BELTED SUM DECOMPOSITION OF FULLY AUGMENTED LINKS

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Abstract. Purcell and Adams have introduced notions of nerves and belted sums for fully augmented links (FALs). We prove that all nerves corresponding to FALs are made up of 3-cycles taking 1 of 3 forms. We show that nerve decomposition along one of these forms corresponds to cutting along a specific pair of thrice-punctured spheres in the nerve's FAL. Furthermore, this corresponds to a belt-sum decomposition such that the decomposition is made up of FALs. Finally, we show that decomposition can continue until reaching a finite set of prime FALs, and that this decomposition is unique for a given FAL. As an application, we take a closer look at octahedral FALs.

1. Introduction

Given two knots K_1 and K_2 we may, by removing a segment from both K_1 and K_2 and connecting their end points, form their *connected sum* $K_1 \# K_2$. The connected sum provides a notion of primeness and factorization for knots, where a knot K is called prime if K is not the connected sum of two others. While the existence of prime decompositions for connected sums is easy enough to see, the uniqueness of prime decompositions was not found until Shubert proved it in 1954. However, there may exist notions of prime decompositions for other classes of links.

Using Purcell's definition [4], *fully augmented links* (FALs) are obtained from links as follows. Starting with any link L, place a single unknotted component, called a *crossing circle*, over all twist regions and reduce the number of twists modulo 2 [4]. Call the surface formed by a single crossing circle a *crossing disk*. A result from Adams' allows us to remove all remaining half-twists so that any non-crossing circle component lies on the *plane of projection*, or reflection plane. Call these components *knot circles*.

Belted sums were first introduced by Adams in 1985 (see [1]). The process of belt-summing two FALs can be observed in Figure 1. Identifying two components of each distinct FAL, connect the strands intersecting each disk and fuse the identified components. For example, in Figure 1, c_1 of L_1 and c_2 of L_2 fuse to become c_{12} of $L_1 \oplus L_2$ where $\oplus$ denotes the process of belt summing two FALs. Our goal is to prove that every FAL decomposes uniquely as the belt-sum of prime FALs where we define an FAL to be *prime* if it is not the belt-sum of two others.

Utilized throughout our paper are thrice punctured spheres. Adams has shown that the process of belt-summing corresponds to the gluing of two thrice punctured spheres along exactly one of their punctures [1]. Furthermore, a rotation of a hemisphere of a thrice punctured sphere may lead to the removal of half-twists. This is the result we have referred to previously. Later on, we will show that, of the thrice punctures spheres possible in FALs, there exists only one pair of thrice punctured spheres that corresponds to a possible belt-sum decomposition.

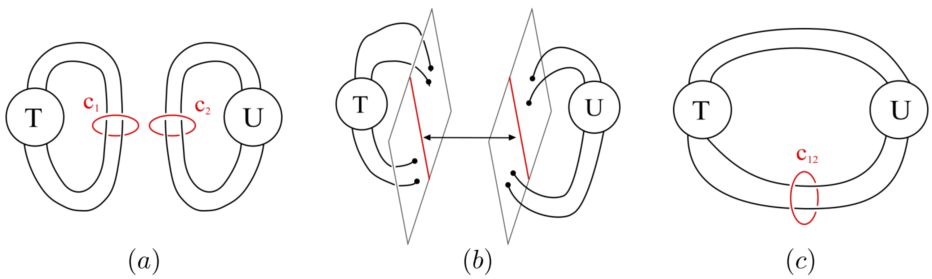

FIGURE 1. In (a), we identify two crossing circles to fuse. In (b), we take the crossing circle to infinity and connect endpoints of strands in L_1 and L_2 respectively. We reform the fused crossing circle in (c), and complete the belt-sum operation.

The other main tool we will use throughout our paper are combinatorial representations of the FALs, called *nerves*. Nerves can be obtained directly from FALs by taking the nerve dual, denoted nerve*, first (see Figure 2). For the rest of the paper we restrict our attention to hyperbolic FALs, since we are hoping to analyze the geometry of hyperbolic links. Starting with any hyperbolic FAL, the work of

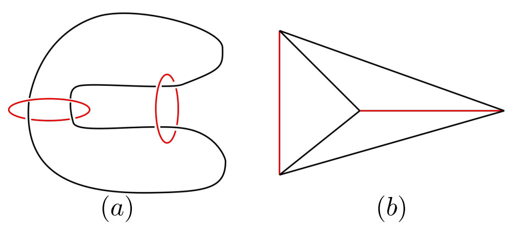

FIGURE 2. Consider the Borromean rings (a). We find nerve* (b) by placing painted edges perpendicular to crossing disk intersections with the projection plane. The knot strands coming out of crossing disks give us information to form unpainted edges. Take the dual of nerve* to arrive to the nerve of the Borromean rings (notice that the nerve* and the nerve are equivalent in this case).

Purcell gives us the following results (see [4]):

(1) Given a hyperbolic FAL, its nerve has no loops and no sets of multiple edges between vertices.
(2) Given any γ a triangulation of S^2 such that each edge has distinct ends and no two vertices are joined by more than one edge, we may choose a collection of edges of γ and paint them such that each triangle of γ meets exactly one red edge. Then this painted nerve corresponds to a hyperbolic FAL.
(3) An FAL is hyperbolic if and only if the associated knot or link is nonsplittable, prime (not a connected sum), twist reduced, with at least two twist regions.

Furthermore, we will say a cycle is nontrivial if there exist edges and vertices on its interior and exterior. Else it is trivial. We define a *central subdivision* of a nerve to be the process of placing a vertex in any face and then drawing single edges connecting the vertex with the three vertices in the boundary of that face. For example, one can obtain the nerve of the Borromean rings by centrally subdividing K_3 once (see Figure 2 (b)).

The paper is organized as follows. In section 2, we reveal the information that nerves possess related to the possible belt-sum decompositions of FALs. In section 3, we recall how to find thrice-punctured spheres in FALs and do casework to find adequate 3p. spheres for FALs and belt-summing. In section 4, we prove our first main theorem regarding the existence of belted sum prime decompositions and list the consequences of this fact. Finally, in section 5, we prove that this decomposition is unique for any given FAL. In section 6, as an application, we take a closer look at prime decompositions within a subfamily, the octahedral FALs.

Acknowledgement

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2. Combinatorics of Triangulations

We call a nerve N *well-painted* if it follows (2) of Purcell's results (see section 1). That is, N is well-painted if it is a triangulation of S^2 with no sets of multiple edges and each triangle has exactly 1 painted edge. Note that these triangles are trivial. We say a triangle is a trivial 3-cycle with exactly 1 painted edge. Since N is a triangulation of S^2, we know that every face is a triangle. Furhtermore, since each triangle has exactly 1 painted edge, and every painted edge borders 2 faces, we conclude that a single red edge borders two triangles. For any 3-cycles, we say it is nth-painted if it has n painted edges. Finally, we can say that if N is well-painted, then N corresponds to a hyperbolic FAL by Purcell.

In this section, we will describe the properties required of a painted nerve if it is to correspond to an FAL. Specifically, we will show that any 3-cycle in a nerve N of an FAL can take one of two forms, either once or thrice painted. Notice that if a 3-cycle is trivial, it must be a triangle, hence it is once painted by Purcell's results.

Theorem 1. *If a nerve N is well-painted, then all 3-cycles are either once-painted or thrice-painted.*

Proof. Consider N, a well-painted nerve. Since it is well-painted, we know it is a triangulation of S^2. Call this triangulation $\mathcal{T}$. Consider a 3-cycle C not bounding a face of $\mathcal{T}$. Hence, there exists two triangulations on either side of C, call these $\mathcal{T}_1$ and $\mathcal{T}_2$ respectively. It is a fact that any triangulation of S^2 has an even number of faces. Each face is a triangle as we have defined them. Hence, $C \cup \mathcal{T}_1$ yields an even number of triangles, so we know there exists an odd number of triangles on the first side of C. Similarly, we know there exists an odd number of faces on the other side of C.

Suppose there are n painted edges in $\mathcal{T}_1$ not including C. By Purcell, we know that each triangle is bounded by exactly 1 painted edge and 2 unpainted edge.

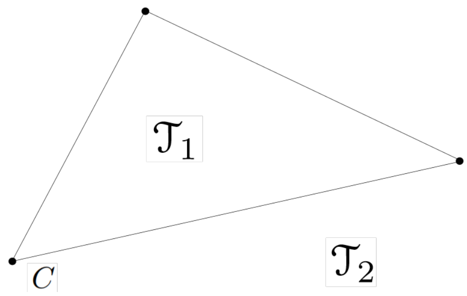


FIGURE 3. 3-cycle C with triangulations $\mathcal{T}_1$ and $\mathcal{T}_2$.

