1-Homothety Curvature Homogeneity and 1-Curvature Homogeneity

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Abstract

This paper examines the 1-homothety curvature homogeneity and 1-weak curvature homogeneity condition for certain families of manifolds. In the case where $R = R_\phi$ for ϕ of rank greater than or equal to three and the kernel of ∇R contains the kernel or R we show that a manifold is 1-homothery curvature homogeneous and 1-weak curvature homogeneous simultaneously only if the manifold in question is 1-curvature homogeneous.

1 Introduction

Let V be a vector space over a field K.

Definition 1.1a An **algebraic curvature tensor** (or ACT for short) is a 4-multilinear tensor $R : V \times V \times V \times V \to R$ that satisfies:

$$(1)\ R(X,Y,Z,W) = -R(Y,X,Z,W)$$
$$(2)\ R(X,Y,Z,W) = -R(Z,W,X,Y)$$
$$(3)\ R(X,Y,Z,W) + R(X,Z,W,Y) + R(X,W,Y,Z) = 0$$

The third of these identities is called the Bianchi identity.

Definition 1.1b An **algebraic covariant derivative curvature tensor** (or ACDCT for short) is a 5-multilinear tensor $\nabla R : V \times V \times V \times V \times V \to R$ that satisfies (1)-(3) in the first four slots and the second Bianchi identity:

$$(4)\ \nabla R(X,Y,Z,U;V) + \nabla R(X,Y,U,V;Z) + \nabla R(X,Y,V,Z;U) = 0$$

Definition 1.2 A **1-model space** is a quadruple $(V, \langle \cdot, \cdot \rangle, R, \nabla R)$ the elements of which are a vector space, an inner product on the vector space, an ACT on the vector space, and a ACDCT on the vector space respectively. A **1-weak model space** is the triple that results from the omission of the inner product, a **0-model space**, or simply a model space for short, is the triple that results from the omission of the ACDCT, and a **0-weak model space** is the pair of the vector space and the ACT.

A semi-Riemannian manifold (M, g) has a natural connection, the levi-civita connection, which is the unique torsion free metric connection[1]. The covariant derivative of X with respect to Y shall be denoted $\nabla_Y X$. This gives rise to a 1-model space at every point defined by $(T_p(M), g_p, R_p, \nabla R_p)$ where R_p is the Riemann curvature tensor defined by the formula:

$$R(X, Y, U, V) = g(\nabla_X \nabla_Y U - \nabla_Y \nabla_X U - \nabla_{[X,Y]} U, V)$$

and its covariant derivative, whose formula is:

$$\begin{gathered}\nabla R(X, Y, U, V; Z) = Z(R(X, Y, U, V) - R(\nabla_Z X, Y, U, V) \\ -R(X, \nabla_Z Y, U, V) - R(X, Y, \nabla_Z U, V) - R(X, Y, U, \nabla_Z V)\end{gathered}$$

Definition 1.3a A manifold (M, g) is said to be **k-curvature homogeneous** if for any $p, q \in M$ there is a linear isometry $\ell : T_p M \to T_q M$ such that

$$\nabla^i R_p(X, Y, Z, U; V_1, ..., V_k) = \nabla^i R_p(\ell X, \ell Y, \ell Z, \ell U; \ell V_1, ..., \ell V_k)$$

for $i = 0, 1, ..., k$.

Definition 1.3b A manifold is said to be **k-weak curvature homogeneous** if we relax the curvature homogeneity condition, allowing l to be any linear map that satisfies the property of curvature preservation above up to level-k.

Definition 1.3c A manifold is said to be **k-homothety curvature homogeneous** if for any $p, q \in M$ there is a linear homothety $\ell : T_p M \to T_q M$ such that

$$l \nabla^i R_p(X, Y, Z; V_1, ..., V_k) W = \nabla^i R_q(lX, lY, lZ, ; lV_1, ..., lV_k) lW (i = 1, ..., k).$$

Here we note that this is the i_{th} covariant derivative curvature *operator* and not the tensor itself.

A result of Dunn and McDonald (also reached independently by Gilkey et. al.) that will be of great use to us in this investigation is that 1-homothety curvature homogeneity is equivalent to the condition that there exists a smooth positive function $\psi(p)$ on M such that the 1-model spaces $(T_pM, g_p, \psi(p)R_p, (\psi(p))^{3/2}\nabla R_p)$ are all isomorphic.

