# NESTED AND FULLY AUGMENTED LINKS

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ABSTRACT. This paper focuses on two subclasses of hyperbolic generalized fully augmented links: fully augmented links and nested links. The link complements of fully augmented links have many nice geometric properties that many generalized fully augmented links do not have. Nested links are a class of generalized fully augmented links that share many qualities with fully augmented links, including cell decomposition properties, cusp properties, and sharpness of a volume bound.

## 1. INTRODUCTION & BACKGROUND

A knot is a closed curve in three dimensions. A link is a collection of multiple knots that can be interconnected. The class of links focused on in this paper are hyperbolic links, which are links whose complement can be described as a complete hyperbolic manifold. The subclass of links explored in this paper are hyperbolic generalized fully augmented links. In order to obtain an augmented link, take a knot or link diagram K and place a trivial component around every twist region. This trivial component is known as a crossing circle (when referring to the knot strand) or crossing disc (when referring to the surface bound by the strand). Remove all full twists from the twist region, leaving behind either no twists or a partial twist. The resulting link is a hyperbolic generalized fully augmented link L.



FIGURE 1. Left: a knot with two twist regions. Center: trivial components added to twist regions. Right: all full twist removed to create an augmented link.

Hyperbolic generalized fully augmented links can have arbitrarily many strands in each twist region. The number of strand in the twist regions affects the geometry of the link complement. For example, a crossing disc with two strands passing through it is a thrice punctured sphere (one puncture for each strand and one for the crossing circle that bounds the disc) and thus creates a triangular face in the link complement. This is important because in hyperbolic space all triangles whose vertices are ideal are totally geodesic and they are all isometric to one another, which is helpful in the creation of the manifold. However, when you add a third crossing strand the crossing disc becomes a four times punctured sphere and thus creates a rectangular face in the link complement. This is inconvenient because rectangles are not always totally geodesic and are not isometric to all other rectangles. Thus, when all of the twist regions have only two strands the hyperbolic complement is easier to define than when the twist regions have more strands.

In this paper, two subclasses of augmented links are explored: fully augmented links and nested links. In a fully augmented link, each twist region has only two strands and thus the crossing disc is twice punctured. In a nested links, the twist regions can have more strands, but the crossing discs can be made coplanar so that each disc is still only twice punctured by a combination of strands and other discs. This is useful because then the crossing discs only form triangular faces in the link complement, which have nice hyperbolic properties.



FIGURE 2. the augmentation of a nested tangle.

In this paper, the similarities between fully augmented and nesting links will be analyzed. The main point of comparison is Purcell's An*introduction to fully augmented links*, where she outlines some of the basic characteristics of fully augmented links. Most of these characteristics are true for nested links as well. Main points of comparison in this paper are cell decomposition, circle packing, cusp shapes and fundamental regions, and volume bounds. Many of the following propositions have analogous propositions in [?], which are noted where appropriate.

### 2. Cell Decomposition

Hyperbolic links are characterized by their link complement. The link complement is a complete hyperbolic manifold that makes up the space surrounding the link. This manifold can be described as a collection of polyhedra glued together with specific gluing instructions. In order to determine the polyhedra of the knot complement, we cut along a cell decomposition of  $S^3$ .

The 0-cells of the complement are the knot strands of the augmented link. The 1-cells are the intersections between the crossing discs and the plane of projection. The 2-cells are the crossing discs and the plane of the original link components. The cell decomposition utilized in this paper is a generalization of the cell decomposition of fully augmented links as outlined by Purcell in Proposition 2.2 of [?]:



FIGURE 3. Cell decomposition of a hyperbolic fully augmented link, with the steps of decomposition labeled.

• Step 1. Cut the complement along the projection plane. This creates two isometric regions,  $P^+$  and  $P^-$  which are symmetric about the projection plane. Half of each crossing disc is in each region.

- Step 2. For each  $P^+$  and  $P^-$ , butterfly and flatten the half crossing discs.
- Step 3. Shrink the link components into ideal vertices.