Furthermore, each painted edge borders exactly two triangles. So, within $\mathcal{T}_1$, there are $2n$ triangles bordered by a single painted edge in $\mathcal{T}_1$. However, since there are an odd number of triangles in $\mathcal{T}_1$, there is an odd number of triangles without painted edges using only edges exclusively within C. Since N is well-painted, this means that we may paint the odd number of triangles without painted edges in $\mathcal{T}_1$ using edges of C. Therefore, either 1 or 3 of the edges of C are painted to obtain a well painted nerve, as desired. □

Knowing that all trivial 3-cycles are once-painted and that all nontrivial 3-cycles are either once-painted or thrice painted in a well-painted nerve N, our goal is to find a way to look at the nerve and extract information, specifically how the corresponding FAL can be decomposed.

Suppose decomposing an FAL corresponds to splitting its nerve into two pieces along a 3-cycle (we will prove this in the following section). Then decomposing along a trivial 3-cycle is of no interest since decomposing will not change the original FAL, and a single triangle with one painted edge does not correspond to an FAL. Furthermore, decomposing along a thrice-painted 3-cycle corresponds to two nerves where a face is bounded by a thrice-painted 3-cycle, so the corresponding nerves are not well-painted. Hence, the only decompositions worth investigating given our definition of decompositions for nerves are along once-painted 3-cycles. We name these as follows.

Definition 2. A *buckle* is defined as a 3-cycle with one painted edge that does not bound a face.

3. Thrice-punctured spheres

Described by Purcell [4], for any FAL, there is a cell decomposition of a link complement into two identical polyhedra. This cell decomposition corresponds to the nerve of the FAL, so we proceed by trying to relate these sets of information. In the FAL complement, there are two 3-cells: the upper and lower-half spaces. If we were to continue the decompose until polyhedra, these two spaces would house the identical polyhedron. The 2-cells, as described by Purcell, are the plane of projection, or reflection plane since it splits the upper and lower half plane, and crossing disks. The 1-cells are the intersections of the 2-cells. There are no 0-cells in the cell-decomposition of an FAL complement.

Simple geodesic surfaces are found in plenty in an FAL. Purcell has proven that 2-cells are totally geodesic, implying that all 1-cells are as well (given they are formed by the intersection of 2-cells). Thrice punctured spheres are also found often in the complement of FALs and are totally geodesic (a result of Adams).

There are unique hyperbolic structures of thrice punctured spheres in the complement of an FAL $\mathcal{F}$. To find punctures, place a sphere S on the link complement. Wherever there curved intersection of the boundary of S and the toroidal Neighborhood of the link complement $N(\mathcal{F})$, the sphere gains a puncture. Note then that these punctures may be modeled in any linear combination given by $S \cap N(\mathcal{F}) = pm + q\ell$ where $p, m \in \mathbb{N} \cup \{0\}$ such that pm is the number of meridians and $q\ell$ is the number of longitudes. Furthermore, note that $N(\mathcal{F})$ is a solid unknotted torus surrounding each link component, since all components in $\mathcal{F}$ are unknotted.

By the nature of thrice punctured spheres in general, we already know much about the geodesics we can form. Given two punctures, we know there may exist a unique simple geodesic between them and that, given a single puncture, there may exists a unique simple geodesic from itself to itself. This implies that there exist 6 possible unique geodesics in a thrice-punctured sphere, 3 from one puncture to another, and 3 from a single puncture to itself. See Figure 4.

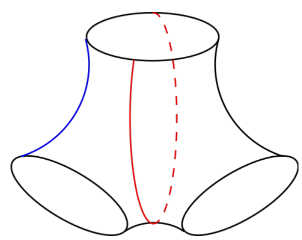


FIGURE 4. In red is a possible geodesic from one puncture to itself. In blue is a possible geodesic from one puncture to another.

The belt sum decomposition of FALs relies on finding pairs of thrice-punctured spheres in the complement of the link. Therefore, before we can belt sum decompose along FALs, we must find all types of thrice-punctured spheres that can live in the complement of a fully augmented link. After that, we must see which of these will correspond to a belt sum. So, given S, a thrice punctured sphere in the complement of an FAL $M_{\mathcal{F}} = S^3 - \mathcal{F}$, we proceed by considering how S intersects the standard cell decomposition of $M_{\mathcal{F}}$. For example, if, for a crossing disk D, $S \cap D$, then the intersection must be a simple geodesic: i.e., the intersection must be some combination of those seen in figure 5 such that the geodesics are simple and non-intersecting. We will use this result often in following work.

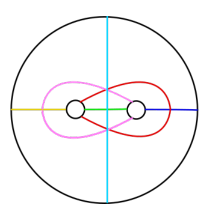


FIGURE 5. Each color denotes a different possible geodesic in the thrice punctured sphere of a crossing circle.

Our first goal will be to put bounds on the possible puncture types. A major tool we will use is the Jordan curve theorem. Recall:

Theorem 3. *(Jordan Curve Theorem) If J is a simple closed curve in $\mathbb{R}^2$, then $\mathbb{R}^2 - J$ has two components, an inside and an outside, with J the boundary of each.*

Since spheres form closed curves on the projection plane, we can apply this to counting knot strands on the plane of projection as follows.

Corollary 4. *If S is a sphere intersecting the projection plane of an FAL $\mathcal{P}$ at more than a single point, then the boundary of $S \cap \mathcal{P}$ will have an even number of intersections with knot strands.*

We start by putting a bound on the number of longitudes for punctures. This will aid in our desired outcome: showing that there exist only three types of punctures in the complement of an FAL.

Proposition 5. *If S is a thrice punctured sphere and C is a crossing circle, then $S \cap N(C)$ can have no more than one longitude, where $N(C)$ is the toroidal neighborhood of C.*

Proof. Suppose for contradiction that $S \cap N(C)$ contains 2 or more longitudes, that is $S \cap N(C) = pm + q\ell$ where $q \geq 2$. Then $S \cap N(C)$ crosses the plane of projection in at least four places. Recall that all two-cells are totally geodesic and all one-cells

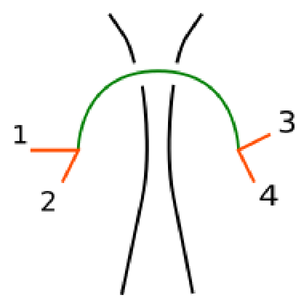


FIGURE 6. Since we have more than one longitude in $S \cap N(C)$, there must exist more than one geodesic exiting both ends of crossing circle C (pictured in green). These geodesics are labeled 1, 2, 3, and 4.

form simple geodesics.

Case 1: Suppose there exists a simple geodesic on the same side of c. In order to be simple, this geodesic must encircle at least one puncture (as shown in figure 7). This puncture must be bounded by a knot circle, so we can say that the geodesic encircles a knot circle. Then it is clear that if the geodesic does not intersect another component of the link, the link must be splittable, and not an FAL. However, if it does intersect another component, then this geodesic produces at least 2 punctures on a single side of C. Together with the boundary of C, this produces 3 punctures. However, this implies that the other side of C cannot produce any punctures. This is impossible, as our previous logic applies to each side of C

Case 2: Suppose that each point of intersection forms a geodesic to a puncture of S. Then each point of intersection forms a geodesic with a different puncture,

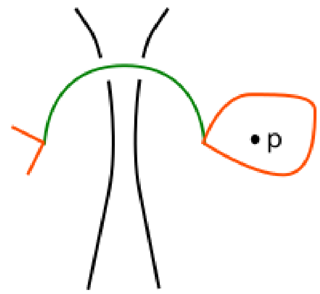


FIGURE 7. Geodesics encircling a puncture p.

since each geodesic must be unique. Since there are at least 4 intersections, there are at least 4 punctures, contradicting the assumption that S is a thrice punctured sphere.

Thus S is not a thrice punctured sphere if $S \cap N(C)$ contains two or more longitudes. So we may conclude that if S is a thrice punctured sphere, then $S \cap N(C) = pm + q\ell$ with $q \in \{0, 1\}$. □

The next lemma will be helpful in the proofs to come. We will prove it by finding compression disks and considering the consequences of their existence.

