A Generalization of Various Theories of Curvature Homogeneity

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Abstract

Since the introduction of the study of curvature homogeneous manifolds, various other theories of curvature-homogeneity have been studied. In this paper we introduce a new type of curvature homogeneity; one that generalizes the previous theories of curvature homogeneity. We also construct two examples of this phenomenon, which do not fall under the previous categories of curvature homogeneity.

1 Introduction

Given a psuedo-Riemannian manifold (M, g) where g is the metric tensor and ∇ is the Levi-Civita connection, the Riemann curvature operator $\mathcal{R} \in (T^*M)^3 \otimes TM$ is defined as

$$\mathcal{R}(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Using $\mathcal{R}$ and the metric tensor, we define the Riemann curvature tensor $R \in \otimes^4(T^*M)$ as

$$R(X, Y, Z, W) = g(\mathcal{R}(X, Y)Z, W).$$

Definition 1.1. We say (M, g) is *k-curvature homogeneous* if for every two points $p, q \in M$ there exists an isometry of tangent spaces, $\Phi : T_pM \to T_qM$ which pulls back the first k covariant derivatives of the Riemann curvature tensor. That is,

$$\Phi^* g_q = g_p \text{ and } \Phi^* \nabla^i R_q = \nabla^i R_p$$

for all $p, q \in M$

It is clear that if M is locally homogeneous, then (M, g) is k-curvature homogeneous for all $k \in \mathbb{N}$, since the Levi-Civita connection is uniquely determined by the metric. With regard to the converse, in 1960, Singer showed that for each $n \in \mathbb{N}$ there is a corresponding $k_n \in \mathbb{N}$, called the Singer number such that if (M, g) is a n-dimensional Riemannian manifold, then k_n-curvature homogeneity implies local homogeneity [1]. The analog for the pseudo-Riemannian case was later proved by Podesta and Spiro [2].

Later, similar notions to curvature homogeneity were explored, such as homothety curvature homogeneity [3,4,5] and weak curvature homogeneity [6]. The concept of a model space greatly simplifies the introduction of these notions.

Definition 1.2. Given a finite dimensional real vector space, V, we say $R \in \otimes^4 V^*$ is an *algebraic curvature tensor* (an ACT) if R satisfies the symmetries:

$$R(x, y, z, w) = -R(y, x, z, w) = R(z, w, x, y)$$

and

$$R(x, y, z, w) + R(x, w, y, z) + R(x, z, w, y) = 0.$$

Notice that these are the symmetries satisfied by the restriction of Riemann curvature tensor to the tangent space of a single point. We similarly define an i^{th} algebraic covariant derivative curvature tensor $R_i \in \otimes^{4+i} V^*$ as a tensor satisfying the same symmetries as the i^{th} covariant derivative of the Riemann curvature tensor at a point.

Definition 1.3. A *model space* is a tuple consisting of a finite dimensional real vector space V, a bilinear symmetric form $\langle \cdot \rangle$ on V, an algebraic curvature tensor R_0 and possibly some algebraic covariant derivative curvature tensors R_I, for some indexing set $I \subset \mathbb{N}$.

Definition 1.4. Given two model spaces of the same type, $\mathcal{M}_1 = (V, g, R_{i_1}, \ldots, R_{i_n})$ and $\mathcal{M}_2 = (W, h, Tj_1, \ldots, T_{j_n})$ where g and h are inner products and R_{i_k} and T_{j_k} are algebraic covariant derivative curvature tensors, we say that $\mathcal{M}_1$ is model space isomorphic to $\mathcal{M}_2$ if there exists a vector space isomorphism $\Phi : V \to W$ such that $\Phi^* T_{j_k} = R_{i_k}$ and $\Phi^* h = g$. If this is the case, we say Φ is a model space isomorphism.

We can rephrase the condition of curvature homogeneity on a manifold as follows. M is k-curvature homogeneous if and only if there is a model space $\mathcal{M} = (V, \langle, \rangle, R_0, R_1, \ldots, R_k)$ such that for every $p \in M$ there is a model space isomorphism $\Phi_p : T_p M \to V$.

Given a manifold, if we take our model space to be $(T_q M, g_q, R_q, \ldots, \nabla^k R_q)$, it is clear that this definition is equivalent to the previous one. Relaxing our model space by suppressing the metric tensor, we obtain a *weak model space.*

Definition 1.5. We say (M, g) is *k-weak curvature homogeneous* if there exists a weak model space $\mathcal{M} = (V, R_0, \ldots, R_k)$ such that for all $p \in M$ there exists a weak model space isomorphism $\Phi_p : T_p M \to V$. Note that there is no requirement of pulling back a metric.

