Volume Comparison of 3-Braided Links with Dehn Fillings

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Abstract

This project focuses on braids, specifically 3-braided links. We start by explaining what braids and closed braids are, followed by defining Dehn fillings. Our goal is to compare the volumes of these 3-braided links under various Dehn fillings.

1 Introduction

A *knot* is a closed knotted strand in space, a projection contains n *crossings.* Crossings are the under and over strands in the knot. See Figure 1.

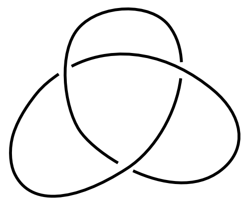


Figure 1: Trefoil knot has three crossings

A *link* contains multiple knots tangled with one another and the knotted loops in the link are called *components*. See Figure 2.

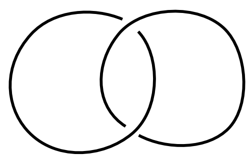
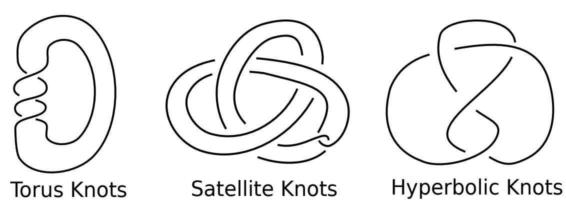


Figure 2: A link of two components

2 Braids

There are three different types of knots: torus knots, hyperbolic knots, and satellite knots.



A *braid* is a collection of n strings, which are attached to a bar at the top and at the bottom. See Figure 3. As you travel along the strings you always go down. Now if we glue the bottom bar onto the top bar of the braid, we will have a *closed braid*. See Figure 3. However, braids are not a subset of any one of three types stated above, but braids can represent knots. Any knot has a closed braid representative, so each type of knot- torus, statellite, and hyperbolic- are realized as closed braids.

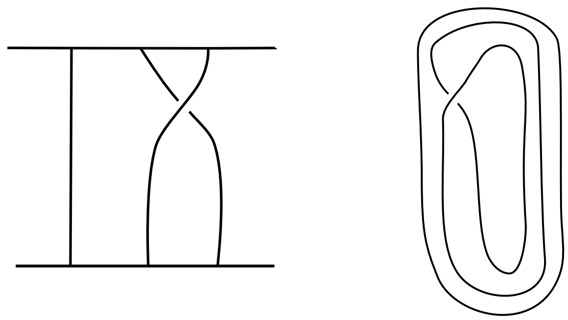


Figure 3: A braid and its closed braid

Throughout this project, we will be focusing on three braids.

3 Dehn Fillings

We'll specifically be focusing on $\frac{1}{n}$ Dehn fillings. Essentially a $\frac{1}{n}$ *Dehn filling* removes the crossing disk so that you can take the strands and create n full twists.

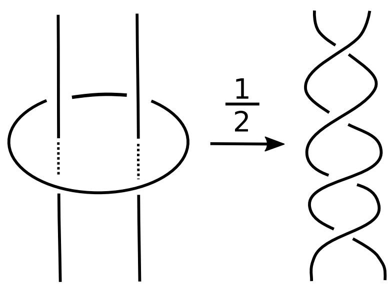


Figure 4: A Dehn filling with 2 full twists

In Figure 4, we have a zoomed in portion of a link with a crossing circle and two strands. If we do a $\frac{1}{2}$ Dehn filling, we will add 2 full twists to our strands. So we will have a braid with a total of 4 crossings. If we were to zoom out, we could have a three braided link such as Figure 5.

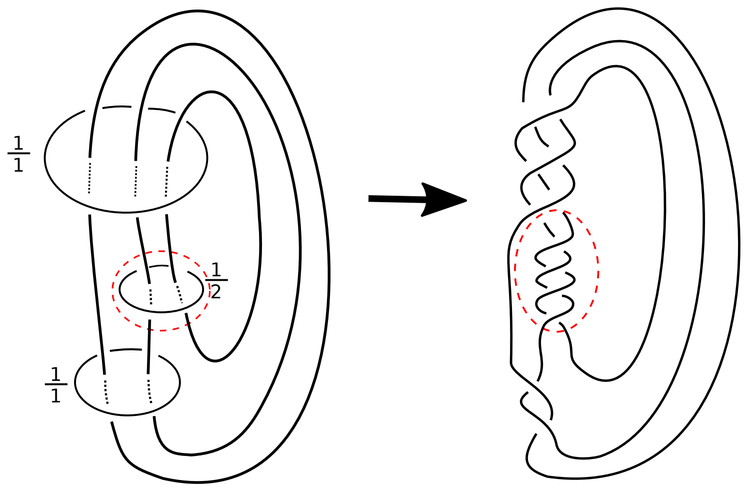


Figure 5: Dehn filling a 3-braid parent

4 Hyperbolic Volume

Remember knots can be classifed into three types: torus knots, satellite knots, and hyperbolic knots. For hyperbolic knots and links, it is a consequence of Mostow-Prasad rigidity that geometric invariants are link invariants. In particular, hyperbolic volume is a link invariant. This means if two hyperbolic links have different hyperbolic volumes, then they are different links. See Figure 5.