Lemma 6. *If S is an embedded thrice punctured sphere where two of the punctures are meridians, then the third must be given by a longitude.*

Proof. Consider a thrice punctured sphere where two meridians give two punctures. Suppose we cap the punctures given by link components K_1 and K_2 corresponding to each meridian with meridinal disks D_1 and D_2. This gives an embedded disk $D = S \cup D_1 \cup D_2$. Consider the manifold $M = S^3 - N(K_3)^{\circ}$ where K_3 is the link component corresponding to the curve c_3 on S. Then2 the boundary of M is $\partial N(K_3)$ and is given by the union of surfaces S, D_1, and D_2. Therefore, D is properly embedded in M since it is both embedded and $D \cap \partial M = \partial D$ where ∂ denotes the boundary of the surface. However, this forces D to be a compressing disk for $N(K_3)$ since ∂D is essential on ∂M, that is, it cannot be taken to a point on ∂M.

Notice that $N(K_3)^{\circ}$ is a solid unknotted torus, hence $M = S^3 - N(K_3)^{\circ}$ is another solid unknotted torus. Let $T(K_3)$ be the surface of $N(K_3)$. Then $\partial M = T(K_3)$. D is a compressing disk for $N(K_3)$, so D is also a compressing disk for $T(K_3)$. So ∂D is a meridian of M, implying that it is a longitude of $N(K_3)$. See figure 8.

□

Corollary 7. *If S is a thrice punctured sphere and C is a crossing circle, then if $S \cap N(C)$ does not contain any longitudes, it can have no more than two meridians.*

From here, we can start looking at the case where a sphere has a knot circle longitude as a puncture. Since knot circles lie on the plain of projection, they are totally geodesic. Because crossing disks are also geodesic, a knot circle intersects crossing disks in simple geodesics. Therefore, if a sphere has a longitudinal puncture along a knot circle, it intersects each crossing disk that knot circle intersects in a

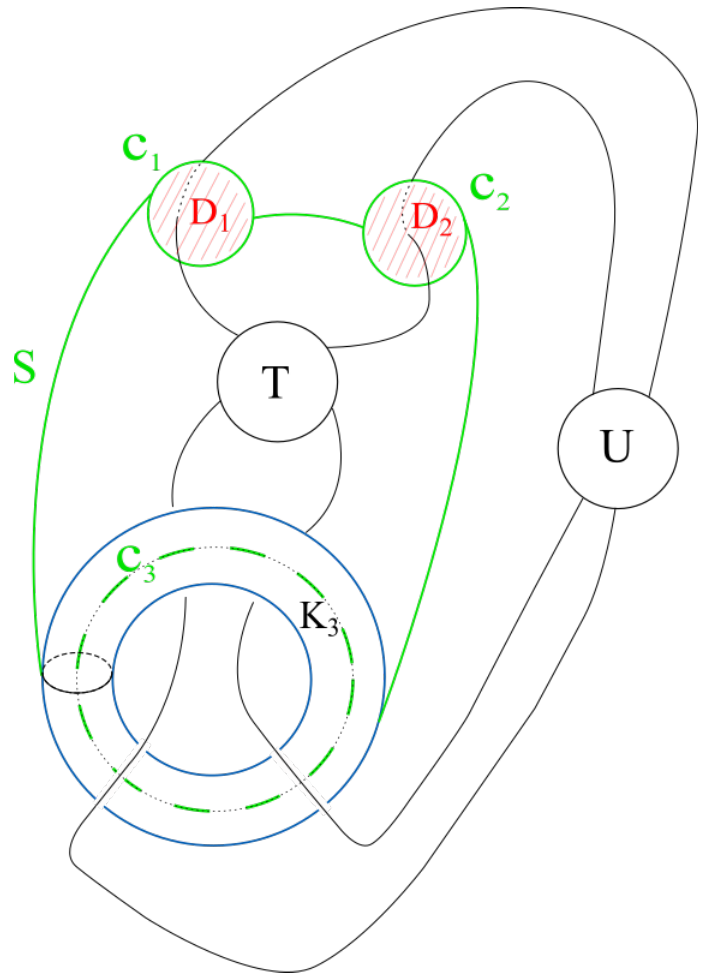
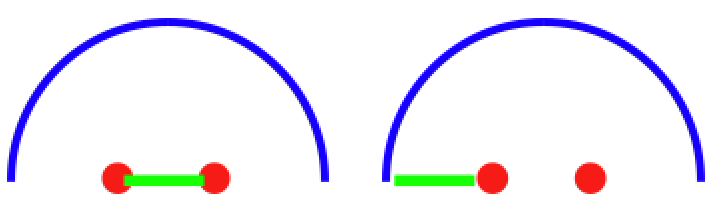


FIGURE 8

simple, unique geodesic. There are two options for how this geodesic may be formed, shown in green in figure 9 (a) and (b).



(a) (b)

FIGURE 9. The two possible intersections between a crossing circle and a punctured sphere that includes a longitude of a knot circle.

In order to determine how many punctures a sphere with a knot circle longitude may contain, we will examine an arbitrary knot circle in an FAL, and consider the possible geodesics that a sphere can form in each of its crossing disks.

Lemma 8. *If a puncture of a sphere includes a longitude along the neighborhood of a knot circle of an FAL, then that sphere has at least $n-1$ punctures, where n is the number of intersections between that knot circle and crossing disks. If that knot circle does not intersect any crossing disk twice, then the sphere has a minimum of $n+1$ punctures.*

Proof. Suppose S is a sphere in the complement of an FAL that has a longitudinal puncture along a knot circle, K. Then each time K intersects a crossing circle, S must intersect it in one of the two ways shown in in Figure 9. Let C be a crossing disk that K intersects.

Case 1: Consider the case where $C \cap S$ is the one given in Figure 9 (a), and the two knot circles passing through the crossing circle are the same. Then this geodesic does not introduce another puncture to the sphere. However, as geodesics are unique, there cannot be another one that connects the knot circle to itself elsewhere along the knot circle.

Case 2: Let the intersection be again the one given in Figure 9(a), but where the two knot circles are different. Then the sphere includes a puncture that is either a meridian in the neighborhood of the second knot circle or one that includes a longitude of that neighborhood. Then there may be further punctures resulting from this second knot circle intersecting other crossing disks. Therefore, at least one puncture is introduced from this intersection with the crossing disk.

Case 3: If $S \cap C$ is the one shown in Figure 9(b), then the sphere has a puncture that forms a meridian in the toroidal neighborhood of C.

Let n be the number of intersections between K and crossing disks. In Cases 2 and 3 above, every intersection between K and a crossing circle produces at least one puncture in the sphere. Case 1 accounts for two intersections between K and crossing circles, but only forms one geodesic. Furthermore, this can only occur for one crossing disk intersecting K. Thus, if K passes through the same crossing circle twice, there will be at least $n-2$ punctures resulting from intersections with crossing circles, and n punctures otherwise. However, as K itself contains a puncture in the sphere, the sphere has at least $n-1$ punctures if it intersects the same crossing disk twice, and $n+1$ punctures if it doesn't.

□

This narrows down the number of thrice punctured spheres that can have a knot circle longitude as a puncture, because there are now very specific numbers of crossing disks that can intersect a knot circle such that the sphere bound by it will have three punctures.

Proposition 9. *If a thrice punctured sphere has a longitude in the neighborhood of a knot circle of an FAL as one of its punctures, the sphere is of one of the forms shown in Figures 10.*

Proof. Let S be a thrice punctured sphere in the complement of an FAL that contains a longitudinal puncture along some knot circle K. Let n be the number of times K intersects a crossing disk. Then by lemma 8, $n < 5$. Furthermore, if $n = 0$

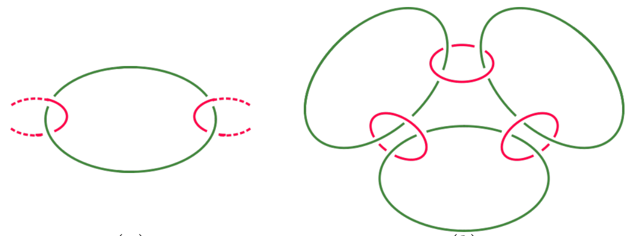


FIGURE 10. The two possible thrice punctured spheres that include a longitude of a knot circle. The green represents the knot circle that has a puncture, and the red represents the crossing circles. Note that the partial crossing circles shown in (a) need not be distinct.

or $n = 1$, then the link is splittable, making it not fully augmented. Therefore $n \in \{2, 3, 4\}$.

Case: $n = 2$. If K passes through the same crossing circle twice, then it does not pass through any other crossing circles. This is clearly splittable, and so not an FAL.

If K passes through two different crossing circles, by Lemma 8, S can be a thrice punctured sphere. Then S forms two geodesics. If one of the geodesics resembles figure 9(a), then S forms either a meridinal or longitudinal puncture along a second knot circle, K_2. Suppose that S forms a meridian on K_2. Then there must exist another simple geodesic in the crossing disk from K_2. Then there exists a geodesic from K_2 to the crossing circle, adding a fourth puncture. Therefore S must form a longitudinal puncture on K_2.