Definition 1.6. We define an *algebraic curvature operator* as an element of $\otimes^3 V^* \otimes V$ which satisfies the same symmetries of the Riemann curvature operator restricted to the tangent space of a single point. An algebraic covariant derivative curvature operator is similarly defined. We use script $\mathcal{R}$ to distinguish a curvature operator from a curvature tensor.

Definition 1.7. We say (M, g) is *k-homothety curvature homogeneous* if there exists a model space $(V, \langle, \rangle, \mathcal{R}_0, \ldots, \mathcal{R}_k)$ such that for each $p \in M$ there exists a linear isomorphism $\Phi_p : T_p M \to V$ and a smooth function $\lambda : M \to \mathbb{R}$ such that $\Phi_p^* \langle, \rangle = \lambda_p g_p$ and $\Phi_p^* \mathcal{R}_i = \nabla^i \mathcal{R}_p$ for $i = 0, \ldots, k$.

The following theorem is due to Corey Dunn and Cullen Mcdonald [3].

Theorem 1.1. *M is k-homothety curvature homogeneous if and only if there exists a smooth positive function $\lambda : M \to \mathbb{R}$ such that for each $p \in M$ there is an isometry $\Phi : T_pM \to V$ and such that $\Phi^* R_i = \lambda(p)^{\frac{i+2}{2}} \nabla^i R_p$.*

2 G-curvature Homogeneity

The aforementioned homogeneity conditions all turn out to be examples of the following more general phenomenon.

If T is a tensor on a vector space V and $A \in GL(V)$, we let A^*T denote the precomposition of T with A in each slot.

The notion of a structure group is useful for the results to follow.

Definition 2.1. We define the *structure group*, $G_{\mathcal{M}}$, of a model space $\mathcal{M} = (V, \langle\cdot\rangle, R_{i_0}, \ldots, R_{i_n})$ as

$$G = \{A \in GL(V) : A^*\langle\cdot\rangle = \langle\cdot\rangle \text{ and } A^* R_{i_k} = R_{i_k} \text{ for } k = 0, \ldots, n\}.$$

The following is the central object of study for this project.

Definition 2.2. Let $G \leq GL(n)$ be a Lie group and let (M, g) be a n-dimensional pseudo-Riemannian manifold. We say M is *G-curvature homogeneous* with respect to a model space $\mathcal{M}$ and index set $S \subset \mathbb{N}$ if for each $p \in M$, there is an $A \in G$ such that there exists a linear isometry $\Phi_p : T_pM \to V$, such that

$$\Phi_p^* A^* R_i = R_p$$

for $i \in S$. We also require that for each $A \in G$, there exists a $p \in M$ and isometry $\Phi_p : T_pM \to V$ such that $\Phi_p^* A^* R_i = \nabla^i R_p$ for all $i \in S$. If $S = \{1, \ldots, k\}$, we say that M is *G-modeled on $\mathcal{M}$ up to order k.*

Theorem 2.1.

1. If M is G-modeled on $\mathcal{M}$ up to order k, then M is k-weak curvature homogeneous.

2. M is k-curvature homogeneous, if and only if M is k-$G_{\mathcal{M}}$-curvature homogeneous where $G_{\mathcal{M}}$ is the structure group of the k-model $\mathcal{M} = (V, \langle,\rangle, R_0, \ldots, R_k)$.

3. Suppose M is k-homothety curvature homogeneous with respect to the model space $(T_{p_0}M, g_{p_0}, \mathcal{R}_{p_0}, \ldots, \nabla^k \mathcal{R}_{p_0})$ where p_0 is some base point of M. Suppose moreover that the homothety scalar function $\lambda : M \to \mathbb{R}^+$ has the property that the set

$$\{A \in GL(T_{p_0}M) : A^* g_{p_0} = c g_{p_0} \text{ for some c} \in \lambda(M)\}$$

forms a group under composition of functions. Then, M is G-modeled up to order k on the model space $\mathcal{M} = (T_{p_0}, \langle\cdot\rangle, \mathcal{R}_0, \ldots, \mathcal{R}_k)$ and $G = H_{\langle\cdot\rangle} = \{A \in GL(V) : A^*\langle\cdot\rangle = \lambda\langle\cdot\rangle\}$, where the action of G on $\mathcal{M}$ is given by $A^*\mathcal{M} = (V, A^*\langle\cdot\rangle, \mathcal{R}_0, \ldots, \mathcal{R}_k)$. The converse also holds.

