Iterative Extension on Volume Altering Hyperbolic Tangle Surgeries

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Abstract

This research is centered on extending methods developed to alter links with given patterns in consecutive crossing circles developed by Harnois and Trapp such that hyperbolicity is maintained and volume is altered by twice that of an ideal octahedron. We will primarily utilize their methods along with those used by Purcell pertaining to augmented links. To this end, we have developed a means of approaching a specific pattern for constructing types of generalized augmented links.

1. Introduction

A mathematical *knot* is an embedding of a circle in three dimensions. In layman's terms, the fundamental difference between a knot as defined for our purposes and an everyday knot is that mathematical knots have the requirement that their ends be glued together. The simplest example of such a knot one may consider is the unknot:

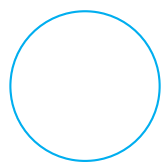


Figure 1.1. 0_1

We must also consider links, which are groups of knotted, non-intersecting knot components. A classic example we will use later is the Borromean Rings:

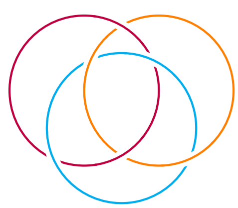


Figure 1.2. 6_2^3

There are three families of knots and links: torus, satellite, and hyperbolic. Our work is based on the last category, of which 1.2 is an example. As such, we will be primarily focused on the connections between knot theory, hyperbolic geometry, and graph theory. For example, using processes described by Purcell and others that will be described in the following section, we can see that the hyperbolic volume of the Borromean rings is the same as that of two ideal hyperbolic octahedra:

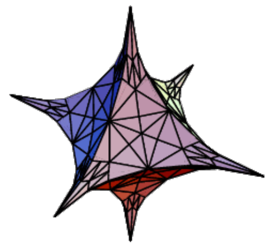
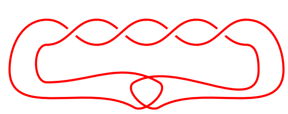


Figure 1.3. A hyperbolic octahedron

2. Background & Notation

Our goal herein is to provide an elementary explanation for how to connect knots and links to graph theory and hyperbolic geometry. We begin by generating the *cell decompostion* of the link on its complement in three-space by slicing the link along the plane of projection and systematically altering the strands above and below in a manner we will describe. From this, we can develop our graph theoretic approach, as these decompositions can be realized using graphs with properties including the degree of the vertices being dependent on the patterns present in the original link. We use this decompostion to generate *circle packings*, a realization in which tangencies between circles are equivalent to vertices in the decomposition. This in turn can be used to describe our *ideal polyhedra*, as the tangencies will also correspond to ideal points on a polyhedron in hyperbolic space.

The fundamental connection between between disciplines used herein is based on work by Purcell on *augmented links*, wherein *twist regions* – sections of the knot or link in which the strands cross over each other – are replaced with trivial components, called *crossing circles*. The twists are then reduced modulo 2, so if there is an even number of crossings, all are replaced with the crossing disk, whereas one half twist is left should there have been an odd number of them. If there are not any twists left after this reduction, we say that the crossing disk is *flat*, while they are said to be *twisted* if there is a crossing left [1]; we are primarily focused on the former variety of crossing disk. We can thus see here that if we fully augment the following link:



We will get:

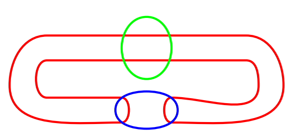


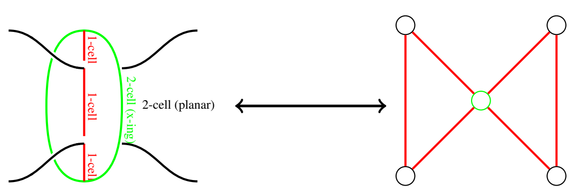
Figure 2.1. The Borromean Rings, re-arranged

Links of this type, wherein all crossing disks intersect two strands, are called *fully augmented links*.

Purcell's work envisions the cell decompositions of augmented links as 4-regular planar graphs – those for which the degree of each vertex is four – as well as their corresponding circle packings. To generate this, we will first cut the link along the plane, which results in the creation of a 3-cell in each of the regions above and below the plane, denoted P^3_+ and P^3_-. As there will not be any sections of this link with dimension zero (i.e., points)[1], we can see that there will not be any 0-cells; this implies that the polyhedron generated will be ideal. The line segments along which the plane and the strands intersect inside the crossing circles will be the 1-cells, and the 2-cells will be the regions between them. Those regions bounded by single strands are called as *planar 2-cells*, and those bounded by multiple strands are *crossing 2-cells* [2].