Now suppose one of the crossing disks resembles figure 9(a) and the other resembles figure 9(b). Then S forms a second longitudinal puncture along a second knot circle, K_2. Then K_2 must intersect at least two crossing disks, so it must form a unique geodesic to another knot circle, thus adding a fourth puncture. Therefore the crossing disks that K intersects cannot have different looking geodesics.

Thus we are left with the case where both of the crossing disks have geodesics resembling figure 9(a), and S forms two other longitudinal punctures on different knot circles, or the case where both geodesics resemble figure 9(b). Then S can be the sphere shown in Figure 10(a) or (b).

Case: $n = 3$. If the knot circle passes through the same crossing circle twice, we get the local picture shown in Figure 11, which is splittable, and so not an FAL.

If the knot circle doesn't pass through the same crossing circle twice, then by Lemma 8, the sphere has four punctures instead of three.

Case: $n = 4$. By Lemma 8, to get a thrice punctured sphere when $n = 4$, the knot circle must pass through the same crossing circle twice. In this case, we get the knot circle shown in Figure 12. However, supposing that the large crossing circle shown is contained elsewhere in the link diagram in Figure 10 (a) allows Figure 12 to be a subcase of the former.

□

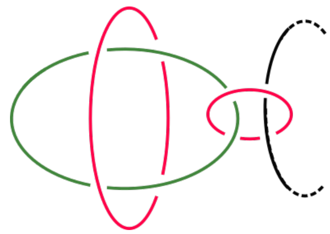


FIGURE 11. The local diagram of a splittable link.

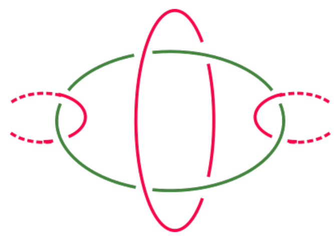


FIGURE 12. The local diagram of an $n = 4$ case.

We proceed to to consider the case where a sphere has a puncture that forms a longitude and one or more meridians on a crossing disk. That is, if S is a sphere and C is a crossing disk, $S \cap N(C)$ forms a curve along the toroidal neighborhood that has both longitudes and meridians. We wish to show that this cannot happen in a thrice punctured sphere. Notice that by proposition 9, if S is a thrice punctured sphere that forms a crossing circle longitude, then S cannot have any knot circle longitudes. This result will come in handy as we consider spheres that form punctures along crossing disks.

Proposition 10. *If S is a thrice punctured sphere and C is a crossing circle for an FAL, and $S \cap N(C)$ contains one longitude, then $S \cap N(C)$ may not contain any meridians.*

Proof. Let S be a thrice punctured sphere, and let L be a fully augmented link with a crossing circle C. Assume for contradiction that $S \cap N(C) = \ell + qm$, where ℓ corresponds to 1 longitude, m corresponds to one meridian, and q is a positive integer.

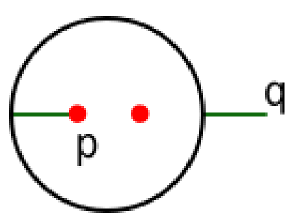
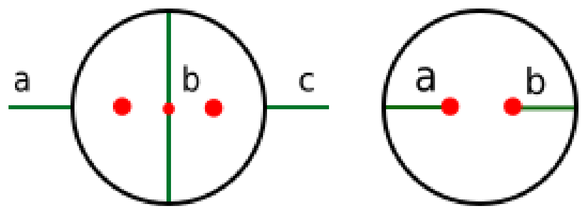


FIGURE 13. Case 1

Case 1: Suppose $S \cap N(C) = \ell + m$. Then the interior of C appears as shown in figure 13. Then p forms a puncture along a knot circle (call this knot circle

K). By proposition 9, S cannot form a longitudinal puncture along K, so p must form a meridian. Then q must form a geodesic to the rest of S that creates at most 1 puncture (otherwise S contains too many punctures). By proposition 9, q cannot form a longitude along a knot circle. Furthermore, by corollary 4, if q forms a longitude along a crossing circle, it would have to form an additional meridian (otherwise the geodesics on the projection plane are not unique). Therefore q must form a meridian. Then by lemma 6, the third puncture must be a longitude. Thus we arrive at a contradiction, so $S \cap N(C) \neq \ell + m$.

Case 2: Suppose $S \cap N(C) = \ell + 2m$. Then there are two possible configurations for how S intersects the toroidal neighborhood of C, each shown in figure 14.



Case 2a   Case 2b

FIGURE 14. In these figures, a, b, and c are geodesics along the plane of projection.

- *Case a:* Suppose the interior of C resembles the first figure. Then S must be connected by unique geodesics a, b, and c. Then, in order for a to connect to the interior of C, there must exist a closed section of the link L between a, b, and C. Therefore, if a forms a longitude along a crossing circle, by corollary 4, there exists an additional puncture in a or b along the meridian of a knot circle, which would form too many punctures in S. Additionally, by proposition 9, a cannot form a longitude on a knot circle, so a must form a meridian by construction. Similarly, c must form a meridian. Then there are two meridians in S. Then by lemma 6, the third puncture must be a longitude. This contradicts our assumption that the puncture around C is equal to to $\ell + 2m$.

- *Case b:* Suppose the interior of C resembles the second figure. Note that a and b must be intersecting two different components, otherwise they would not be unique geodesics. Furthermore, by proposition 9, a and b form meridians. Then by lemma 6, the third puncture must be a longitude. This contradicts our assumption that the puncture around C is equal to to $\ell + 2m$.
  Thus in either case we arrive at a contradiction. Therefore we may conclude that $S \cap N(C) \neq \ell + 2m$.

Case 3: Suppose $S \cap N(C) = \ell + 3m$. Then the interior of S appears as in figure 15.

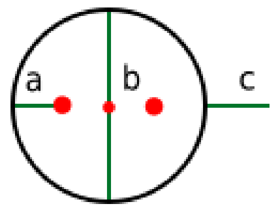


FIGURE 15. Case 3

Using a similar argument as in Case 1, a must form a meridian on the knot circle. By a parallel argument to the one used in case 2a, c must form a meridian along its geodesics. Then there exist two meridians, so by lemma 6 the third puncture of S must be a longitude. Thus we arrive at a contradiction, so $S \cap N(C) \neq \ell + 3m$.

Case 4: Suppose $S \cap N(C) = \ell + 4m$. Then the interior of S is described by figure 16.

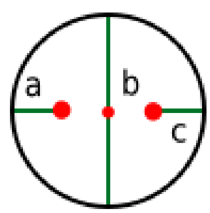


FIGURE 16. Case 4

Using a similar argument to the one used in case b of Case 2, a and c must be meridians, so the third puncture of S must be a longitude. This contradicts our assumption, so it cannot be true that $S \cap N(C) = \ell + 4m$.

Case 5: Suppose $S \cap N(C) = \ell + qm$ where $q > 4$. Then there do not exist enough unique geodesics on the interior of C such that they don't intersect. So this cannot be true.

Thus $S \cap N(C) \neq \ell + qm$ where $q > 0$. Therefore we may conclude that if $S \cap N(C)$ contains one longitude, then $S \cap N(C)$ may not contain any meridians. □

Corollary 11. *If $S \cap N(C) = \epsilon\ell + \delta m$, either ϵ, δ, or both must be zero with $\epsilon \in \{0, 1\}$ and $\delta \in \{0, 1, 2\}$. That is, punctures are given by 1 longitudinal curve, 1 meridinal curve, or 2 meridinal curves in $S \cap N(C)$.*

With this important corollary, we are almost ready to conclude the exact number of thrice punctured spheres in an FAL that don't have a knot circle longitude. Before we can do this, we need to eliminate the case where a thrice punctured sphere forms a single meridian along a crossing circle (although this can happen, it requires a knot circle longitude as a puncture).

Lemma 12. *If S is thrice punctured sphere and C is a crossing circle, and $S \cap N(C) = m$, then another puncture of S must be a longitude of a knot circle.*

Proof. Let S be a thrice punctured sphere and let C be a crossing circle for a FAL L. Suppose S has a puncture along a $1m$ curve. Then S must form a geodesic on the interior of C. Since S only intersects the interior of $N(C)$ once, the only possible geodesic, up to symmetry, is described by figure 17.

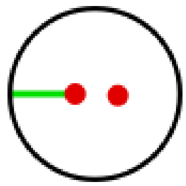


FIGURE 17

Therefore, S must intersect a knot circle through C (let this knot circle be called K). Then one of the punctures of S is either a meridian or a longitude along K. If the second puncture is a longitude, then this proof holds. Now consider the case where the second puncture forms a meridian on K. Then by lemma 6, the third puncture of S must be a longitude.