FIGURE 4. Cell decomposition of a hyperbolic nested link, with the steps of decomposition labeled.

The cell decomposition can be transformed into a circle packing. In a circle packing, all of the faces that come from the plane of projection are transformed into circles that have the same tangencies as the cell decomposition.

Using the circle packing of a cell decomposition, the nerve  $\Gamma$  is the graph obtained by placing a vertex at the center of each circle and adding edges between vertices when the faces they correspond to are tangent. The dual  $\Gamma^*$  to the nerve can be obtained by doing the same thing with the shaded faces. The dual can also be obtained by placing a vertex in every face of  $\Gamma$  and adding edges between vertices when the faces share edges.

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FIGURE 5. Clockwise from upper left: 1. The cell decomposition from Figure 4, 2. The circle packing of the cell decomposition, 3. The dual to the nerve in blue, 4. The nerve in green.

# 3. PROPERTIES OF CELL DECOMPOSITION

It can be shown that nested links have the same cell decomposition qualities of fully augmented links from Proposition 2.2 in [?].

**Proposition 1.** Let L be a hyperbolic nested link. There is a decomposition of the complement into two isometric polyhedra and these polyhedra have the following properties:

- Faces of the polyhedra can be checkerboard colored. Shaded faces are triangles that come from the twice punctured crossing discs. Unshaded faces come from the projection plane.
- Ideal vertices are all 4-valent.
- The dihedral angle at each edge is  $\pi/2$ .

*Proof.* The edges of the knot decomposition come from the intersections of the crossing discs with the projection plane. Just like with fully augmented links, all of the edges of nested links bounds one shaded face and one unshaded face. This allows for checkerboard coloring. Since the crossing discs of nested links are composed entirely of 2-punctured discs, each 2-punctured disc has exactly three edges and is thus a triangle.

The ideal vertices in the decomposition come from the link components of L. In a nested link, the vertices from the link component embedded in the projection plane have four edges as described by Purcell in [?]: two for each of the crossing discs the component touches. The outermost crossing discs have four edges as well: two for each of the two triangles formed from the 2-punctured disc. The inner crossing discs also have four edges: two from the triangle formed on that face of the 2-punctured disc and two from the triangle of the crossing disc immediately above. Thus, all ideal vertices are 4-valent.

The dihedral angle is  $\pi/2$  at each edge because each shaded face is a triangle. If you observe the circle packing of the link complement, each shaded triangle is bound by three mutually tangent circles. If you send the ideal vertex between two of the circles to infinity, the circle packing is displayed as two parallel lines both tangent to the third circle. Since the circle is tangent to two parallel lines, it must be tangent along a diameter d and d must be perpendicular to all three circles. The dome on which the triangular face lies ends up presented as that diameter d of the circle because it must pass through the points where the circles intersect. Thus, the shaded faces must intersect the unshaded faces at an angle of  $\pi/2$ . Since the polyhedra are checkerboard colored, the dihedral angle at each edge is  $\pi/2$ 



FIGURE 6. Left: section of a circle packing, where the bound of the hemisphere face the triangle lies on is in blue. Right: the same circle packing with one ideal vertex sent to infinity.

Since the decomposition of a nested links satisfies the above proposition and results in polyhedra with totally geodesic faces and the shaded faces are triangles, nested links also have the same properties as Lemma 2.3 of [?], shown below in Lemma 2.

**Lemma 2.** Let L be a hyperbolic nested link. Then the polyhedral decomposition of the link complement corresponds to a circle packing

on  $S^2$  whose nerve is a triangulation of  $S^2$ . The nerve also has the following properties:

- Each edge of the nerve has distinct endpoints.
- No two vertices are joined by more than one edge.

*Proof.* The proof is identical to Purcell's proof of Lemma 2.3 in [?].  $\Box$ 

Purcell provides a converse to the above lemma for fully augmented links. When you start with a triangulation on  $S^2$  and paint a group of edges red such that each vertex only has one colored edge it correlates to a hyperbolic fully augmented link whose nerve is the triangulation. It is shown that performing the same painting on the dual graph of the triangulation is also associated with the hyperbolic fully augmented links. There is an analogous converse to the above lemma for nested links, which involves painting the dual graph.