To decompose the link, we slice along the plane and flatten the halves of the crossing circles. For the time being, we will focus exclusively on the actions on the positive side of the plane, bearing in mind that actions in the other side are the same up to reflection. From this, we will get two copies of the 1-cell and the (top) boundary of each crossing disk. These boundaries will be shrunk to points, and the components bounding the 2-cells will be pulled together.

In this construction, the 0-cells will be the points left from the constriction of the arcs, and the 1-cells are the copies of the 1-cells from the intersections of the planar 2-cells with the crossing disks in the link. Meanwhile, each planar 2-cell provides a 2-cell in the decomposition, with two crossing 2-cells given by each crossing 2-cell from the link. Thus, viewing this as a graph, each 0-cell is 4-valent, and every edge will form a boundary between a crossing 2-cell and at most n planar 2-cells, where n is the number of punctures in the disk. We will henceforth refer to the crossing 2-cells as *shaded* faces, and the planar 2-cells as *unshaded* [2]. This process can be seen via the following tangle:



We should note here that in fully augmented links, the shaded faces will always be triangular, as they are bounded by the three 1-cells present in the link.

From these, we generate the *nerve* (γ) – the graph with vertices corresponding to the unshaded faces of the decompositions and edges representing points of tangency between faces – and its dual (Γ). We will focus more on the latter graph. One fundamental connection made therein is centered on the relation between 3-regular planar graphs, polyhedra, and links. Purcell has made note of the fact that the medial graph of a given 3-regular planar graph will be the cell decomposition of an ideal polyhedron – one whose vertices are points at infinity in hyperbolic space – and if the edges of the original graph are colored such that each vertex is adjacent to one and only one colored edge, then a fully augmented link can be generated such that the colored edges correspond to crossing circles. Purcell calls such edges *dimers*, and any admissable graph does not necessarily generate a unique link [1].

[1]It is worth mentioning here that the strands themselves can be thought of as being 0-dimensional as they are topologically equivalent to points. For clarity, though, we shall forgo this notation.

For example, consider the complete graph on four vertices with the following coloring pattern:

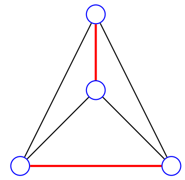


Figure 2.2. K_4

According to Purcell, we can generate a fully augmented link from this by taking the colored edges as crossing disks and the black edges as the strands connecting them like so:

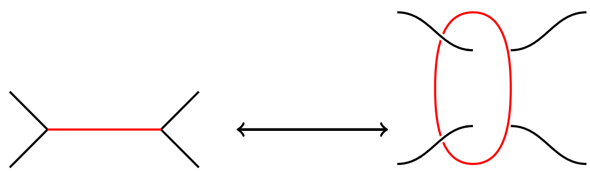


Figure 2.3. Going from a graph to a link

Thus, from Figure 2.2, we get a copy of the Borromean Rings, as in Figure 2.1.

These cell decompositons also correspond to circle packings, wherein unshaded faces are represented as circles. Points of tangency in these circles will correspond to vertices in the cell decomosition, and the regions between the circles will represent the shaded faces in the decomposition. To generate an ideal polyhedron from this, the points at which shaded faces intersect with tangencies between unshaded faces become points at infinity – that is, *ideal points* – and the faces are glued together in patterns to be described later. We must also make note of the fact that adding another circle – i.e., unshaded face – to a circle packing is equivalent to appending a regular ideal octahedron to the polyhedron the diagram had corresponded to. The validity of this construction is given as a result of Andreev's Theorem [2].

Another piece of information to bear in mind is the ease with which ideal polyhedrons with triangular faces can be utilized relative to those whose faces are other n-gons. We know that such triangles are isometric to each other; moreover, for fully augmented links, the angles between the faces are right [2], [1]. This can be seen by sending vertices of these triangles to infinity via Moebius transformation. These properties are not guaranteed for more general classes of links.

The motivation for the ideal polyhedra to be regular stems from our goal of gluing the polyhedron generated above and below the plane together. There are two copies of each edge on either side of the plane, so there will be four in total when the faces are glued together. In order for us to glue these faces, the angles around each edge must be 2π, which would imply the dihedral angle in each polyhedron must have been $\frac{\pi}{2}$, or, equivalently, that the polyhedra be regular.

The work of Harnois and Trapp on hyperbolic surgeries in augmented links also bears mentioning. From this, we can describe an admissible gluing pattern for the polyhedra generated from the decomposition of an augmented link: corresponding unshaded faces – the regions between the shaded faces – are glued together across the plane of reflection, while shaded faces – those corresponding to copies of the crossing disks in the link – are not glued in this manner. For these faces, if a given shaded face F is glued to another face F' where they are not each other's reflection across the plane, then the corresponding faces across the reflection are glued together. If this first condition is met, sufficient evidence will have been given for the admissibility of a given gluing as a hyperbolic gluing pattern on that identifies corresponding unshaded faces must be admissible. As a result, the polyhedron P, together with an admissible gluing pattern $\mathcal{A}$, is a fundamental polyhedron for a complete hyperbolic manifold [2].