If the third puncture of S is a longitude along a crossing circle, then a closed region is created along the plane of projection between the two crossing disks that S forms longitudes on and S. Then, in order for all geodesics to be unique, S must intersect another component on the projection plane, adding a fourth puncture. Thus we arrive at a contradiction and may conclude that the third puncture of S must be a longitude along a knot circle.

□

With this elimination, we have now narrowed down enough cases to consider all possible ways a thrice punctured sphere can be formed on a FAL, assuming a puncture is not a knot circle longitude.

Proposition 13. *All possible thrice punctured spheres in an FAL that aren't punctured along a longitude of a knot circle can be represented by the forms shown in Figure 18.*

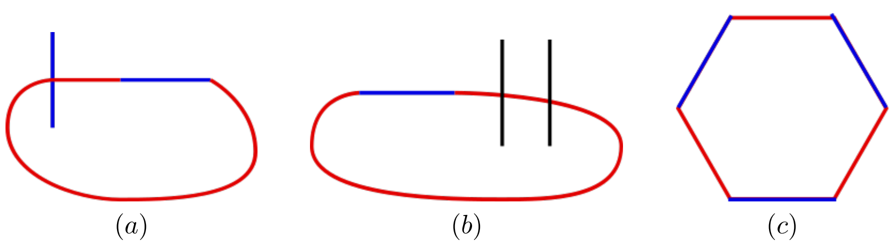


FIGURE 18. Three possible thrice punctured spheres. Red represents the spheres, blue for crossing circles, and black for knot circles.

Proof. Consider a thrice punctured sphere in an FAL that does not contain a longitude of a knot circle. By Lemma 12, this sphere also doesn't have a puncture as a $1m$ curve along a crossing circle. Then, by corollary 11, there are three possible ways for this sphere to be punctured: as a $2m$ curve along a crossing circle, a 1ℓ curve along a crossing circle, and a $1m$ curve along a knot circle (see figure 19). Note that Figure 19(a) creates two punctures, while the other two have one puncture each. The knot subdiagrams shown in Figure 20 are cases in which a thrice punctured sphere could conceivably be constructed using these punctures. These, as well as those in Figure 18 are the only ways to construct a sphere in an FAL with exactly three punctures. We will show that the thrice punctured spheres in Figure 20 cannot be constructed.

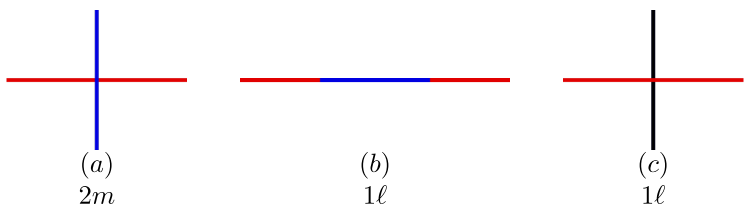


FIGURE 19. Ways to puncture a sphere.

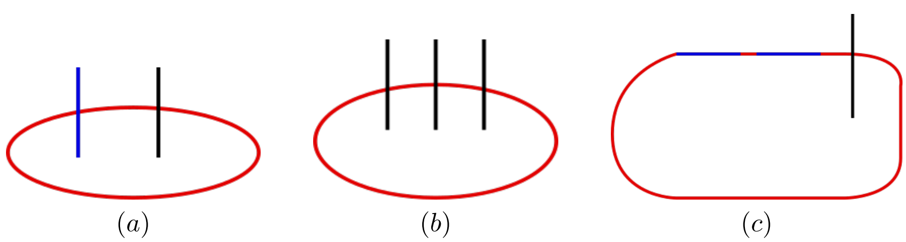


FIGURE 20. Conceivable thrice punctured spheres

Consider figure 20(a). The thrice punctured sphere partitions the plane into two sections, one inside and one outside, with one knot circle crossing this boundary. By Corollary 4, this cannot happen. As such, figure 20(a) does not represent a possible thrice punctured sphere.

Similarly, if we look at figures 20(b) and (c), we will see that the thrice punctured spheres there partition the planes into inside and outside regions. In both cases, there are an odd number of knot circle components which cross between the inside and outside. This contradicts Corollary 4.Thus, the spheres in figure 20 are all impossible to fully construct in an FAL complement.

Notice that in each case in figure 18, when a thrice punctured sphere partitions the plane, there are an even number of knot circle strands that cross the boundary from outside to inside, as is allowed by Corollary 4. This means that these three cases are the only possible thrice punctured spheres that can be found in FALs that do not have punctures along the longitudes of knot circles. □

With this result, we now know all of the thrice punctured spheres that can exist with a knot circle longitude, and the ones without a knot circle longitude. Therefore, we have found all possible spheres in an FAL.

Corollary 14. *By propositions 9 and 13, there are exactly 5 different ways to form a thrice punctured sphere on an FAL.*

The five ways to form a thrice punctured sphere are described in Figures 10 and 18.

Definition 15. Let $M = S^3 - F$, where F is an FAL. Suppose S_1 and S_2 are two embedded, disjoint thrice punctured spheres who share the boundary of one puncture. If $M - (S_1 \cup S_2)$ creates two disconnected manifolds, then S_1 and S_2 yield a belt sum decomposition of F.

We proceed to determine which of the thrice punctured spheres we have found can be belt sum decomposed along. The following lemma is an important eliminator of cases when considering all of our possibilities.

Lemma 16. *Let S_1 and S_2 be thrice punctured spheres that share a puncture in $S^3 - F$ where F is an FAL. If $\partial(S_1 \cup S_2)$ contains the longitude in any neighborhood $N(\kappa)$ of knot or crossing circle κ, then the combination of S_1 and S_2 does not correspond to a belted sum.*

Proof. Suppose S_1 and S_2 share a boundary at a single puncture. Furthermore, $\partial(S_1 \cup S_2)$ contains the longitude in $N(\kappa)$. S_1 retracts onto the eyeglasses graph (see figure 21). Hence, S_1 retracts to a 1-manifold. Furthermore, if one slices along S_2, $M - S_2$ remains connected, since no thrice punctured sphere will disconnect a 3-manifold. So, $M - (S_1 \cup S_2)$ remains connected. By definition 15, this means that the combination of thrice punctured spheres S_1 and S_2 does not correspond to a belted sum. □

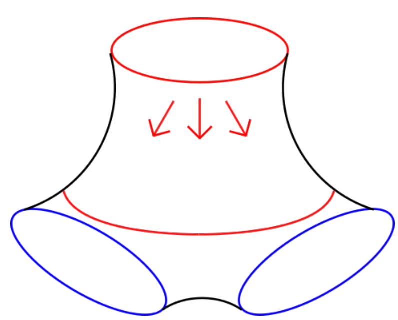


FIGURE 21. A retraction of one puncture towards the curves within the torus boundary of the other two.

Lemma 16 tells us that if S_1 and S_2 are thrice punctured spheres, and there is a knot circle longitude on $S_1 \cup S_2$ that is not a shared boundary, then S_1 and S_2 don't correspond to a belt sum decomposition.

Proposition 17. *There are exactly three ways to belt sum decompose along a thrice punctured sphere in a fully augmented link.*

Proof. Let S_1 be a thrice punctured sphere in a FAL L. Let $M = S^3 - L$. By corollary 14, there are exactly 5 forms S_1 could take.

Case 1: Suppose S_1 corresponds to figure 22.

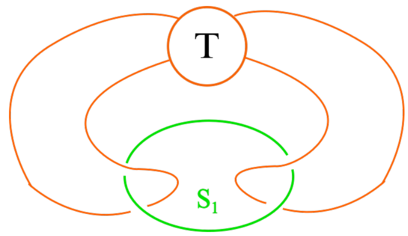


FIGURE 22. Case 1

Suppose there exists a thrice punctured sphere S_2 that shares a puncture with S_1. Then by lemma 16, S_2 must share a longitude with S_1. Hence, it must be one of (a) or (b). If $S_2 =$(a), then S_2 would be the same thrice punctured sphere as S_1, yielding a contradiction. Therefore $S_2 =$(b), as shown in figure 23.

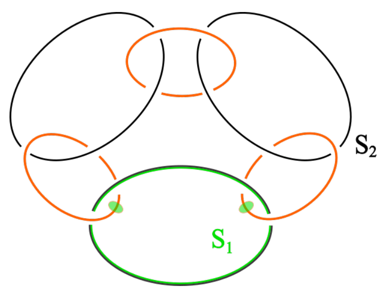


FIGURE 23. Case 1(a)

However, because S_2 has two other longitudinal punctures, by lemma 16, S_1 and S_2 do not correspond to a belt sum decomposition.