Proof.

1. Suppose there is a model space $\mathcal{M} = (V, \langle\cdot\rangle, R_0, \ldots, R_k)$ and that for each $p \in M$ there exists an isometry $\Phi_p : T_pM \to V$ and some $A_p \in G$ such that $\Phi_p^* A^* R_i = \nabla^i R_p$.
   The map $\Psi = A_p \circ \Phi_p$ is to be our weak model space isomorphism. Pulling back R_i by Ψ, we obtain that $\Psi^* R_i = \Phi_p^* A_p^* R_i = R_p$ as desired.

2. This follows immediately from the definition of the structure group and homogeneity conditions.

3. Suppose M is k-homothety curvature homogeneous with respect to the model space $(T_{p_0}M, g_{p_0}, \mathcal{R}_{p_0}, \ldots, \nabla^k \mathcal{R}_{p_0})$ where p_0 is some base point of M.

   Then by definition of homothety curvature homogeneity, given any $q \in M$, there is a model space isomorphism between $(T_qM, g_q, \mathcal{R}_q, \ldots, \nabla^i \mathcal{R}_q)$ and $(T_{p_0}M, \lambda_q g_{p_0}, \mathcal{R}_{p_0}, \ldots, \nabla^i \mathcal{R}_{p_0}) = (T_{p_0}M, A^* g_{p_0}, \mathcal{R}_{p_0}, \ldots, \nabla^i \mathcal{R}_{p_0})$ for some A in the homothety group of g_{p_0}. If the homothety scalar function $\lambda : M \to R^+$ has the property that the set $\{A \in GL(T_{p_0}M) : A^* g_{p_0} = c g_{p_0}$ for some c $\in \lambda(M)\}$ forms a group under composition of functions, then M is G-curvature homogeneous.
   For the converse, the existence of such model space isomorphisms gives us k-homothety curvature homogeneity directly.

□

Although our theory falls under the category of weak curvature homogeneity, it serves to distinguish various types of weak curvature homogeneity. We conjecture, moreover, that if M is a connected pseudo-Riemannian manifold, that k-weak curvature homogeneity implies k-G-curvature homogeneity.

3 Construction One

We will now construct a manifold which is 0-G-curvature homogeneous, but neither curvature homogeneous nor homothety curvature homogeneous. By convention, we write only the non-zero entries of the metric and curvature tensors. We let our model space $\mathcal{M} = (V, \langle\cdot\rangle, R_0)$ where $V = Span(\{x, y, z\})$,

$$R_0(x, z, z, x) = R_0(y, z, z, y) = 1 \ , \ R_0(x, y, z, x) = 1$$

and

$$\langle x, x\rangle = \langle y, z\rangle = 1.$$

We let our Lie group $G \leq GL(V)$ be the following set of matrices on the basis (x, y, z):

$$G = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & 1 \end{pmatrix} : t \in \mathbb{R}^+ \right\}$$

Let (x, y, z) be the standard coordinates on $\mathbb{R}^3$ and let $M = \{(x, y, z) \in \mathbb{R}^3 : x > 0\}$. Let the metric, g, be defined by

$$g(\partial_x, \partial_x) = 1 \ , \ g(\partial_y, \partial_z) = e^{2f(x)} \ , \ g(\partial_z, \partial_z) = h(x).$$

Theorem 3.1. *For certain choices of f, the manifold (M,g) is 0-G-curvature homogeneous with respect to the model $\mathcal{M}$. Moreover, (M,g) is neither curvature homogeneous, nor homothety curvature homogeneous.*

The non-zero covariant derivatives of vector fields are:

$$\nabla_{\partial_x}\partial_y = f'(x)\partial_y \ , \ \nabla_{\partial_x}\partial_z = \frac{1}{2}e^{-2f(x)}(-2h(x)f'(x)+h'(x))\partial_y + f'(x)\partial_z \ ,$$

$$\nabla_{\partial_y}\partial_z = e^{-2f(x)}f'(x)\partial_x \ , \ \nabla_{\partial_z}\partial_z = -\frac{1}{2}h'(x)\partial_x.$$