We must also consider their work on surgeries, specifically those that alter volume. Of particular focus is the surgery on disks in series, which takes the following tangle:

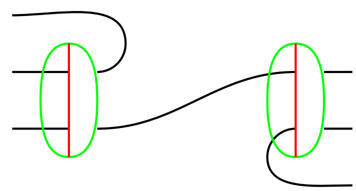


FIGURE 2.4. Strand Pattern 1 ($\mathcal{P}_1$)

and associates it with:

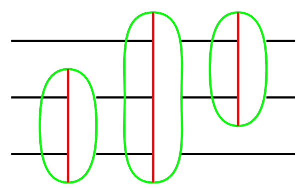


FIGURE 2.5. Strand Pattern 2 ($\mathcal{P}_2$)

The reverse mapping is also valid. Their work guarantees that this surgery will maintain the hyperbolicity of a link while altering the volume by that of two ideal regular hyperbolic octahedron, a value henceforth notated $2v_8$ [2]. The primary goal of this work is to show that these surgeries can be enacted countably many times while maintaining hyperbolicity and the known effect on volume, provided that the link's crossing disks have certain properties to be described in the next section.

Of less significance to us is the parallel tangle surgery, which connects:

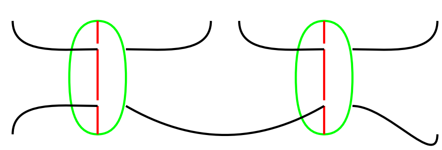


FIGURE 2.6. Strand Pattern 3 ($\mathcal{P}_3$)

with:

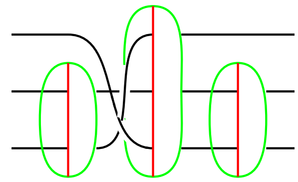


Figure 2.7. Strand Pattern 4 ($\mathcal{P}_4$)

by a change in volume of $2v_8$. We will also present a small corollary on volume altering surgeries with parallel tangles.

3. Subsumption

Before discussing our main result, we must explore a construction integral to its veracity. The fundamental concern with working beyond fully augmented links is in the shape of the faces formed. Whereas a fully augmented link will form purely triangular shaded faces – which Purcell and others have shown to be right-angled, and thus ideal, as previously mentioned – generalized augmented links can result in faces that are not, for which the desired properties are not guaranteed to translate.

With this in mind, there are certain constructions for which ideal polyhedra can be realized. We must first provide new terminology to fit the given scenario:

- A pair of crossing disks are said to be *edge identified* if a pair of its punctures connect to another disk such that in the corresponding cell decomposition, their 1-cells share an edge
- A pair of component strands in a link are said to be *edge connecting* if they form edge identifications between pairs of crossing disks
- A pair of component strands are *k-edge connecting* if they are edge connecting between any pair of edge identified crossing disks in a group of k such disks
- A group of k disks are *sequentially edge identified* if they are connected by a pair of k-edge connecting strands
- A thrice-puntured disk can *subsume* others if it is sequentially edge connected to 2 or more disks with four punctures

The clarity of this construction can be best expressed via an example; consider the following link, BR_3 in Figure 3.1:

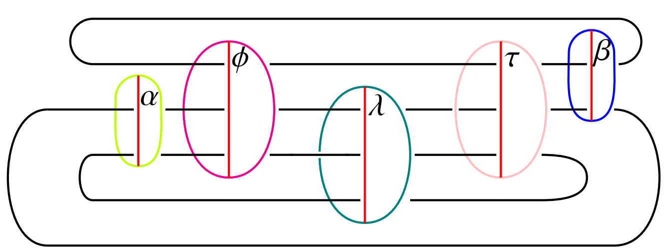


Figure 3.1. BR_3 as a link

we can see that α is edge identified to ϕ, the strands between the back of α and the front of ϕ are edge connecting, and are further 4-edge connecting between α, ϕ, λ, and τ. So, these disks are sequentially edge identified, and α can subsume each of them.

We should note here that α and β can be nested in ϕ and τ, respectively. Here, though, we are not particularly interested in such presentations, and, in fact, keeping them separate results in a cleaner pattern in the subsumption through α via circle packing.

