \frac{\alpha+\gamma}{-(\beta+\gamma)}} - (\sqrt{\frac{\alpha+\gamma}{-(\beta+\gamma)}} \pm y\sqrt{\frac{\alpha+\gamma}{-(\beta+\gamma)}})]^2) \\ &= (b-y)^2(\frac{\alpha+\gamma}{\beta+\gamma} - 1 + 1 - \frac{\alpha+\gamma}{\beta+\gamma}) = 0.\end{aligned}$$

Thus $\kappa(span\{v, w\})$ is undefined for *all* $w \in V$ so M cannot have $cvc(\gamma)$.

Next, show: for $-\alpha < \gamma < -\beta$, M does not have $cvc(-\alpha)$.

Consider vectors $v = e_1 \pm \sqrt{\frac{\alpha-\beta}{-(\beta+\gamma)}}e_3$. Let $z = \pm\sqrt{\frac{\alpha-\beta}{-(\beta+\gamma)}}$

By way of contradiction, suppose $\exists\ w = ae_1 + be_2 + ce_3$ s.t. $\kappa(span\{v, w\}) = -\alpha$.

Then

$$\frac{\alpha(b-ay)^2+\beta(c-az)^2+\gamma(yc-bz)^2}{-(b-ay)^2-(c-az)^2+(yc-bz)^2}=-\alpha$$
$$\implies \alpha[(b-ay)^2-(b-ay)^2-(C-az)^2+(yc-zb)^2]+\beta(c-az)^2+\gamma(yc-zb)^2=0$$
$$\implies (\beta-\alpha(c-az)^2+(\alpha+\gamma)(yc-bz)^2=0 \implies \frac{\beta-\alpha}{-(\alpha+\gamma)}=\frac{(yc-bz)^2}{(c-az)^2}.$$

Case 1: $a=0$

Then $\frac{\alpha-\beta}{\alpha+\gamma}=\frac{z^2b^2}{c^2} \implies c^2=z^2b^2\frac{\alpha+\gamma}{\alpha-\beta}$. Consider the denominator of $\kappa(span\{v,w\})$:

$$\begin{aligned}-b^2-c^2+z^2b^2 &= -b^2+z^2b^2\frac{\alpha+\gamma}{\beta-\gamma}+z^2b^2\\ &= b^2[z^2(\frac{\alpha+\gamma}{\beta-\alpha}+1)-1]\\ &= b^2[-\frac{\alpha-\beta}{\beta+\gamma}(\frac{\alpha+\gamma}{\beta-\alpha}+1)-1]=b^2(\frac{\alpha+\gamma}{\beta+=\gamma}-\frac{\alpha-\beta}{\beta+\gamma}-1)\\ &= b^2(\frac{\alpha+\gamma-\alpha+\beta-\beta-\gamma}{\beta+\gamma})=0.\end{aligned}$$

Case 2: $a\neq 0$

Without loss of generality, say $a=1$. Then $\frac{\alpha-\beta}{\alpha+\gamma}=\frac{z^2b^2}{(c-z)^2} \implies (c-z)^2=z^2b^2\frac{\alpha+\gamma}{\alpha-\beta}$, as in Case 1. Thus, considering the denominator of $\kappa(span\{v,w\})$ results in the same calculation as above, so $\kappa(span\{v,w\})$ is undefined for all w. So M cannot have $cvc(-\alpha)$.

Next, show: for $-\alpha<\gamma<-\beta$, M does not have $cvc(-\beta)$.

Consider vectors $v=e_1\pm\sqrt{\frac{\alpha-\beta}{\alpha+\gamma}}e_2$. Let $y=\pm\sqrt{\frac{\alpha-\beta}{\alpha+\gamma}}$

By way of contradiction, suppose $\exists\ w=ae_1+be_2+ce_3$ s.t. $\kappa(span\{v,w\})=-\beta$.