There are canonical constructions of ACTs and CDACTs. Given a symmetric bilinear form ϕ on a vector space V there is an algebraic curvature tensor R_ϕ on V given by the formula

$$R_\phi(X, Y, U, V) = \phi(X, V)\phi(U, Y) - \phi(X, U)\phi(Y, V)$$

given a symmetric bilinear form ϕ and a totally symmetric trilinear form ψ there is an algebraic covariant derivative curvature tensor defined by the formula:

$$\nabla R_{\phi\psi}(X, Y, U, V; Z) = \phi(X, V)\psi(Y, U, Z) + \phi(U, Y)\psi(X, Y, Z)$$

$$-\phi(X, U)\psi(Y, V, Z) - \phi(Y, V)\psi(X, U, Z)$$

Remark: At the zero level, homothety curvature homogeneity implies weak curvature homogeneity. Weak curvature homogeneity is the condition that for $p, q \in M$ there is a map isometry $\ell : T_pM \to T_qM$ such that

$$R_p(X, Y, Z, U) = R_q(\ell X, \ell Y, \ell Z, \ell U)$$

. By the construction mentioned earlier, a manifold is 0-homothety curvature homogeneous if all of the model spaced $(T_pM, g_p, \psi(p)R)$ are all isomorphic. Let $\ell : T_pM \to T_qM$ be one of the isomorphisms which is guarunteed to exist guarunteed to exist. One has that $\ell^*\psi(q)R_q(X, Y, U, V) = \psi(q)R_q(\ell X, \ell Y, \ell U, \ell V) = \psi(p)R_p(X, Y, U, V)$. If one sets $h = (\frac{\psi(q)}{\psi(p)})^{1/4}\ell$, then h is an isomorphism of the weak zero models.

Definition 1.4 We define the **kernel of an ACT** R (denoted $ker(R)$) as the set of X such that the trilinear form given by inner multiplication $\iota_X R(\cdot, \cdot, \cdot) = R(X, \cdot, \cdot, \cdot)$ is identically 0. Similarly, we shall define the **kernel of an ACDCT** ∇R as the set of X such that the 4-linear forms given by inner multiplication $\iota_X \nabla R(\cdot, \cdot, \cdot; \cdot) = \nabla R(X, \cdot, \cdot, \cdot; \cdot)$ and $\nabla R(\cdot, \cdot, \cdot, \cdot; X)$ are both identically 0.

We shall set out to prove the results concerning the constructions above in the following order:

Theorem 2.1 Let Mbe a manifold whose curvature is of the form $R = R_\phi$ where ϕ is of rank greater than or equal to three, M if M satisfies the condition that $ker(R) \subseteq ker(\nabla R)$ then:

- ϕ, R, and ∇R give rise to well defined tensors at every point on the vector space $T_pM/ker(R_p)$. Furthermore, on $T_pM/ker(R_p)$, ϕ gives rise to a nondegenerate inner product.

- Contractions of ∇R by ϕ are invariant under 1-weak model space isomorphism.

From this result, we prove:

Theorem 2.2 Let M be 1-homothety curvature homogeneous. Suppose ϕ induces a positive definite bilinear form on $T_pM/ker(R)$ for every $p \in M$. Then M is 1-weak curvature homogeous if and only if M is 1-curvature homogeneous.

2 Results

Proof of theorem 2.1: For the first assertion, one simply has to consider the tenors characterized by the equations $\pi^*\overline{\nabla R} = \nabla R$, $\pi^*\bar{R} = R$, and, because ϕ is of rank greater than or equal to three, R_ϕ and ϕ share the same kernel, so the third equation $\pi^*\overline{\phi} = \phi$ can be added without changing the system of equations. Suppose now that $\overline{\phi}$ had a nontrivial kernel containing a nonzero vector v. Because $\overline{\phi}$ is symmetric, by graham-schmidt pseudo-orthonormalization, it is readily apparent that by negating the component of v where the signature entry is negative in ϕ, call this vector $\overline{v}$. We conclude that if $\overline{\phi}(v, \overline{v})$ is zero, then v is zero.

Furthermore, if there is a 1-weak curvature isomorphism of a model space, the form ϕ and the space $\ker(R)$ must be preserved. Likewise the form ∇R and therefore $\overline{\nabla R}$ must also be preserved. Therefore any full contraction of R by $\overline{\phi}$ is preserved as well since contractions are independent of basis.
Proof of theorem 2.2: Consider the contaction:

$$||\overline{\nabla R}||^2_\phi = (\overline{\phi})^{i_1 i_2}(\overline{\phi})^{j_1 j_2}(\overline{\phi})^{k_1 k_2}(\overline{\phi})^{\ell_1 \ell_2}(\overline{\phi})^{m_1 m_2}\overline{\nabla R}_{i_1 j_1 k_1 \ell_1 m_1}\overline{\nabla R}_{i_2 j_2 k_2 \ell_2 m_2}.$$

There exists a basis in which the form ϕ is diagonal as ϕ is symmetric. In this diagonal basis one has R with coefficients of the form (using Dunn and McDonald's construction):

$$\overline{R_{ijji}} = \lambda_{ijji}\Delta = \overline{\phi}_{ii}\overline{\phi}_{jj}.$$

This set of equations gives rise to the relations:

- $\overline{\phi}_{ii} = \overline{\phi}_{jj}\frac{\lambda_{1ii1}}{\lambda_{1jj1}}$ for $i, j > 1$
- $\overline{\phi_{11}} = \overline{\phi_{33}}\frac{\lambda_{1221}}{\lambda_{2332}}$.