We can begin to see the advantage of subsumption when we consider the cell decomposition of this link:

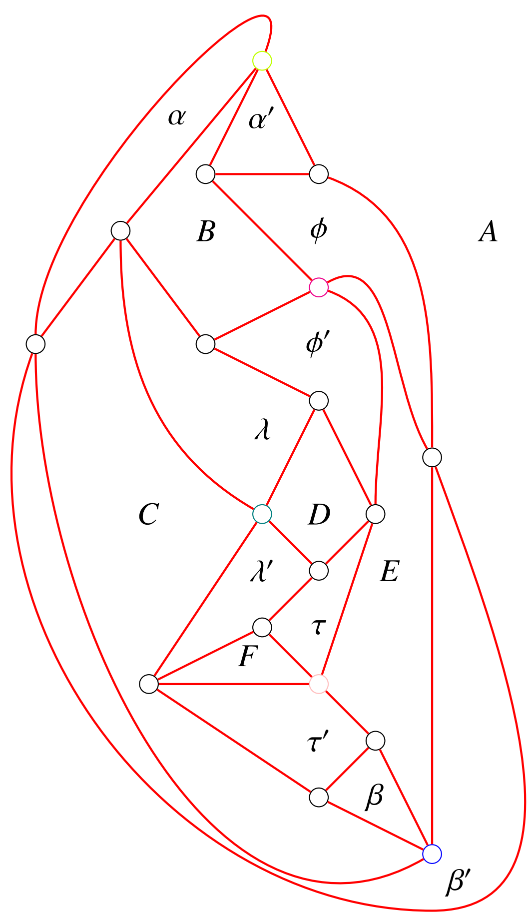


Figure 3.2. The cell decomposition of BR_3

From this, we can get a circle packing:

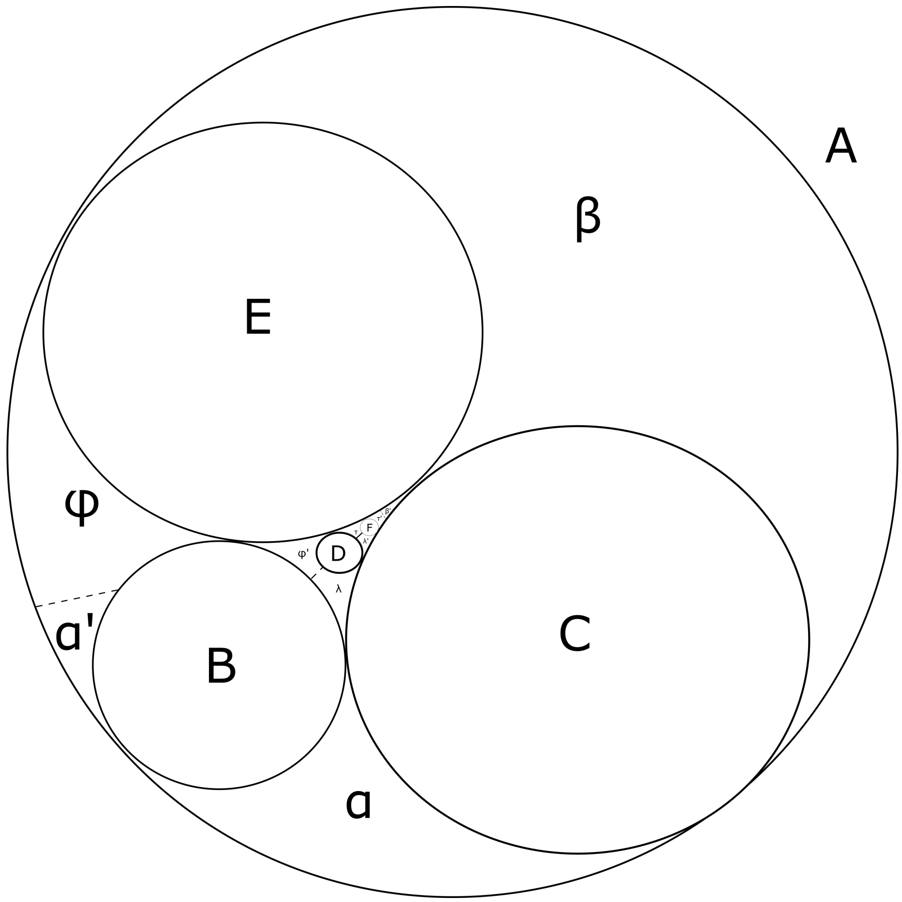


Figure 3.3. The circle packing corresponding to BR_3

In each of these representations, we can see shaded faces that will be rectangular. Thus, the hyperbolicity of the corresponding polyhedron is not trivially guaranteed, as it would be with triangular faces. We can see that edge identified in the first subsumption corresponds to a second tangency between the unshaded faces A and B in the circle packing. This in turn separates a region that would correspond to a single triangular shaded face into an ideal triangle – α – and an ideal rectangle – ϕ – that is tangent to another ideal rectangle – ϕ'. So, by subsuming ϕ, we "return" the region to a triangle – Φ –, and split the rectangle into two ideal triangles by forcing the unshaded face D down, as it gains a tangency to B. Thus, where we had two ideal rectangles and one ideal triangle, we now have three ideal triangles. Moreover, in adding the tangency between B and D, the volume lost by the rectangle that had been between them is exchanged to the region between D and F. So, looking at the circle packings before and after the first subsumption, focused especially on the region described, we have:

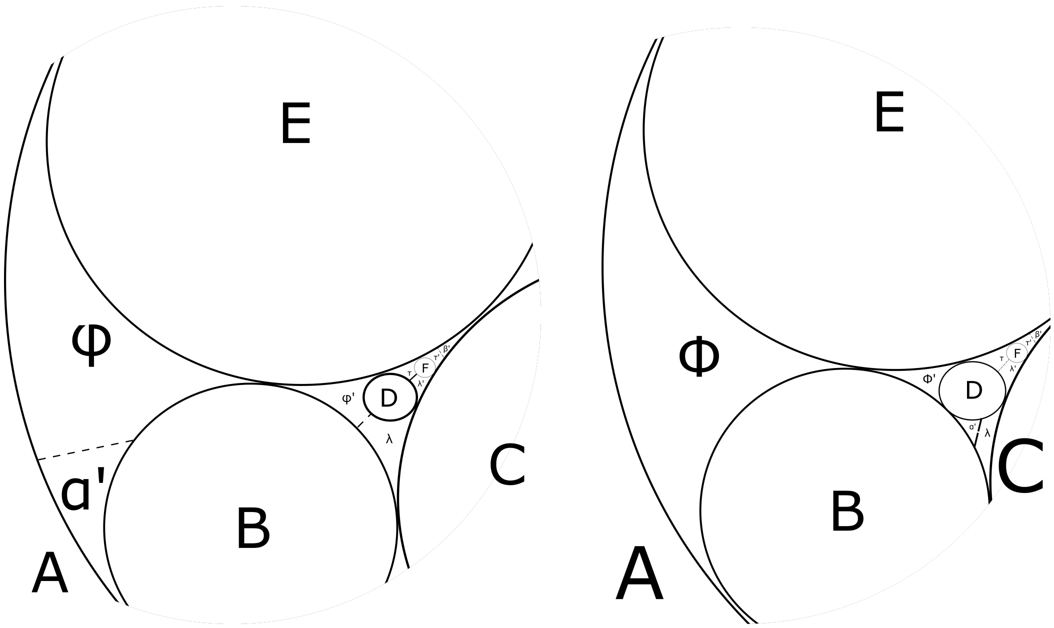


FIGURE 3.4. The circle packing before and after the first subsumption

and if we also consider the new cell decomposition and full circle packing, we now have:

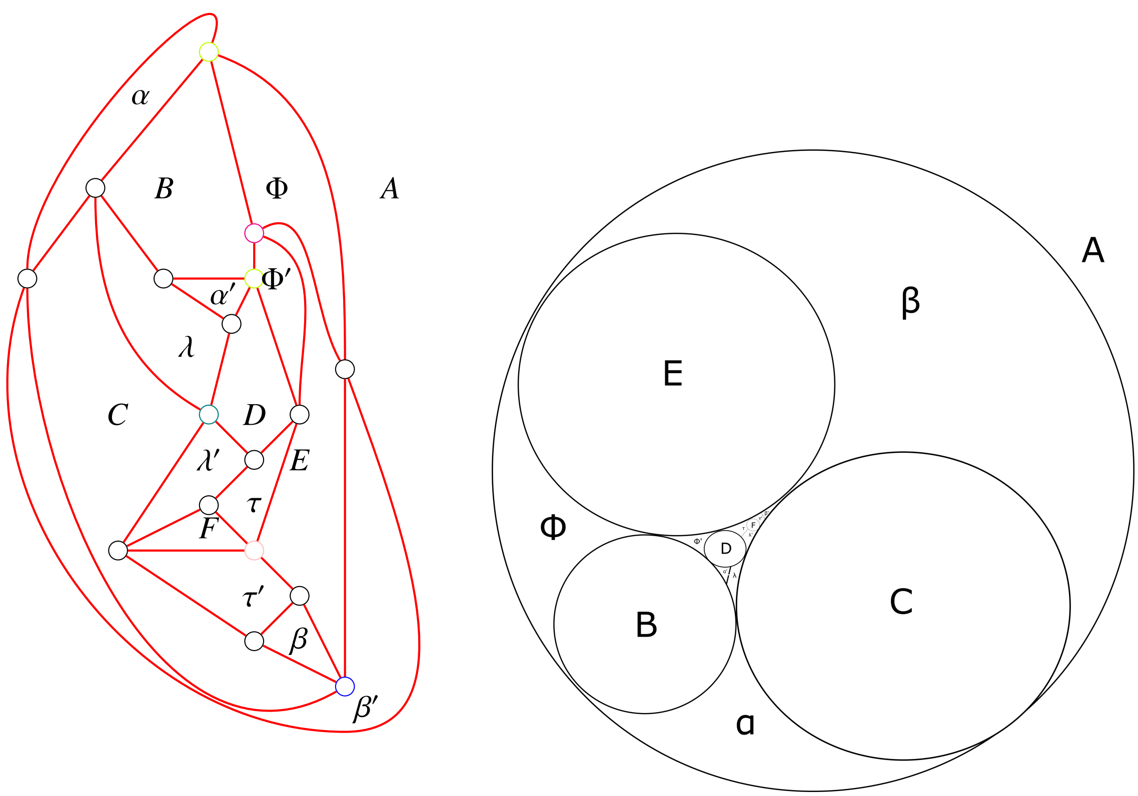
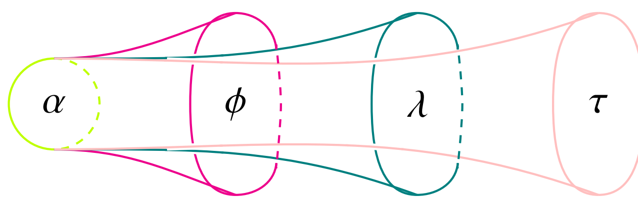