Then

$$\frac{\alpha(b-ay)^2+\beta(c-az)^2+\gamma(yc-bz)^2}{-(b-ay)^2-(c-az)^2+(yc-bz)^2}=-\beta$$
$$\implies \alpha(b-ay)^2+\beta[(c-az)^2-(b-ay)^2-(c-az)^2+(yc-zb)^2]+\gamma(yc-zb)^2=0$$
$$\implies (\alpha-\beta)(b-ay)^2+(\beta+\gamma)(yc-bz)^2=0 \implies \frac{\beta-\alpha}{\beta+\gamma}=\frac{(yc-bz)^2}{(b-ay)^2}.$$

Case 1: $a=0$

Then $\frac{\alpha-\beta}{\beta+\gamma}=\frac{y^2c^2}{b^2} \implies b^2=y^2c^2\frac{\beta+\gamma}{\beta-\alpha}$.Consider the denominator of $\kappa(span\{v,w\})$:

$$\begin{aligned}-b^2-c^2+y^2c^2 &= y^2c^2-c^2+\frac{\beta+\gamma}{\alpha-\beta}\\ &= c^2[y^2(\frac{\beta+\gamma}{\alpha-\beta}+1)-1]\\ &= c^2[\frac{\alpha-\beta}{\gamma+\alpha}(\frac{\beta+\gamma}{\alpha-\beta}+1)-1]\\ &= c^2[\frac{\beta+\gamma}{\gamma+\alpha}+\frac{\alpha-\beta}{\gamma+\alpha}-1]=c^2[\frac{\alpha+\gamma}{\alpha+\gamma}-1]=0.\end{aligned}$$

Case 2: $a \neq 0$

Without loss of generality, say $a = 1$. Then $\frac{\alpha-\beta}{\beta+\gamma} = \frac{y^2c^2}{(b-y)^2} \implies (b-y)^2 = y^2c^2\frac{\beta+\gamma}{\beta-\alpha}$ as in Case 1. Thus, considering the denominator of $\kappa(span\{v,w\})$ results in the same calculation as above, so $\kappa(span\{v,w\})$ is undefined for all w. So M cannot have $cvc(-\beta)$.

Next, show: For $\boldsymbol{\alpha > \beta}$, when $\boldsymbol{\gamma = -\alpha}$, $\boldsymbol{M}$ does not have $\boldsymbol{cvc(\epsilon)}$ for any $\boldsymbol{\epsilon \in \mathbb{R}}$.

By Theorem 2.1, if M has $cvc\epsilon$), then $\epsilon = -\alpha = \gamma$.Consider $v = e_1 + ye_2 + e_3$.

By way of contradiction, suppose $\exists\ w = ae_1 + be_2 + ce_3$ s.t. $\kappa(span\{v,w\}) = -\alpha = \gamma$. Then

$$\begin{aligned}
\kappa(span\{v,w\}) &= \frac{\alpha(b-ay)^2 + \beta(c-a)^2 - \alpha(yc-b)^2}{-(b-ay)^2 - (c-a)^2 + (yc-b)^2} = -\alpha \\
&\implies \alpha[(b-ay)^2 - (yc-b)^2 - (b-ay)^2 - (c-a)^2 + (yc-b)^2] + \beta(c-a)^2 = 0 \\
&\implies (\beta-\alpha)(c-a)^2 = 0 \\
&\implies \text{ c=a because } \beta \neq \alpha \text{ by assumption.}
\end{aligned}$$

Considering the denominator of $\kappa(span\{v,w\})$: $-(b-ay)^2 - (c-a)^2 + (yc-b)^2 = -(b-cy)2 - (c-c)^2 + (yc-b)^2 = 0$. So $\kappa(span\{v,w\})$ is undefined for all $w \in V$.

Finally, show: For $\boldsymbol{\alpha > \beta}$, when $\boldsymbol{\gamma = -\beta}$, $\boldsymbol{M}$ does not have $\boldsymbol{cvc(\epsilon)}$ for any $\boldsymbol{\epsilon \in \mathbb{R}}$.

By Theorem 2.1, if M has $cvc\epsilon$), then $\epsilon = -\beta = \gamma$.Consider $v = e_1 + e_2 + ze_3$.

By way of contradiction, suppose $\exists\ w = ae_1 + be_2 + ce_3$ s.t. $\kappa(span\{v,w\}) = -\beta = \gamma$. Then

$$\begin{aligned}
\kappa(span\{v,w\}) &= \frac{\alpha(b-a)^2 + \beta(c-az)^2 - \alpha(c-bz)^2}{-(b-a)^2 - (c-az)^2 + (c-bz)^2} = -\beta \\
&\implies \alpha(b-a)^2 + \beta[(c-az)^2 - (c-bz)^2 - (b-a)^2 - (c-az)^2_(c-bz)^2] = 0 \\
&\implies (\alpha-\beta)(b-a)^2 = 0 \\
&\implies \text{b=a because } \beta \neq \alpha \text{ by assumption.}
\end{aligned}$$

Considering the denominator of $\kappa(span\{v,w\})$: $-(b-a)^2 - (c-az)^2 + (c-bz)^2 = -(a-a)2 - (c-az)^2 + (c-az)^2 = 0$. So $\kappa(span\{v,w\})$ is undefined for all $w \in V$.