The curvature entries on the coordinate vector fields are:

$$R(\partial_x,\partial_y,\partial_z,\partial_x) = e^{-2f}(f'(x)^2+f''(x)) \ , \ R(\partial_y,\partial_z,\partial_z,\partial_y) = e^{4f(x)}f'(x)^2 \ ,$$

$$R(\partial_x,\partial_z,\partial_z,\partial_x) = -h(x)(f'(x))^2 + f'(x)h'(x) - \frac{h''(x)}{2}.$$

On the basis $X = \partial_x$, $\tilde{Y} = \frac{1}{e^{2f}}\partial_y$, $\tilde{Z} = \partial_z - \frac{h}{2e^{2f}}\partial_y$, we have

$$g(X,X) = 1 \ , \ g(\tilde{Y},\tilde{Z}) = 1$$

and

$$R(X,\tilde{Y},\tilde{Z},X) = -((f')^2+f'') \ , \ R(\tilde{Y},\tilde{Z},\tilde{Z},\tilde{Y}) = (f')^2 \ , \text{and}$$
$$R(X,\tilde{Z},\tilde{Z},Z) = \Delta = -h(f')^2 + f'h' + h((f')^2+(f'')) - \frac{1}{2}h''.$$

If we rescale $Z = 1/\sqrt{\Delta}\tilde{Z}$ and $Y = \sqrt{\Delta}\tilde{Y}$, we make $R(X,Z,Z,X) = 1$ and do not change the values of the metric or the other curvature entries.

In order for M to be 0-G modeled on $\mathcal{M}$, we solve the differential equation

$$f' = -((f')^2 + f'')$$

for then if $t = f'$, then on the frame (X,Y,Z) we would have

$$R(X,Y,Z,X) = t \ , \ R(X,Z,Z,X) = 1 \ , \text{and} \ R(Y,Z,Z,Y) = t^2.$$

$f' = \frac{1}{e^x-1}$ is a solution to this equation, so we let $f(x) = -x - Log(e^x-1)$. Note that the range of f' for $x > 0$ is all of $\mathbb{R}^+$ which means that every $A \in G$ is realized at some point on M. It is also interesting to note that the solution is independent of h (so long as $\Delta \neq 0$).

Scalar invariants: To show that M is not 0-homothety curvature homogeneous, and therefore also not curvature homogeneous, we check to see that $\frac{||R||^2}{\tau^2}$ is non-constant [5]. The formulas for these contractions are:

$$||R||^2 = g^{i_1i_2}g^{j_1j_2}g^{k_1k_2}g^{l_1l_2}R_{i_1j_1k_1l_1}R_{i_2j_2k_2l_2}$$
$$\tau^2 = g^{i_1i_2}g^{j_1j_2}g^{k_1k_2}g^{l_1l_2}R_{i_1j_1j_2i_2}R_{k_1l_1l_2k_2}$$

Computing these invariants on our frame we have,

$$\begin{aligned}||R||^2 &= 8R(X,Y,Z,X)R(X,Z,Y,X) + 4R(Y,Z,Y,Z)R(Z,Y,Z,Y)\\ &= 8R(X,Y,Z,X)^2 + 4R(Y,Z,Y,Z)^2\\ &= 8t^2 + 4t^4\\ \tau^2 &= 4R(X,Y,Z,X)(4R(X,Y,Z,X) + 2R(Y,Z,Y,Z) + 2R(Y,Z,Y,Z)(4R(X,Y,Z,X) + 2R(Y,Z,Y,Z))\\ &= 16R(X,Y,Z,X)^2 + 16R(X,Y,Z,X)R(Y,Z,Y,Z) + 4R(Y,Z,Y,Z)^2\\ &= 4t^4 - 16t^3 + 16t^2\end{aligned}$$

Hence, since

$$\frac{||R||^2}{\tau^2} = \frac{4t^2(2+t^2)}{4t^2(t^2-4t+4)} = \frac{t^2+2}{(t-2)^2}.$$

and $t = f'$ is non-constant on M, it follows that M is neither homothety curvature homogeneous, nor curvature homogeneous. Thus, our example departs from the previous theories of curvature homogeneity.

4 Construction Two

In this example, we construct a 1-G-curvature homogeneous manifold where the action of G on the space of algebraic curvature tensors is rank two at each point. Moreover, this manifold is neither curvature homogeneous nor homothety homogeneous.