**Definition 3.** An edge-symmetric graph is a connected graph such that for one edge a, each edge  $b_1$  is related to one edge  $b_2$  by symmetry about a. An edge-symmetric spanning forest is a spanning forest such that each tree is an edge-symmetric graph.



FIGURE 7. The formation of a nested link from the dual graph of a triangulation.

**Lemma 4.** Let  $\Gamma$  be a triangulation of  $S^2$  and  $\Gamma^*$  is the dual graph to  $\Gamma$ . Every edge-symmetric spanning forest of  $\Gamma^*$  is associated with a hyperbolic nested link.

*Proof.* This is proven by Harnois and Trapp in [?].

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## 4. Cusps

In [An intro], Purcell characterizes the cusps of fully augmented links. Cusps of nested links have similar characteristics to fully augmented links, with a slight difference in the way the meridian of the cusp is presented in the cell decomposition. The propositions in this section are analogous to Lemma 3.1 and Proposition 3.2 in [?].

**Definition 5.** A cusp is the neighborhood of an ideal vertex. It is the toroidal region of the link complement around a link component.



FIGURE 8. Left: a fully augmented tangle, the crossing circle is blue, the crossing disc is shaded, and the cusp is green. Right: a nested tangle, the crossing circles are blue, the crossing discs are shaded, and the cusps are in green and purple

**Proposition 6.** Any cusp of a nested link is tiled by rectangles, each determined by a circle packing corresponding to a vertex of the ideal polyhedra.

*Proof.* According to Proposition 1, every vertex is 4-valent and all dihedral angles are  $\frac{\pi}{2}$ . Thus, the cusp of every vertex must form a rectangle.



FIGURE 9. Left: the link from Figure 4. Right: the locations of the cusps in the cell decomposition.

**Proposition 7.** The circle packing when any ideal vertex is sent to infinity consists of two parallel white lines each tangent to a pair of

white circles such that the two lines and two circles bound the rest of the circle packing.

*Proof.* Looking at the circle packing of a nested link, when you send a vertex to infinity the faces tangent at that vertex become the rectangular cusp. The two unshaded circles tangent at that point become two faces parallel to one another and perpendicular to the plane – these create the white lines in the circle packing. Since the shaded faces tangent at that point are triangles, they become two faces parallel to one another and perpendicular to the plane. Also, they are bounded below by only one semicircular line where they intersect the unshaded faces because two of their edges bound the parallel faces so there is only one additional edge to each triangle. Thus, the circle packing is presented as two parallel white lines (the unshaded faces) both tangent to a pair of white circles (the planes of the lower bounds on the shaded faces), with the rest of the packing between them.



FIGURE 10. Left: the cusp with the vertex corresponding to crossing circle A is sent to infinity. Right: the cusps when each of the vertices corresponding to to crossing circle B are sent to infinity.

**Proposition 8.** Let  $C_1$  be a cusp corresponding to an outermost crossing circle in a hyperbolic nested link and let  $C_2$  be a smaller nested crossing circle. Then  $C_1$  and  $C_2$  satisfy the following:

- (1) A fundamental region for  $C_1$  is formed from two rectangles: one from an ideal vertex in  $P^+$  and one from the corresponding ideal vertex in  $P^-$ .
- (2) A fundamental region for  $C_2$  is formed from four rectangles: two from ideal vertices in  $P^+$  and two from the corresponding ideal vertices in  $P^-$ .

- (3) Longitudes of  $C_1$  and  $C_2$  are parallel to the curve given by a shaded face intersected with the cusp boundary, and it intersect white faces twice.
- (4) If  $C_1$  has no partial twists, a meridian is parallel to the curve given by a white face intersected with the cusp boundary.
- (5) If  $C_2$  has no partial twists, then a meridian is parallel to the curve given by a white face intersected with the cusp boundary, and it intersects shaded faces twice.