FIGURE 3.5. The cell decomposition & circle packing of BR_3 after one subsumption

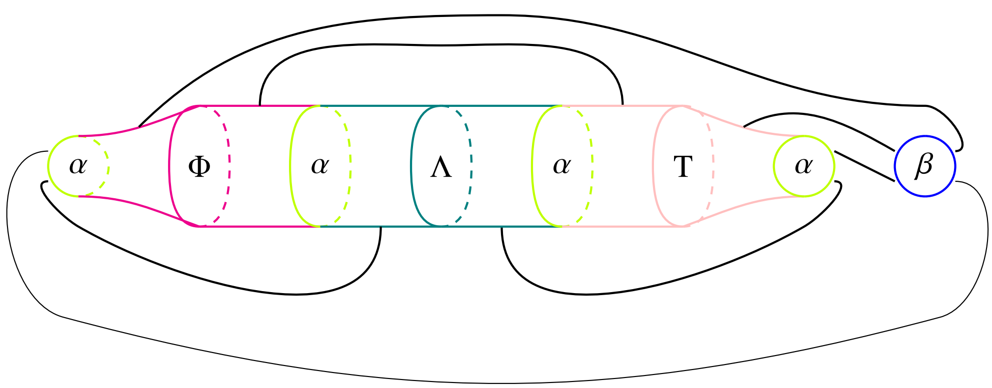
instead. This process will now repeat, as another triangular shaded region will now be split at the edge identification between α and λ.

We can further realize this construction geometrically via cutting the polyhedron the decomopsition corresponds to, and altering the gluing pattern such that ϕ is glued to ϕ'. From this, we would get two ideal triangles, α and the face originally split by α and ϕ, which we denote Φ. The third triangle – Φ' – is generated from the manner in which the edges of ϕ and ϕ' align.

Due to α's position relative to the disks it subsumes, it takes two punctures from each of them. If we consider a local picture of only α, the disks it subsumes, and the strands connecting them during this process, we will see the following relation:



and, considering the whole link after subsumption:



which corresponds to:

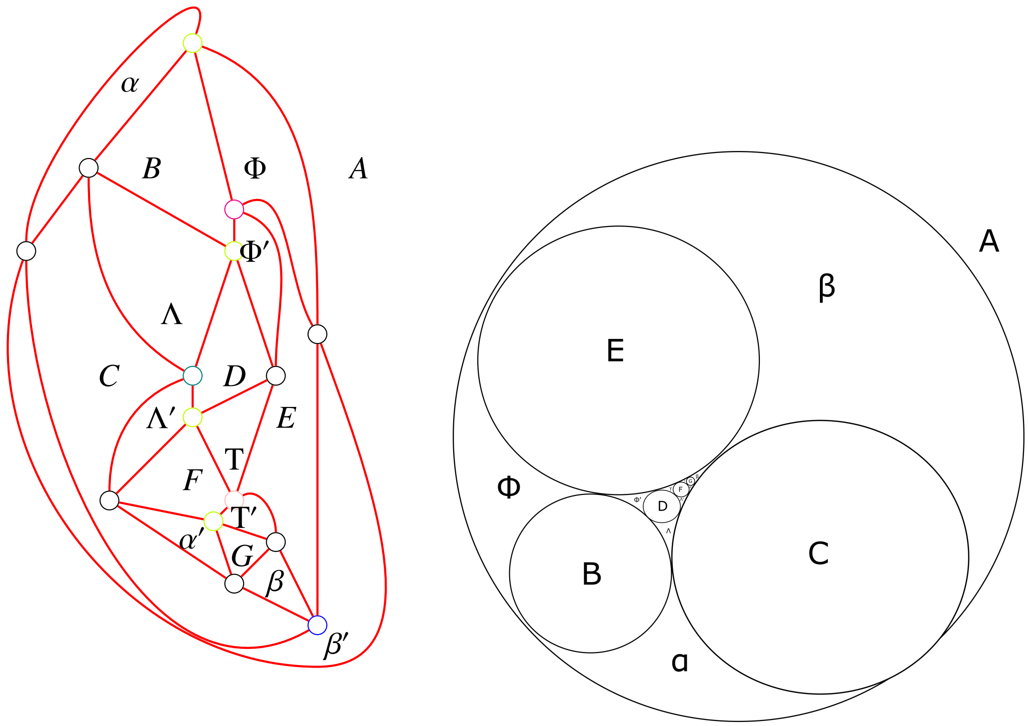
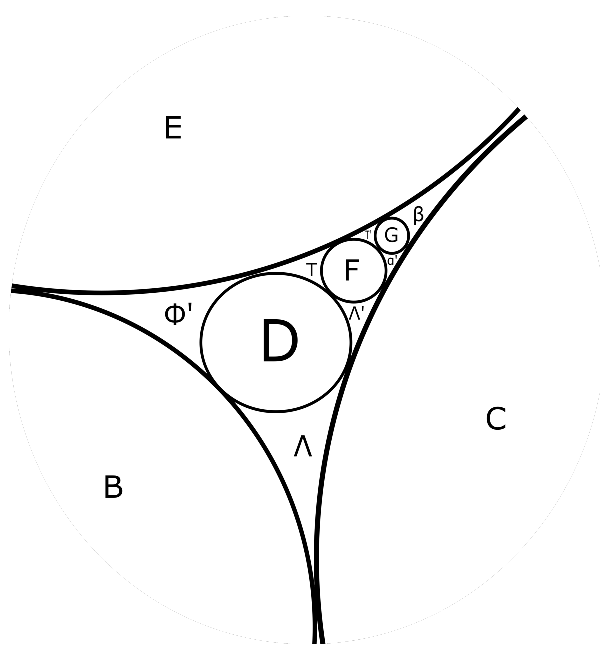


Figure 3.6. The cell decomposition & circle packing corresponding to BR_3 post-subsumption

Looking at the local picture or the region between B, C, and E:



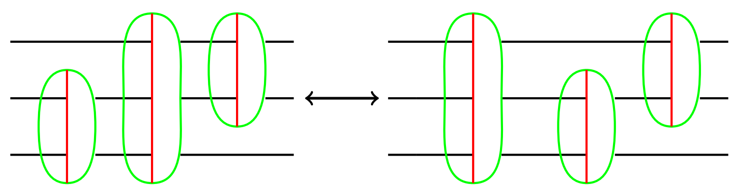
we can see that every face is now triangular, which guarantees that this link will correspond to a regular polyhedron. The coloring of the vertices in the cell decomposition implies a specific gluing pattern between α and the other faces, though. We must also make note of the fact that there is a new unshaded face, G; this comes from the edge shared by τ' and β in this example, and is an artifact of the particular representation used here.

4. Repeated Surgery

Our primary result is as follows:

Theorem 4.1. *The volume altering surgery on series tangles can be done countably many times while maintaining hyperbolicity and altering the volume by* $2v_8$ *each time.*

Proof. Let us begin with an arbitrary augmented link that contains a $\mathcal{P}_2$ tangle, as in Figure 2.5. We will assume throughout this process that only this tangle will be changed by these surgeries; that is to say, all properties including hyperbolicity and volume will be assumed to remain as such outside of the given tangle. Speaking topologically, the ability to enact surgeries repeatedly can be seen as an immediate result of the construction used in the first surgery. If we use $\mathcal{P}_2$ as a reference, we can see that the middle strand will connect the top of one thrice-punctured disk to the bottom of the other through the middle of the 4-punctured disk. So, if we move either thrice-punctured disk through the four-punctured disk, the smaller disks will be arranged in arranged in $\mathcal{P}_1$, with the caveat that their relative placement has been flipped:



As such, we will call this specific arrangement $\mathcal{P}'_1$.