Our proposed 0-model is $\mathcal{M} = (V, \langle,\rangle, R_0)$ where $V = Span(\{x,y,z\})$,

$$\langle x,x\rangle = \langle y,y\rangle = \langle z,z\rangle = 1$$

and

$$R_0(x,y,x,y) = R_0(x,y,x,z) = R_0(x,z,x,z) = 1.$$

The Lie group we will realize (on the basis (x,y,z)) is

$$G = \left\{ \begin{pmatrix} 1 & 0 & 0\\ 0 & t & 0\\ 0 & 0 & s\end{pmatrix} |s,t \in \mathbb{R}+ \right\}$$

Let (x,y,z) be coordinates on $\mathbb{R}^3$ and let $M = \{(x,y,z) \in \mathbb{R}^3 : y > 0, z < 0\}$. Let the metric, g, be defined by

$$g(\partial_x,\partial_x) = e^{2f(y)+2h(z)}\ ,\ g(\partial_y,\partial_y) = g(\partial_z,\partial_z) = 1.$$

Theorem 4.1. *For certain choices of f and h the manifold (M,g) is 1-G-curvature homogeneous with respect to the model $\mathcal{M}$. Moreover, (M,g) is neither curvature homogeneous, nor homothety curvature homogeneous.*

First, we will prove (M, g) is 0-G-curvature homogeneous. The covariant derivatives of the coordinate vector fields are:

$$\nabla_{\partial_x}\partial_z = h'(z)\partial_x \ , \ \nabla_{\partial_x}\partial_y = f'(y)\partial_x$$

$$\nabla_{\partial_x}\partial x = -e^{2(f(y)+h(z))}(f'(y)\partial_y + h'(z)\partial_z).$$

Let $X = e^{-(f(y)+h(z))}\partial_x$, $Y = \partial_y$, and $Z = \partial_z$. Then (X, Y, Z) forms an orthonormal frame. On this frame, we have:

$$R(X, Y, X, Y) = ((f')^2 + f'') \ , \ R(X, Y, X, Z) = f'h' \ , \text{ and}$$

$$R(X, Z, X, Z) = ((h')^2 + h'')$$

In order that M be 0-G-modeled on M we solve the differential equation:

$$(f'(y))^2(h'(z))^2 = ((f'(y))^2 + f''(y))((h'(z))^2 + h''(z)).$$

One particular solution to this is $f(y) = 2\log(\frac{y}{2})$ and $h(z) = -Log(-z)$. With this solution, the corresponding $A \in G$ at $(x, y, z) \in M$ is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}}f' & 0 \\ 0 & 0 & \sqrt{2}h' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{y} & 0 \\ 0 & 0 & \frac{-\sqrt{2}}{z} \end{pmatrix}$$

Moreover, since the since the images of $f' = \frac{2}{y}$ and $h' = -\frac{1}{z}$ are all of $\mathbb{R}^+$ on M (recall $y > 0$ and $z < 0$ on M), G does indeed form a group.

As polynomials in $t = \frac{1}{\sqrt{2}}f'$ and $s = \sqrt{2}h'$, the non-zero entries of the first covariant derivative of the curvature tensor up to the usual symmetries are:

$$\nabla R(X, Y, Y, X, Y) = \sqrt{2}t^3 \qquad \nabla R(X, Y, Z, X, Y) = \frac{1}{\sqrt{2}}t^2 s$$

$$\nabla R(X, Y, Z, X, Z) = \frac{1}{\sqrt{2}}s^2 t \qquad \nabla R(X, Y, Z, Y, X) = \frac{1}{\sqrt{2}}t^2 s$$

$$\nabla R(X, Z, Z, X, Z) = -\sqrt{2}s^3 \qquad \nabla R(X, Z, Z, Y, Z) = \frac{1}{\sqrt{2}}s^2 t$$

Hence, by adding to our model space the algebraic covariant derivative curvature tensor $R_1 \in \otimes^5 V^*$ whose entries are $R_1(x, y, y, x, y) = \sqrt{2}$, $R_1(x, y, z, x, y) = \frac{1}{\sqrt{2}}$, $R_1(x, y, z, x, z) = -\frac{1}{\sqrt{2}}$, $R_1(x, y, z, y, x) = \frac{1}{\sqrt{2}}$, $R_1(x, z, z, x, z) = -\sqrt{2}$, and $R_1(x, z, z, y, x) = \frac{1}{\sqrt{2}}$, we can use the same moving frame (X,Y,Z) and the same selection of $A \in G$ for each p to obtain a model space isomorphism from each TpM to $(\mathbb{R}^3, \langle,\rangle, A_p^* R_0, A_p^* R_1)$. Therefore, M is 1-G-curvature homogeneous.