FIGURE 11. The fundamental regions from the cusps in Figure 9. Left: the fundamental region for crossing circle A, an example of a  $C_1$  cusp from Proposition 8. Right: the fundamental region for the crossing circle B, an example of a  $C_2$  cusp from Proposition 8.

Proof. For nested links, the cusps  $C_1$  and  $C_2$  are intersected by the crossing disc. For cusp  $C_1$ , the disc intersects the cusp along a single longitude. For cusp  $C_2$ , the cusp is completely split in half and intersected along two longitudes. Since the cusps are intersected by the crossing discs, the longitudes reside on the shaded faces of the crossing discs in the cell decomposition. For  $C_1$  and  $C_2$  the longitude is split in two, half in  $P^+$  and half in  $P^-$ . Thus, in order to realize the entire longitude two cusps are needed: one from  $P^+$  and one from the corresponding vertex in  $P^-$ . The full longitude intersects the unshaded planar components twice, once for each time it passes through the plane of projection.

The cusps are also intersected along two meridians by the plane of projection. Thus, the meridian lies on the unshaded planar faces. For cusp  $C_1$ , the meridian lies on a single unshaded face and thus is completely present in just one rectangular cusp. However, for cusp  $C_2$ the meridian is split in two, half on one side of the crossing discs and

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half on the other side. For  $C_2$  the entire meridian is presented on two different cusps.

For  $C_1$  the full longitude requires two cusps and the meridian needs only one cusp, so the fundamental region is formed from two rectangles. For  $C_2$ , the full longitude and the flu meridian each require two cusps, so the fundamental region is formed from two rectangles.

## 5. Volumes

In [?], Purcell presents a volume bound on augmented links. She says that for any knot or link K in  $S^3$  which has a diagram D and a maximal twist region selection such that the corresponding augmentation yields a link L in  $S^3$  whose complement is hyperbolic, the volume satisfies:

$$vol(S^3 - L) \ge 2v_8(tw(D) - 1),$$

where  $v_8$  is the volume of a regular hyperbolic octahedron and tw(D) is the number of generalized twist regions of the maximal twist region selection of D. Thus, tw(D) is equivalent to the number of crossing discs in L. In context of the fully augmented and nested links, this means the number of crossing discs determines the volume bound.

Purcell has shown that octahedral fully augmented links are sharp on this bound. However, she noted that for links with more than two strands per twist region the volume tends to be far from sharp on this bound. Nested links can have more than two strands per twist region and there is a class of nested links that are sharp on this bound.

In order to show that a class of nested links is sharp on this bound, first we will show that the nerve can determine not only that the link is formed from octahedra glued together but also that it determines the specific number of octahedra needed. Then we will show that the nerve also determines the number of crossing discs in the link. Finally, we will combine this information to show that the volume is equal to  $2v_8(c-1)$  where c is the number of crossing discs in the link.

**Proposition 9.** Let L be a nested link with polyhedral decomposition into two polyhedra isometric to P. Then P is obtained by gluing regular ideal octahedra if the nerve is obtained by central subdivision of the complete graph on four vertices.

*Proof.* In [?], Purcell shows that when all shaded faces are triangles in the cell decomposition the link complement of a fully augmented link is obtained by gluing regular octahedra if and only if the nerve is obtained by a central subdivision of the complete graph on four vertices. This is proven by showing that when ideal octahedra are glued together along unshaded faces, the nerve must be a centrally subdivided  $K_4$  graph.

Then by showing that every time the nerve is centrally subdivided it is equivalent to adding an ideal octahedron to the face of a polyhedron.

Purcell then associates these nerves to fully augmented links by creating a spanning forest on the dual to the nerve  $\Gamma^*$  where each tree is an edge. Each of the trees corresponds to one of the crossing discs. The nerve can also be associated with nested links. An edge symmetric spanning forest on  $\Gamma^*$  correlates to a nested link whose cell decomposition has triangular shaded faces. Since the link's nerve is a centrally subdivided  $K_4$  its complement must be built by gluing regular octahedra, it will just have different gluing instructions than the fully augmented link.