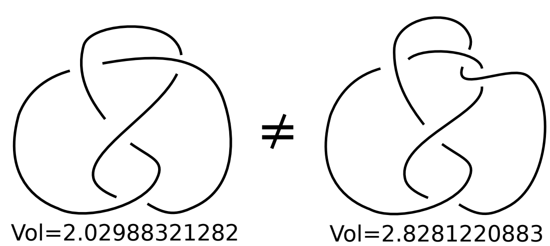


Figure 6: Different volumes of links

While many different links can have the same volume, a large number of links can be distinguished by their volumes. Now this leads us to an important question. Are the volumes of hyperbolic links the same after Dehn filling? The answer is no. According to Thurston's Hyperbolic Surgery Theorem, after Dehn filling a link, the volume of that link will be reduced.

5 Slopes Lengths

Consider a torus neighborhood of a component, known as a *cusp*, on the 3-braided parent. We will define s to be slopes on the cusp.

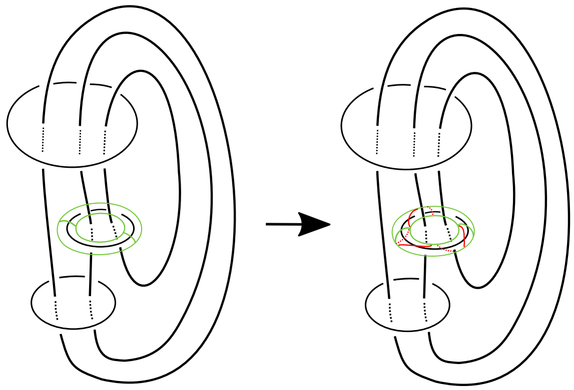
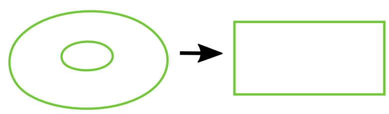


Figure 7: Slopes on a cusp

In the universal cover, cusps lift to a rectangle.



Also, a curve on a torus lifts to a slope. For instance, in Figure 7, a $\frac{1}{2}$ Dehn filling corresponds to 1 meridian and 2 longitudes. So, if we start at a point and we travel along the red slope line until we've hit the green line, then we've traveled one longitude and half a meridian. If we continue along the red slope line, we will have traveled another longitude and the final half of the meridian. In total we've traveled, 1 meridian and 2 longitudes. See Figure 7.

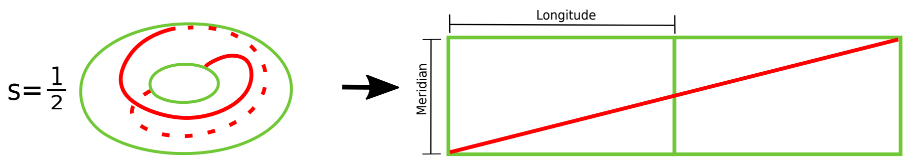


Figure 8: Meridians and Longitudes

With some work we get the rectangle to be a euclidean plane and find the length and width to be 4. Given this information, it is easy to find the slope length to be $\sqrt{80}$ of a $\frac{1}{2}$ Dehn filling using the Pythagorean theorem. See Figure 8.

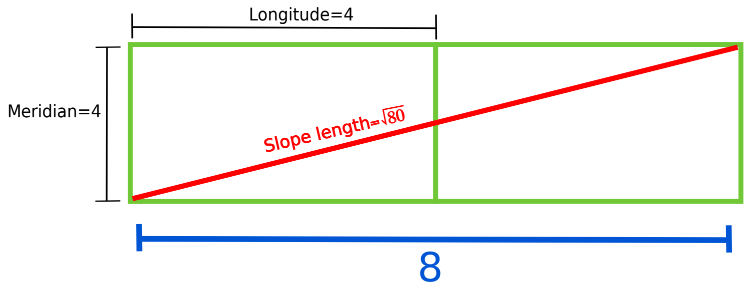


Figure 9: Slope length of a $\frac{1}{2}$ Dehn filling

More generally, we can create a general formula for slope length for $\frac{1}{n}$ Dehn fillings. That is, $4\sqrt{n^2+1}$. See Figure 9.