□

Proof. (4 of 5)

Case 1: $v = e_1 + ye_2 + ze_3$ and only one of y, z or neither equal zero. Consider $w = ye_2 + ze_3$. Then

$$\begin{aligned}
\kappa(span\{v,w\}) &= \frac{\alpha y^2 + \alpha z^2 + \gamma(yz-yz)^2}{-y^2 - z^2 + (yz-yz)^2} \\
&= \frac{\alpha(y^2+z^2)}{-1(y^2+z^2)} \\
&= -\alpha
\end{aligned}$$

By assumption, the denominator will never equal zero.

Case 2: $v = e_1$.

Choose $w = e_2$ ($w = e_3$ would also work).
Then

$$\kappa(span\{v,w\}) = \frac{R_{1221}}{(-1)(1)-(0)} = -\alpha$$

$\square$

Proof. (5 of 5)
Let $v \in V$ s.t. $v = xe_1 + ye_2 + ze_3$; $x, y, z \in \mathbb{R}$.
Let $w \in V$ s.t. $w = ae_1 + be_2 + ce_3$; $a, b, c \in \mathbb{R}$.
Then

$$K(span\{v,w\} = \frac{R(v,w,w,v)}{< v,v >< w,w > - < v,w >^2}$$

Numerator:

$$R(xe_1 + ye_2 + ze_3, ae_1 + be_2 + ce_3, ae_1 + be_2 + ce_3, xe_1 + ye_2 + ze_3)$$
$$= xbayR_{1212} + x^2b^2R_{1221} + xcazR_{1313} + x^2c^2R_{1331} + a^2y^2R_{2112} + yabxR_{2121} + ycbzR_{2323} + y^2c^2R_{2332} + z^2a^2R_{3113}$$
$$+ zacxR_{3131} + z^2b^2R_{3223} + zbcyR_{3232}$$
$$= xbay(\gamma) + x^2b^2(-\gamma) + xcaz(\gamma) + x^2c^2(-\gamma) + a^2y^2(-\gamma) + yabx(\gamma) + ycbz(-\gamma) + y^2c^2(\gamma) + z^2a^2(-\gamma) + zacx(\gamma) + z^2b^2(\gamma)$$
$$+ zbcy(-\gamma)$$
$$= \gamma(2xbay - x^2b^2 + xcaz - x^2c^2 - a^2y^2 - 2ycbz + y^2c^2 - z^2a^2 + z^2b^2)$$

Denominator:

$$< xe_1 + ye_2 + ze_, xe_1 + ye_2 + ze_> < ae_1 + be_2 + ce_3, ae_1 + be_2 + ce_3 > - < xe_1 + ye_2 + ze_, ae_1 + be_2 + ce_3 >^2$$
$$= (-x^2 + y^2 + z^2)(-a^2 + b^2 + c^2) - (-xa + by + cz)^2$$
$$= 2xbay - x^2b^2 + xcaz - x^2c^2 - a^2y^2 - 2ycbz + y^2c^2 - z^2a^2 + z^2b^2$$

So

$$K(spanv,w) = \frac{\gamma(2xbay - x^2b^2 + xcaz - x^2c^2 - a^2y^2 - 2ycbz + y^2c^2 - z^2a^2 + z^2b^2)}{2xbay - x^2b^2 + xcaz - x^2c^2 - a^2y^2 - 2ycbz + y^2c^2 - z^2a^2 + z^2b^2} = \gamma$$

Because v, w were chosen arbitrarily, M has $csc(\gamma)$.