FIGURE 12. Left: a complete graph on four vertices  $K_4$ . Center left: a  $K_4$  graph with one central subdivision. Center right: the dual graph added in blue. Right: the dual graph the  $K_4$  with one central subdivision.

**Proposition 10.** Given a graph  $\Gamma^*$  that is the dual graph of a triangulation on  $S^2$ , any edge symmetric spanning forest on  $\Gamma^*$  will correspond to a nested link with the number of crossing circles being equal to half the number of vertices in  $\Gamma^*$ .

*Proof.* Since any triangulation on  $S^2$  has an even number of triangles,  $\Gamma^*$  has an even number of vertices. An edge symmetric spanning forest on  $\Gamma^*$  can yield two results: if each tree is a single edge then the forest corresponds to a fully augmented link and if at least one of the trees is larger than a single edge then the forest corresponds to a nested link.

Case 1. Fully Augmented Links:

According to Purcell in [?], any spanning forest on  $\Gamma^*$  such that every tree is a single edge will produce a hyperbolic fully augmented link. In the forest, each tree corresponds to one crossing disc and each edge connects two vertices. Thus, the number of crossing discs is equal to half the number of vertices.

Case 2. Nested Links:

According to Harnois and Trapp in [?], every edge symmetric spanning forrest on  $\Gamma^*$  corresponds to a hyperbolic nested link. In any edge



FIGURE 13. Left: a section of a painted  $\Gamma^*$  graph with 8 vertices, the edge symmetric tree is in purple. Right: The 4 crossing discs corresponding to the tree in purple and the locations of planar strands that will puncture the discs in black.

symmetric tree, the two vertices that are isometric by symmetry correspond to the same crossing disc and are the only two vertices that correspond to that crossing disc. Thus, there are half as many crossing circles as there are vertices in  $\Gamma^*$ .

**Theorem 11.** Let  $\Gamma$  be a graph obtained by central subdivision of the complete graph on four vertices and  $\Gamma^*$  is the dual graph to  $\Gamma$ . Let L be a fully augmented or nested link generated from  $\Gamma^*$  by an edge symmetric spanning forest. The volume of L satisfies:

$$vol(S^3 - L) = 2v_8(c - 1),$$

where  $v_8$  is the volume of a hyperbolic ideal octahedron and c is the number of crossing discs in L.

*Proof.* According to Purcell in [?], the complete graph on four vertices corresponds to one octahedron in  $P^+$  and each subdivision equates to adding one additional octahedron. The  $\Gamma^*$  of the complete graph on four vertices is also the complete graph on four vertices. As stated above, the number of crossing discs in L is equal to half the number of vertices in  $\Gamma^*$ . Thus, the complete graph on four vertices has two crossing discs.

Every subdivision of  $\Gamma$  adds one vertex and three edges to  $\Gamma$ , which turns what was once one triangle into three triangles, a gain of two triangles. This results in adding two vertices and three edges to  $\Gamma^*$ . Thus, the number of octahedra n in  $P^+$  and the number of vertices vin  $\Gamma^*$  satisfies  $n = \frac{v-2}{2}$ . Since  $c = \frac{v}{2}$ , it follows that n = c - 1. Since the polyhedral decomposition creates two polyhedra isometric to P, the total volume is dictated by n = 2(c - 1). Thus, the volume of Lsatisfies  $vol(S^3 - L) = 2v_8(c - 1)$ .

## 6. Open Questions

This paper only touches on finitely few characteristics of fully augmented and nested links. It only highlights one very small difference (the presentation of the meridian in the fundamental region) and a few similarities. Further research could determine how many more characteristics are either the same or different between these classes of links.

Another continuation of this work would be to find other classes of generalized fully augmented links that share some of these properties with fully augmented and nested links, such as links that have triangular shaded faces in the cell decomposition or links that can be formed by gluing together ideal octahedra.

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