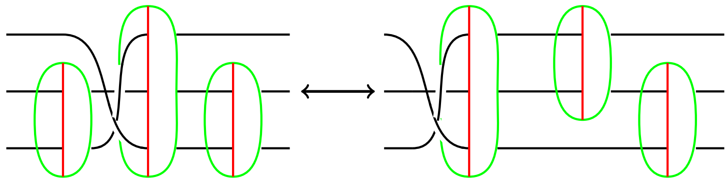
With this, we can now do the surgery again, getting two thrice punctured disks and two four-punctured disks. We can further see that this process of moving the smaller disks can be done countable many times, as one thrice-punctured disk will always be sequentially edge identified to those with four punctures. The only difference case by case will be if the disks are in $\mathcal{P}_1$ or $\mathcal{P}'_1$. Thus, we can always re-arrange the crossing disks such that there are consecutive crossing disks in series.

In terms of hyperbolicity and volume, we will call on the fact that every surgery either adds or removes an unshaded face. From here, we can guarantee via construction that the 4-punctured disks can be subsumed within one of the thrice-punctured disks. As a result, we can re-arrange the faces such that all shaded faces are triangular, which ensures both hyperbolicity and the desired volume. □

Due to a pattern resulting from the parallel surgery, we also get the following corollary:

Corollary 4.2. *If consecutive crossing disks are present in parallel, then their volume can be altered repeatedly using series surgery.*

Proof. From $\mathcal{P}_4$, we can re-arrange the thrice-punctured disks such that they are in series:

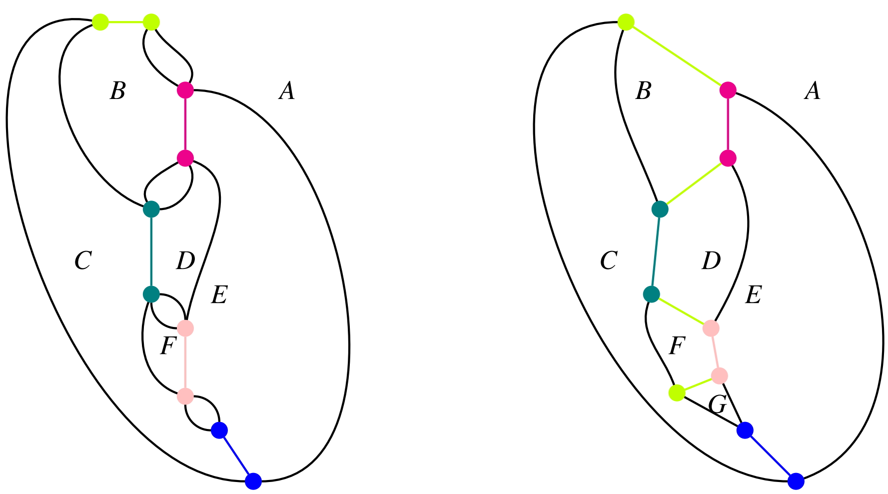


So, the result follows directly from Theorem 4.1. □

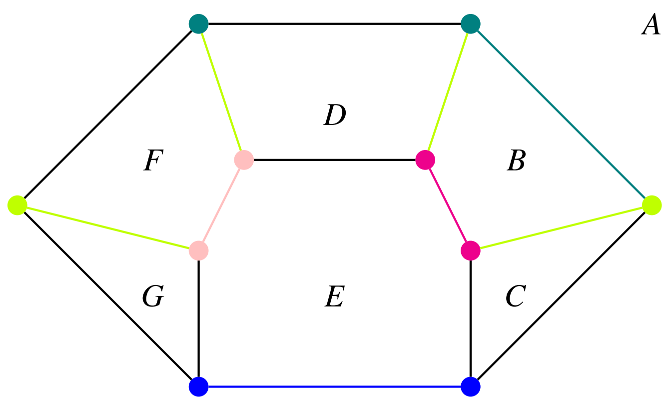
5. Open Questions

There is still a great deal we would like to learn about how subsumption can be generalized and utilized for cases where series tangles are not present. We would also like to know whether or not it can be extended for cases where larger disks get edge identified, and what effect more shared strands could have.

There also appears to be a connection with octahedral links using the dual graphs to those for which subsumption is an option[2]. If we consider Γ from the worked example before and after:



we can see that the second Γ is isomorphic to the dual of a central subdivision of K_4:



[2]This is based on work done with Hayley Olson during a brief period in which our research covered similar ground.

and graphs of this type have proven central to the generation of certain families of octahedral links. What, if anything, this connection may reveal is unknown at this point.

6. Acknowledgments

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7. References

[1] Jessica S. Purcell. An introduction to fully augmented links. In *Interactions between hyperbolic geometry, quantum topology and number theory*, volume 541 of *Contemp. Math.*, pages 205–220. Amer. Math. Soc., Providence, RI, 2011.

[2] John Harnois and Rolland Trapp. Hyperbolic tangle surgeries on augmented links.