Moreover, since f' and h' are smooth, we obtain a smooth map $\Phi : M \to G$ defined by

$$(x, y, z) \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}}f' & 0 \\ 0 & 0 & \sqrt{2}h' \end{pmatrix}$$

We can endow G with the coordinates

$$\phi : G \to \mathbb{R}^2 \;\; \phi(\begin{pmatrix} 1 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & s \end{pmatrix}) = (t, s)$$

. Then $\phi \circ \Phi) : \mathbb{R}^3 \to \mathbb{R}^2$ has derivative,

$$D(\phi \circ \Phi) = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} f'' & 0 \\ 0 & 0 & \sqrt{2} h'' \end{pmatrix}.$$

Since f'' and h'' never vanish on M, $D(\phi \circ \Phi)$ always has rank two.

We can also identify the space of algebraic curvature tensors on a 3-dimensional vector space with $\mathbb{R}^6$. Given a basis $x_1, \ldots, x_n$ for a vector space V, let $R[dx_i, dx_j, dx_k, dx_l]$ denote the algebraic curvature tensor R such that $R(x_i, x_j, x_k, x_l) = 1$.

Then, the tensors $R[dx_1, dx_2, dx_2, dx_1]$, $R[dx_1, dx_3, dx_3, dx_1]$, $R[dx_2, dx_3, dx_3, dx_2]$, $R[dx_1, dx_2, dx_3, dx_1]$, $R[dx_2, dx_3, dx_1, dx_2]$, $R[dx_1, dx_3, dx_3, dx_2]$, form a basis for the space of algebraic curvature tensors on $\mathbb{R}^3$, call this space $(\mathcal{R}, 3)$. In this manner, $(\mathcal{R}, 3)$ is vector space isomorphic to $\mathbb{R}^6$. If we define $\Psi : G \to (\mathcal{R}, 3)$ as the map $A \mapsto A^* R_0$, then its the derivative of $\Psi \circ \phi^{-1}$ is

$$\begin{pmatrix} 2t & 0 \\ 0 & 2s \\ s & t \\ 2ts^2 & 2st^2 \\ 2ts & t^2 \\ s^2 & 2ts \end{pmatrix}.$$

Then by the chain rule,

$$D(\Psi \circ \phi^{-1} \circ \phi \circ \Phi) = D(\Psi \circ \phi^{-1}) D(\phi \circ \Phi) = \begin{pmatrix} 0 & \sqrt{2} t f'' & 0 \\ 0 & 0 & 2\sqrt{2} s h'' \\ 0 & \frac{1}{\sqrt{2}} s f'' & \sqrt{2} t h'' \\ 0 & \sqrt{2} t s^2 f'' & 2\sqrt{2} s t^2 h'' \\ 0 & \sqrt{2} t s f'' & \sqrt{2} t^2 h'' \\ 0 & \frac{1}{\sqrt{2}} s^2 f'' & 2\sqrt{2} t s h'' \end{pmatrix} = \begin{pmatrix} 0 & f' f'' & 0 \\ 0 & 0 & 4h' h'' \\ 0 & h' f'' & f' h'' \\ 0 & 2(h')^2 f' f'' & 2(f')^2 h' h'' \\ 0 & \sqrt{2} h' f' f'' & \frac{1}{\sqrt{2}} (f')^2 h' \\ 0 & \sqrt{2} (h')^2 f'' & 2\sqrt{2} f' h' h'' \end{pmatrix}.$$

Since f', f'', h', h'' are all non-vanishing on M, this map is rank two at each point on M. In fact, this implies that the image of $\Psi \circ \Phi$ is a two-dimensional manifold [6].
This has the corollary that M is neither curvature homogeneous, nor homothety curvature homogeneous, for in the case of curvature homogeneity, the image of $\Psi \circ \Phi$ is a single point, and in the case of homothety curvature homogeneity, the image of $\Psi \circ \Phi$ is an open subset of a radial line.

5 Open Questions

- We have yet to prove whether or not the first example is 1-G-curvature homogeneous and whether or not the second example is 2-G-curvature homogeneous.
- In general, the theory of invariants for G-curvature homogeneity seems rich, as there is no "uniform" sort of action on the tangent space at each point, so more novel ways of contracting tensors and finding invariant subspaces are required for showing a certain manifold is *not* G-curvature homogeneous.
- The Singer number theory of G-curvature homogeneity might also be interesting to explore.

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