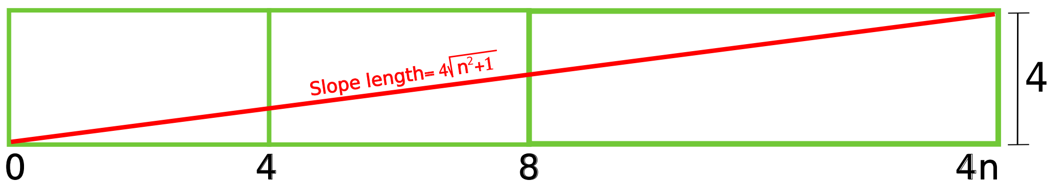


Figure 10: General slope length formula

Note that a $\frac{1}{3}$ Dehn filing has slope length $4\sqrt{10}$. Now that we know what slope lengths are, we can finally state a helpful theorem by Futer, Purcell, and Kalfagianni. This gives specific bounds on how much volume can drop under Dehn fillings, using slope lengths.

Theorem 5.1 (Futer,Purcell, Kalfagianni). *If M is a hyperbolic manifold, and $s_1, ..., s_k$ are slopes on cusps of M with minimum length $\ell_{\min}$ at least 2π, then the Dehn filled manifold $M(s_1, ..., s_k)$ is hyperbolic with volume bounded below by*

$$vol(M(s_1, ..., s_k)) \geq \left(1 - \left(\frac{2\pi}{\ell_{\min}}\right)^2\right)^{\frac{3}{2}} vol(M).$$

6 Results

Theorem 6.1. *If $|s_i| \leq \frac{1}{2}$ for all i=1, ..., 5 , then $vol(M(s_1, ..., s_5)) > vol(L(t_1, t_2, t_3))$ for any choice of t_i.*

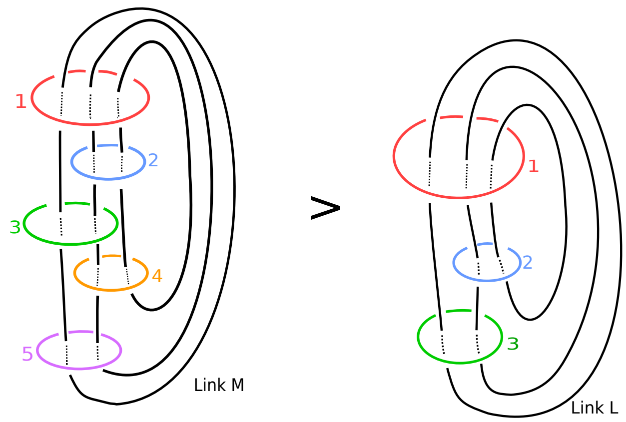


Figure 11: 3-Braid Parent

Proof. We will prove this result by cases. Link M can be made with 8 regular ideal octahedra, so $vol(M) = 8v_8 \approx 29.310899014$.Similarly Link L can be made with 4 regular ideal octahedra, and has volume $vol(L) = 4v_8 \approx 14.6$. Note that since $vol(L) > vol(L(t_1, t_2, t_3))$, the theorem follows if we show $vol(M(s_1, ..., s_5)) > vol(L(t_1, t_2, t_3))$. So, we will only focus on Dehn filling the link M.

Case 1. For $n = 2$. Up to symmetry, there are only five different Dehn fillings where $|n| = 2$.

Signage of Dehn Fillings	Volume
(+, +, +, +, +)	19.7107463818
(+, +, +, +, -)	19.879618055
(+, -, -, -, +)	19.066251260
(+, -, -, +, +)	19.511793846
(+, +, -, +, -)	19.9056421506

Figure 12: Different Dehn Fillings

Thus, these five volumes are greater than the volume of link L. See Figure 12.

Case 2. For $n \geq 3$. We have $|s_i| \leq \frac{1}{3}$. This implies for each crossing circle there is at least a Dehn filling of $\frac{1}{n}$. To prove that

$$vol(M(s_1, ..., s_5)) > vol(L(t_1, t_2, t_3)),$$

we will apply Futer,Purcell, and Kalfagianni's theorem

$$vol(M(s_1, ..., s_k)) \geq \left(1 - \left(\frac{2\pi}{\ell_{\min}}\right)^2\right)^{\frac{3}{2}} vol(M).$$

Since each $|s_i| \leq \frac{1}{3}$, every slope length will be at least $4\sqrt{10}$. So,

$$vol(M(s_1, ..., s_5)) \geq \left(1 - (\frac{2\pi}{4\sqrt{10}})^2\right)^{\frac{3}{2}} vol(M) = 19.162245632399.$$

Since

$$vol(M(s_1, ..., s_5)) > 19.162245632399 > 14.6 = 4v_8$$

and $4v_8 > vol(L(t_1, t_2, t_3))$, we have

$$vol(M(s_1, ..., s_5)) > vol(L(t_1, t_2, t_3)).$$

□

7 Acknowledgments

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8 References

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