$\square$

2.3 What does the set of vectors, that prevents M having $cvc(\epsilon)$ for some $\epsilon \in \mathbb{R}$, look like?

Proposition 2.3. *The vectors that prevent M from having $cvc(-\alpha)$, $cvc(\gamma)$ or $cvc(-\beta)$ when $-\alpha \le \gamma < -\beta$ or $-\alpha < \gamma \le -\beta$ are all space-like or null.*

Sketch of Proof:
First consider $-\alpha < \gamma < -\beta$.
We know by Theorem 2.1 that M can have $cvc(\gamma)$, $cvc(-\alpha)$ or $cvc(-\beta)$. By taking two arbitrary vectors $v, w \in V$ and supposing $(span\{v,w\}) = \gamma$, we then see what vectors w will force $(span\{v,w\})$

to be undefined (i.e. the denominator will equal zero).
It was determined that only vectors of the form

$$v_1 = e_1 + ye_2 \pm (\sqrt{1 - \frac{\alpha+\gamma}{\beta+\gamma}} + y\sqrt{\frac{\alpha+\gamma}{-(\beta+\gamma)}})e_3$$

force the denominator to equal zero for any $y \in \mathbb{R}$.
A similar process can be done by evaluating when $(span\{v, w\}) = -\alpha$ will be undefined. In this case it was determined that vectors of the form

$$v_2 = e_1 \pm (\sqrt{\frac{\alpha-\beta}{-(\beta+\gamma)}})e_3$$

fore the denominator to equal zero for any $y \in \mathbb{R}$.
Similarly, when $(span\{v, w\}) = -\beta$, vectors of the form

$$v_3 = e_1 \pm (\sqrt{\frac{\alpha-\beta}{+\gamma}})e_2$$

fore the denominator to equal zero for any $y \in \mathbb{R}$.

Next, consider $-\alpha = \gamma \neq -\beta$.
Then the vectors that prevent M from having $cvc(-\alpha = \gamma)$ (as determined by Theorem 2.1) are of the form $v_4 = e_1 + ye_2 + e_3$ for any $y \in \mathbb{R}$

Finally, consider $-\alpha < -\beta = \gamma$.
Then the vectors that prevent M from having $cvc(-\beta\gamma)$ (as determined by Theorem 2.1) are of the form $v_5 = e_1 + e_2 + ze_3$ for any $z \in \mathbb{R}$

Is is clear to see that vectors of the form v_4 and v_5 are always space-like. For example, $< v_4, v_4 >= -1 + y^2 + 1 = y^2 \geq 0$.

Although the other vectors involve more complicated calculations, we also find that they are space-like or null. For example,

$$\begin{aligned} < v_1, v_1 > &= -1 + y^2 + (\sqrt{1 - \frac{\alpha+\gamma}{\beta+\gamma}} + y\sqrt{\frac{\alpha+\gamma}{-(\beta+\gamma)}})^2 \\ &= -1 + y^2 - y^2\frac{\alpha+\gamma}{(\beta+\gamma)} + 1 - \frac{\alpha+\gamma}{\beta+\gamma} \pm 2y\sqrt{1 - \frac{\alpha+\gamma}{\beta+\gamma}}\sqrt{\frac{\alpha+\gamma}{-(\beta+\gamma)}} \\ &= y^2(1 - \frac{\alpha+\gamma}{\beta+\gamma}) \pm 2y\sqrt{1 - \frac{\alpha+\gamma}{\beta+\gamma}}\sqrt{\frac{\alpha+\gamma}{-(\beta+\gamma)}} - \frac{\alpha+\gamma}{\beta+\gamma} \\ &= (1 - \frac{\alpha+\gamma}{\beta+\gamma})[y^2 \pm \sqrt{\frac{\alpha+\gamma}{\alpha-\beta}} + \frac{\alpha+\gamma}{\alpha-\beta}] \\ &= (1 - \frac{\alpha+\gamma}{\beta+\gamma})(y \pm \sqrt{\frac{\alpha+\gamma}{\alpha-\beta}})^2 \geq 0 \end{aligned}$$

It is clear that $(y \pm \sqrt{\frac{\alpha+\gamma}{\alpha-\beta}})^2 \geq 0$. Considering $-\alpha < \gamma < \beta$, we know that $-\frac{\alpha+\gamma}{\beta+\gamma} > 0$ so $(1 - \frac{\alpha+\gamma}{\beta+\gamma}) > 0$. And v_1 is null when $y = \pm\sqrt{\frac{\alpha+\gamma}{\alpha-\beta}}$.

Also of interest is what vectors prevent a model space M from having (ϵ) for any $\epsilon \in \mathbb{R}$. This research did not consider this question

3 VISUALIZING *constant vector curvature*

As both Thompson and Peng have done, I have generated a visualization of the *constant vector curvature* condition. To visualize constnat vector curvature, we assign sectional curvature values to each vector in the model space and project an associated color for that value onto the unit pseudo-spheres in Lorentzian 3-space (for Reimannian 3-space we project the colors onto the unit sphere).