Case 2: Now suppose S_1 corresponds to figure 24.

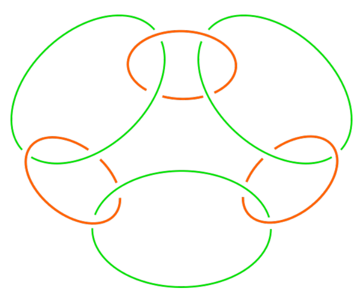


FIGURE 24. Case 2

Let S_2 be a thrice punctured sphere sharing a puncture with S_1. Then S_1 contains two other longitude punctures not shared with S_2. Then by lemma 16, S_1 and S_2

do not correspond to a belt sum decomposition.

Case 3: Suppose S_1 resembles the sphere in figure 25. Let S_2 be a thrice-punctured sphere sharing a boundary with S_1. Then S_2 must share a crossing circle boundary with S_1, but S_2 cannot share a crossing circle boundary with the other two longitude crossing circles in S_1. Then by lemma 16, S_1 and S_2 do not correspond to a belt sum decomposition.

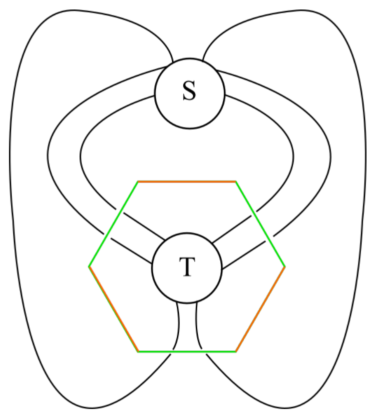


FIGURE 25. Case 3

Case 4: Suppose S_1 resembles the sphere in figure 26. Then let S_2 be a thrice punctured sphere sharing a boundary with S_1. Then by lemma 16, S_2 must share a crossing circle longitude with S_1. Then S_2 may resemble any of the spheres described in figure 18. However, by case 3, S_2 cannot resemble figure 18(c). Then S_2 may either be 18(a) or 18(b).

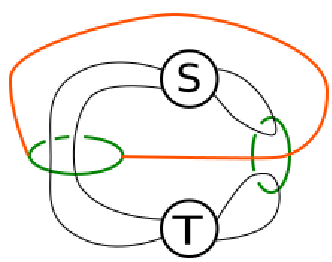


FIGURE 26. Case 4

- *Case (a)* Suppose $S_2 = 18(a)$. Then S_2 must form two meridians along a different crossing circle than the one S_1 forms meridians along. This can be seen in figure 27. Because the boundary of $S_1 \cup S_2$ creates a closed, nontrivial region of M, $M - (S_1 \cup S_2)$ is divided. Therefore this is a way to belt sum decompose along a FAL.

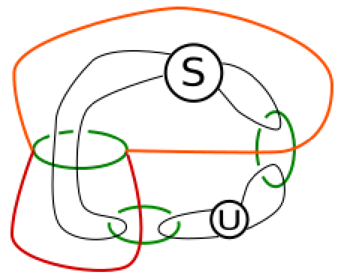


FIGURE 27. Case 4(a), where S_2 forms two meridians along another crossing circle.

- *Case (b)* Now consider the case where S_2 = 18(b). Then S_2 forms two meridians along a knot circle. This can be described by figure 28. Because the boundary of $S_1 \cup S_2$ creates a closed, nontrivial region of M, $M - (S_1 \cup S_2)$ is divided. Therefore this is a way to belt sum decompose along a FAL.

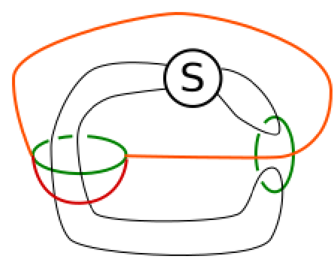


FIGURE 28. Case 4(b), where S_2 forms two meridians along a knot circle.

Case 5: Suppose S_1 resembles the sphere in figure 29. Then let S_2 be a thrice punctured sphere sharing a boundary with S_1. Then by lemma 16, S_2 must share a the crossing circle longitude with S_1. Then, similar to in case 4, S_2 =18(a) or S_2 =18(b).

- *Case (a)* Consider the case where S_2 =18(a). Then $S_1 \cup S_2$ is structurally identical to case 4(b), so this is the same way to belt sum along a FAL.

- *Case (b)* Now suppose S_2 =18(b). Then the behavior of these two spheres is shown in figure 30. Because the boundary of $S_1 \cup S_2$ creates a closed, nontrivial region of M, $M - (S_1 \cup S_2)$ is divided. Therefore this is a way to belt sum decompose along a FAL.

Thus cases 4(a), 4(b), and 5(b) illustrate all possible ways to belt sum along L. So we may conclude that there are exactly three ways to belt sum decompose along a FAL. □

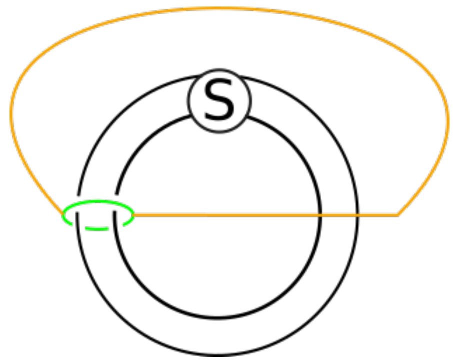


FIGURE 29. Case 5

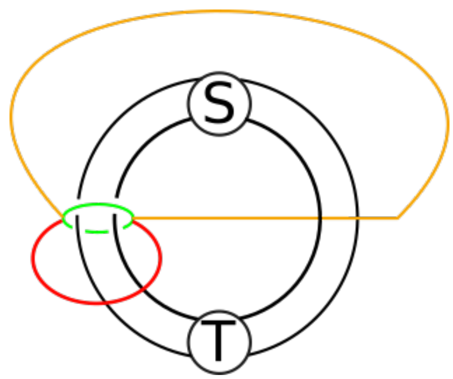


FIGURE 30. Case 5(b), where S_1 and S_2 both form two meridians along knot circles.

4. Existence of Belted Sum prime decompositions

Now that we have found the ways to belt sum decompose along a fully augmented link, we will show the existence of a belt sum decomposition in all FALs. This means that given any FAL, we can determine if it is the belt some of other FALs, and what prime links' belt sums create it. In order to prove this existence, we will rely on one specific way to belt sum decompose (of the three proven in proposition 17). We will also draw important parallels between the existence of a belt sum decomposition and the existence of buckles in the nerve of a FAL. We will start by briefly revisiting types of thrice punctured spheres.

Lemma 18. *If an FAL L has a thrice punctured sphere S_1 with punctures given by 1 crossing circle longitude and 2 crossing circle meridians, then there exists a different S_1' that has punctures given by 1 crossing circle longitude and 2 knot circle meridians.*

Proof. If we have a thrice punctured sphere given by 1 crossing circle longitude and 2 crossing circle meridians, then we must have S_1 given by the solid orange line in figure 31 by our previous work. Now, replace the two punctures given by crossing circle meridians with knot circle meridians, shown by the dotted orange line in figure 31. This corresponds to S_1'. $\square$

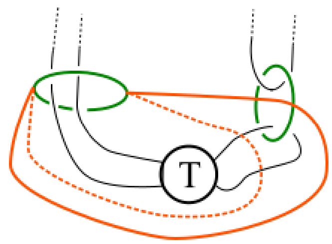


FIGURE 31. Whenever there exists a thrice punctured sphere given by the solid orange line, there exists a thrice punctured sphere given by the dotted orange line.

Proposition 19. *If an FAL can be belt sum decomposed, then there exists a pair of thrice punctured spheres that each contain one crossing circle longitude and two knot circle meridians.*

Proof. Suppose an FAL L may be belt sum decomposed. Then, by proposition 17, L has three possible configurations (see figures 27, 28 and 30). However, by Lemma 18, figure 27 and figure 28 also have different thrice punctured spheres, shown in figure 32.

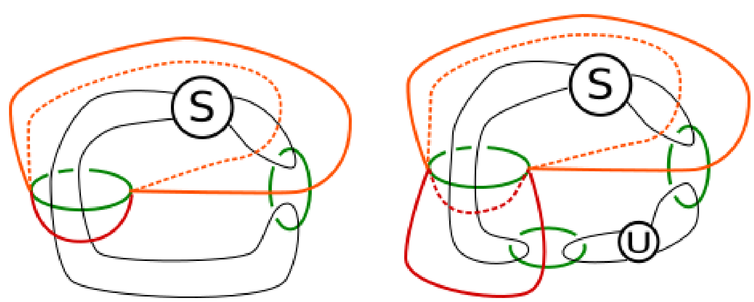


FIGURE 32. Thrice punctured spheres from cases 4(a) and 4(b) of proposition 17 imply the existence of the dashed line thrice punctured spheres.