Since ever coefficient is a constant multiple of $\overline{\phi}_{11}$, we conclude that we must have all coefficients in this basis of the form $\overline{\phi}_{ii} = k_i\Delta^{1/2}$.

The contraction $||\overline{\nabla R}||^2_\phi$ becomes:

$$||\overline{\nabla R}||^2_\phi = \Sigma(\overline{\phi})^{ii}(\overline{\phi})^{jj}(\overline{\phi})^{pp}(\overline{\phi})^{\ell\ell}(\overline{\phi})^{mm}\overline{\nabla R}^2_{ijp\ell m}.$$

If 1-homothety curvature homogeneity is satisfied, this is equal to:

$$\Sigma\frac{1}{k_i k_j k_p k_\ell k_m}\Delta^{-5/2} \cdot \Delta^3 = K \cdot \Delta^{1/2}$$

for nonzero K. By the positive definiteness of $\overline{\phi}$, we know that this is zero only when the 1-curvature coefficients are zero. Since this construction is a 1-weak curvature invariant, it must be constant. Therefore 1-weak curvature homogeneity renders Δ constant. However, this means that there exists a basis $\{e_1, ..., e_n\}$ at every point which satisfies the conditions:

- $\langle e_i, e_j\rangle = g_{ij}(p)$
- $R(e_i, e_j, e_k, e_l) = (R_p)_{ijkl}$
- $\nabla R(e_i, e_j, e_k, e_l; e_m) = (\nabla R_p)_{ijkl;m}$

This, as a matter of fact, means that M is 1-curvature homogeneous.

Example

Let $F : \mathbb{R}^5 \to \mathbb{R}^6$ be defined by $F(x, x_1, x_2, y_1, y_2) = (x, x_1, x_2, y_1, y_2, x_1^2/2 + x_2^2/2 + f(x))$ Give $\mathbb{R}^6$ the metric defined by: $g(\partial_x, \partial_x) = g(\partial_{x_1}, \partial_{y_1}) = g(\partial_{x_2}, \partial_{y_2}) = g(\partial_6, \partial_6) = 1$. This gives rise to the metric g on M whose matrix is:

$$\begin{pmatrix} 1+(f'(x))^2 & x_1 f'(x) & x_2 f'(x) & 0 & 0 \\ x_1 f'(x) & x_1^2 & x_1 x_2 & 1 & 0 \\ x_2 f'(x) & x_2 x_1 & x_2^2 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Consider the frame given by:

$$X = \frac{\partial_x}{\sqrt{1+(f'(x))^2}} - \frac{f'(x)}{\sqrt{1+(f'(x))^2}}(x_1\partial_{y_1} + x_2\partial_{y_2})$$
$$X_1 = \partial_{x_1} - \frac{x_1^2\partial_{y_1}}{2} - x_1x_2\partial_{y_1}$$
$$X_2 = \partial_{x_2} - \frac{x_1^2\partial_{y_2}}{2}$$
$$Y_i = \partial_{y_i}.$$

In this frame the metric entries are constant with the values $g(X, X) = g(X_i, Y_i) = 1$. Furthermore, other than a scaling in the x direction, everything is changed by a combination of ∂_{yi}'s which does not affect the value of the curvature tensor, whose new coefficients are:

- $R_{1221} = \frac{f''(x)}{(1+(f'(x))^2)^2}$
- $R_{1331} = \frac{f''(x)}{(1+(f'(x))^2)^2}$
- $R_{2332} = \frac{1}{1+(f'(x))^2}$

Scaling X_i by $\lambda = \sqrt{\frac{f''}{1+(f')^2}}$ and Y_i by $1/\lambda$, one establishes homothery curvature homogeneity at the zero level, with all coefficients set to $\tilde{R}_{ijji} = \frac{(f''(x))^2}{(1+(f'(x))^2)^3}$. In the original basis, the coefficients of the curvature and its covariant derivative are (up to symmetry):

- $R_{1221} = R_{1331} = \frac{f''(x)}{1+(f'(x))^2}$
- $R_{2332} = \frac{1}{1+(f'(x))^2}$

and

- $\nabla R_{1221;1} = \nabla R_{1331;1} = \frac{4(f'')^2 f' - (1+(f')^2)f'''}{(1+(f')^2)^2}$
- $\nabla R_{2332;1} = \frac{2f'f''}{(1+(f')^2)^2}$
- $\nabla R_{2132;3} = \nabla R_{3123;2} = \frac{f'f''}{(1+(f')^2)^2}$.

The value of the invariant described above would, in this case be:

$$||\nabla R||^2_\phi = \frac{8}{(f'')^3(1+(f')^2)^{3/2}}[20(f')^2(f'')^4 + (f')^4(f''')^2 + (f''')^2$$

$-8(f')^3(f'')^2 f''' - 8f'(f'')^2 f''' + 2(f')^2(f''')^2]$. This shows that even in specific cases, it is difficult to find manifolds that satisfy 1-weak curvature homogeneity if they are not readily 0-curvature homogeneous.

3 Bibliography

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