Furthermore, I have generated an animation that presents the pseudo-spheres with the light-cone (the null space) and the set of bad vectors. The animation shows the set of bad vectors tangent to and rotating around the light-cone as the value of γ moves between $-\alpha$ and $-\beta$. The plane disappears otherwise.

Start by considering the equation for the space-like pseudo-sphere: $x^2 + y^2 - z^2 = 1$ and the equation for the time-like pseudo-sphere: $x^2 + y^2 - z^2 = -1$

Note z is the time-like direction here. For every point (x, y, z) on one of the pseudo-spheres, we know the perp space for the vector to be $span\{v, w\}$ where $v = -ye_1 + xe_2$ and $w = yze_1 + xze_{22}xye_3$ (this is because the inner product of each point with both v and w is zero). We then consider

$$
\begin{aligned}
&\kappa(span\{v, w\}) \\
&= \frac{R(-ye_1 + xe_2, yze_1 + xze_{22}xye_3, yze_1 + xze_{22}xye_3, -ye_1 + xe_2)}{< -ye_1 + xe_2, -ye_1 + xe_2 >< yze_1 + xze_{22}xye_3, yze_1 + xze_{22}xye_3 > - < -ye_1 + xe_2, yze_1 + xze_{22}xye_3 >^2} \\
&= \frac{4x^2y^2(z^2\gamma + y^2\beta + x^2\alpha)}{4x^2y^2(z^2 - y^2 - x^2)} \\
&= \frac{z^2\gamma + y^2\beta + x^2\alpha}{z^2 - y^2 - x^2} \\
&= \frac{\gamma(x^2 + y^2 - 1) + y^2\beta + x^2\alpha}{(-1)} \\
&= x^2(-\alpha - \gamma) + y^2(-\beta - \gamma) + \gamma
\end{aligned}
$$

To create the animation using Maple Software pick values for α and β and leave "A" as is. Below, $\alpha = 5$ and $\beta = -3$ were input. Then execute the following commands to generate the animation:

$with(plots)$

$trappcolorA := (x, y) \rightarrow (-5 - A) \cdot x^2 + (3 - A) \cdot y^2 + A$

```
display([animate(plot3d, [{[cos(u)*sinh(v), sin(u)*sinh(v), cosh(v)], [cos(u)*sinh(v), sin(u)*sinh(v), -cosh(v)]}, v = -1..1.25, u = 0..2·π, style = patchnogrid, color
    = trappcolorA(cos(u)·sinh(v), sin(u)·sinh(v))], A = -12..10), animate(plot3d, [[cosh(v)·cos(u), cosh(v)·sin(u), sinh(v)], v = -1..1.25, u = 0..2·π, color = trappcolorA(cosh(v)·cos(u),
    cosh(v)·sin(u), sinh(v)), style = patchnogrid], A = -12..10), animate(plot3d, [[s·(1 - (5 + A)/(-3 + A))^(1/2) + t·s·((5 + A)/(-(-3 + A)))^(1/2), s·t, s], s = -2..2, t = -1..1, color = black], A = -12..10),
    animate(plot3d, [height, angle = 0..2·π, height = -2..2, coords = cylindrical, color = "SkyBlue", style = wireframe], A = -12..10)])
```

4 Conclusion

We now know that not all three-dimensional model spaces have $cvc(\epsilon)$ for some $\epsilon \in \mathbb{R}$. In the Riemannian setting, we alwyas know what ϵ is. In the Lorentzian setting, it is known exactly what cvc value a model space should have, and if the space turns out not to have this cvc value, then we know the set of vectors that prevent it from having cvc. This set of bad vectors is tangent to the null space. Furthermore, we have a way to visualize constant vector curvautre as well as the way the set of “bad vectors" interacts with the null space.

5 Open Questions

- Does the fact that the vectors preventing model spaces from having cvc are space-like or null warrant the definition of “*time-like cvc*" and “*null cvc*?"
- We know the vectors that prevent the model spaces from having cvc the only values it could have, but what are the bad vectors for **any** ϵ that prevent a model space from having $cvc(\epsilon)$?
- How would we approach determining cvc when the Ricci Operator takes one of the other three Jordan-Normal forms?

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