□

Proposition 19 informs us that whenever an FAL can be belt sum decomposed, there exist a pair of thrice-punctured spheres where each contains a crossing circle longitude and two knot circle meridians. This means that we can disregard the other two circumstances for belt sum decomposition described in proposition 17, because both of them imply the existence of the decomposition shown in proposition 19.

Proposition 20. *If an FAL is belt sum decomposed along a thrice punctured sphere such as the one described in Proposition 19, it will decompose into two FALs.*

Proof. Let L be a fully augmented link containing a pair of thrice punctured spheres, each containing a crossing circle longitude and two knot circle meridians. Then L

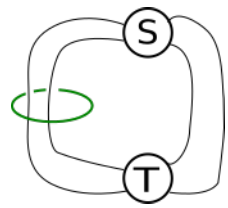


FIGURE 33. A fully augmented link with possible belt-sum decomposition. Note that S and T must be non-trivial fully augmented tangles, otherwise the two spheres belt-summed along are identical.

can be represented by the link drawn in figure 33. We may belt sum decompose along the crossing circle and the two knot circles to get a decomposition (see figure 34). When re-glued, it becomes clear that this pair of thrice-punctured spheres

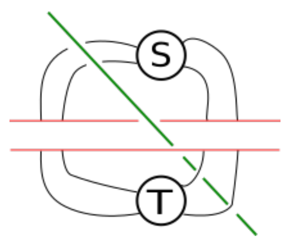


FIGURE 34. Decomposing L by taking a point of the crossing circle to infinity.

corresponds to a belt sum of the FALs shown in figure 35. □

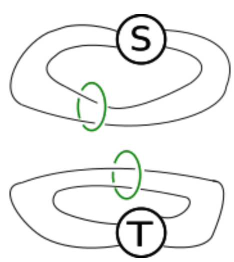


FIGURE 35. L decomposed into two links. As S and T are non-trivial, fully augmented tangles, both of these links are FALs.

From proposition 20, we know that if L is an FAL that can be belt sum decomposed, it contains a thrice punctured sphere as described in Proposition 19. Decomposing along that sphere will result in two link that must be FALs. Although proposition 20 is not used directly when proving the existence of belt sum decompositions, it is an important result because it shows exactly how an FAL is belt sum decomposed when it is able to be. We now have sufficient information about belt sum decompositions to prove their correlation to buckles in the nerves of FALs.

Theorem 21. *An FAL is the Belted sum of two others if and only if its painted nerve has a buckle.*

Proof. $\Rightarrow$ Suppose a FAL L is the belt sum of two others. Then, by proposition 19, L can be described by figure 33. Then the dual for L is shown in figure 36(a), and its corresponding nerve in figure 36(b). Then there exists a buckle in the nerve of L.

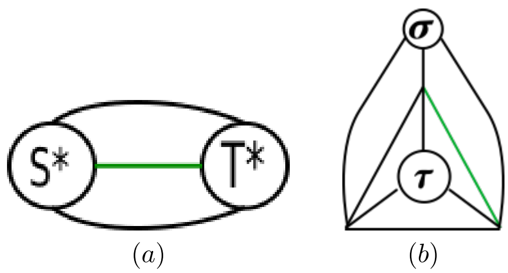


FIGURE 36. Dual and nerve for link L, where $S*$ and $T*$ represent dual forms of S and T. Similarly, σ and τ are nerve forms of S and T.

$\Leftarrow$ Suppose L is the nerve of an FAL that contains a buckle. Then L can be represented by Figure 37.

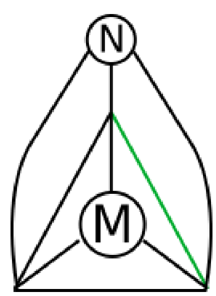


FIGURE 37. The nerve for L , where M and N are arbitrary subnerves.

If we split the nerve of L into along the buckle, we get the two separate nerves shown in Figure 38. One of the nerves includes everything that was inside the buckle, while the other is everything that was outside the buckle. Note that the edges that made up the buckle are included in each of these nerves (although this 3-cycle becomes trivial in each new nerve).

From these two new nerves, we can construct their duals, as shown in Figure 39.

These nerve duals correspond to the FALs constructed in Figure 40.

Now that we have these two separate FALs that were formed from breaking up the nerve of L, we will show that their belt sum has the same nerve as L. The belt sum of these two links is shown in Figure 41.

From here, we simply reverse the process we used earlier, constructing the nerve dual from this link (Figure 42) and then the corresponding nerve (Figure 43).

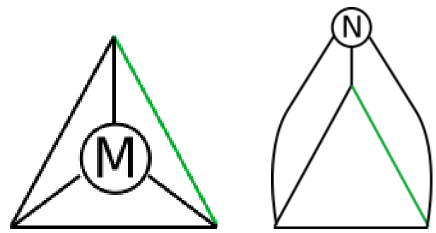


FIGURE 38. The nerve for L, when split along the buckle, is expressed by two separate nerves. Notice that the subnerves M and N are in different nerves.

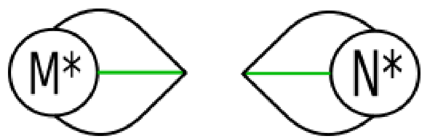


FIGURE 39. Duals for the nerves of the two links, where $M*$ and $N*$ represent dual forms of M and N, respectively.

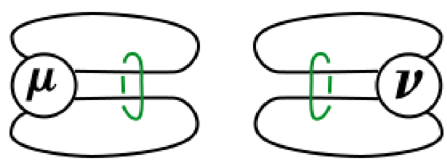


FIGURE 40. Fully augmented links, where μ and ν are sublinks corresponding to M and N.

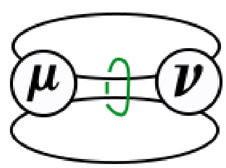


FIGURE 41. This link is the belt some of these two links along their crossing circles.

As the nerve shown in Figure 43 is also the nerve of L, we have demonstrated that as L has a buckle in its nerve, it is the belt sum of two other links.

□

Corollary 22. *If there is no buckle in the nerve of an FAL, then it is prime.*

From theorem 21, we no longer need to look at the topology of a link to determine if it is prime or composite, but can use the nerve instead. Since this applies to all fully augmented links, we will proceed to prove the existence of a belt sum

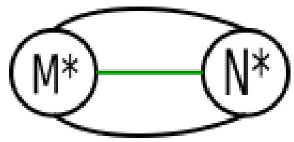


FIGURE 42. The nerve dual of this belt summed link.

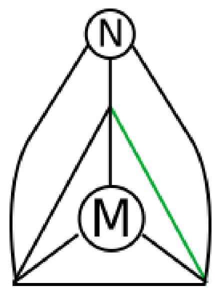


FIGURE 43. The nerve of this belt summed link.

decomposition in any FAL. In order to do this, we will induct on the process described in theorem 21, as decomposition relies on iterating the process of finding and decomposing along buckles.

Theorem 23. *Each FAL has a belt-sum decomposition into prime FALs.*

Proof. Let L be an FAL, and let Λ be the nerve of L. If Λ contains no buckles, then by corollary 22, L is prime. Now suppose Λ has a buckle. Then by theorem 21, L has a belt sum decomposition into two FALs. Call these L_1 and L_2.

Suppose for some $n \in \mathbb{N}$, Λ decomposes into n FALs, where the k^{th} decomposed FAL is denoted L_k for each $k \in \{1, 2, ..., n\}$.

Suppose for all $k \in \{1, 2, ..., n\}$, the nerve for L_k contains no buckles. Then each L_k is prime. Therefore, $L_1 \oplus L_2 \oplus ... \oplus L_n$ is a prime decomposition of L.

Now suppose that there exists some $k \in \{1, 2, ..., n\}$ such that the nerve of L_k contains a buckle. Then by theorem 21, L_k decomposes into two FALs. Let one of these FALs inherit the name L_k and call the other FAL L_{n+1}.

This process repeats for each buckle found in nerve of each L_k. However, because the number of edges in Λ is finite, there are a finite number of buckles in Λ. Therefore, there are a finite number of times Λ may be decomposed. Then for some $m \in \mathbb{N}$, for all $k \in \{1, 2, ..., m\}$, the nerve for L_k contains no buckles. Then L decomposes into m prime FALs.

□

5. Uniqueness of Belted Sum prime decompositions

We have found that having the same pieces of a nerve does not necessarily correspond to the same FAL – for example, one can take the belt sum of two FALs differently by simply identifying a different crossing circle to belt sum along and occasionally achieve a different belted sum. However, we prove that given the same FAL, its prime decomposition will result in unique pieces.

We know that given identical nerves N and M corresponding to L_1 and L_2 respectively, if $N = M$, then $L_1 = L_2$. However, we want to check that given

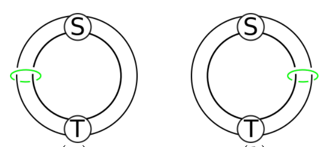
$N \neq M$, and $L_1 = L_2$, N and M decompose into the same prime FALs. We will accomplish this by (1) finding all cases of FALs with multiple fully augmented structures (call these FASs), and (2) finding that differing nerves for the same FAL correspond to the same decompositions.

Definition 24. If L and M are fully augmented links, they are *fully augmented structures (FASs)* of the same FAL if L and M are the same link, but with different components on their planes of projection.

This means that if an FAL L has multiple FASs, then there is at lease one component in L that is a crossing circle in one FAS of L and a knot circle in another FAS of L. An example of an FAL with multiple FASs is the six-chain; notice that if you take a six-chain, and rotate the chain inwards such that all of the crossing circles become knot circles and vice versa, this operation yields a second FAS for the six-chain.

We wish to show that if an FAL has multiple FASs, those FASs will be identical, which implies that they have the same nerve. We also need to consider the case where an FAL can have multiple nerves. If an FAL has multiple nerves, there are components in a link that can be rearranged in such a way that preserves the fully augmented properties of the link and also changes its nerve. We will prove that the only way a link can have multiple nerves is through flype moves.

Definition 25. If an FAL L has multiple twist regions, a *flype move* corresponds to moving a crossing disk from one twist region to another. This process is shown in figure 44.



(a) (b)

FIGURE 44. A Flype Move consists of moving a crossing circle to another twist region in the link. In order for there to be multiple twist regions, S and T must be nontrivial tangles.

In order to prove the uniqueness of decomposition, we will prove that flype moves are the only way that an FAL can have multiple nerves and that, in this case, the different nerves will have the same decomposition. Additionally, we must show that given a nerve, there is only one way to decompose it into prime FALs. This means you cannot decompose a nerve into two different sets of prime FALs.

We will start by proving that if an FAL has multiple FASs, then those FASs are identical.

Lemma 26. *If L is an FAL with a knot circle k, where k is also a crossing circle in a different FAS of L, then k can only be intersected by crossing circles.*

Proof. Let k be a knot circle in an FAL L such that k is a crossing circle in a different FAS of L. (Call the FAS where k is a knot circle FAS_1 and the FAS where k is a crossing circle FAS_2.) This can be seen in figure 45. Then k must have

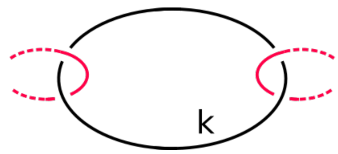


FIGURE 45

exactly two punctures. Suppose for contradiction that one of these punctures is an intersection with another knot circle in FAS_1.

Then there must exist two crossing circles between k and this crossing circle (see figure 46). Then k has four punctures, so we arrive at a contradiction. Therefore k can only be intersected by crossing circles. $\square$

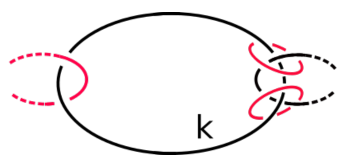
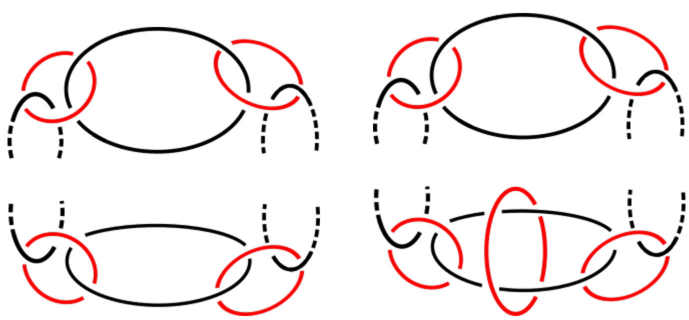


FIGURE 46

Proposition 27. *There exist two classes of FALs with multiple FASs: $2n$-chains ($n \geq 3$) and $2m$-chains ($m \geq 1$) with 1 crossing circle as shown in figure 47(b).*



(a) (b)

FIGURE 47. Two types of FALs that have multiple FASs.

Proof. Suppose L is an FAL with multiple FAS's. Then there exist at least two FAS's. Call them FAS_1 and FAS_2. Then there exists a knot circle in FAS_1 that is a crossing circle in FAS_2. Call this component k (see figure 45). Then, since k is a crossing circle, k must have exactly two punctures. Then by Lemma 26, k is intersected by either one or two crossing circles.

Case 1: Suppose k is punctured by one crossing circle (call this crossing circle j).

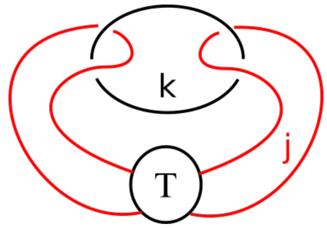


FIGURE 48. Case 1

Then, since j and k both act as crossing circles in different FAS's, and they both have two punctures, they both cannot have any more punctures in them. However, in order for L to be fully augmented, there must exist a crossing circle around one of these components, and it must be around k, since a crossing circle cannot wrap around another in an FAL. Since this crossing circle already has two punctures, there cannot exist any other component intersecting it. Additionally, there cannot be more than one such crossing circle, otherwise the link has an annulus. Then L is the Borromean rings, which is the same as a two-chain with one crossing circle (see figure 49).

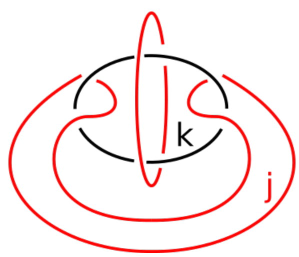


FIGURE 49. Case 1 results in the construction of the Borromean rings.

Case 2: Suppose k is punctured by two different crossing circles (call these m and n).

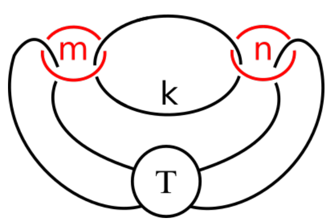


FIGURE 50. Case 2

Then, since they are crossing circles, they each have exactly two components going through them and can only be intersected by knot circles. Furthermore, m and n are knot circles in FAS_2. So by lemma 26, m and n are intersected by one or two crossing circles in FAS_2. If they are intersected by the same component, then L is a four-chain. If they are different components, then this process repeats, and a new pair or knot circles is found. However, since FALs have a finite number of components, there must exist some $n \in \mathbb{N}$ such that after n new pairs of components are found on each side k, the last component found is not part of a pair. Rather, it connects the $n-1^{st}$ pair of components, forming a $2n$-chain. Then all of the components of the chain are crossing circles in either FAS_1 or FAS_2, so none of them can have any additional punctures. Then the only additional component that can be added to L that preserves the properties of an FAL is one crossing circle wrapped around a knot circle in the chain. There cannot be more than one, however, otherwise there in an annulus in L. Both of these options are shown in figure 47.

Therefore, the only types of FALs that L can be are a $2n$-chain or a $2n$-chain with a crossing circle wrapped around it. So we may conclude that there are exactly two classes of FALs with multiple FAS's.

□

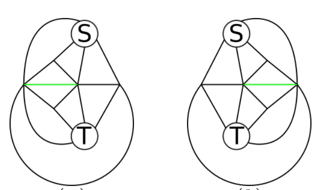
Corollary 28. *If an FAL has multiple FASs, then each FAS of the FAL corresponds to the same nerve.*

This our objective of showing that multiple FAS's for an FAL have the same nerve. We now examine other circumstances where FALs might have multiple nerves, starting with flype moves. Notice that nontrivial flypes on a twist reduced link L corresponds to moving a crossing circle over nontrivial regions on a fully augmented L.

Lemma 29. *For a given FAL L, the only way to move a crossing circle and change the nerve of L is by flype movement.*

Proof. Consider moving a crossing circle in an FAL L. Both before and after this move, the crossing circle must cross over exactly two knot circle strands and not over another crossing circle. The crossing circle may pass over an arbitrary tangle so long as it wouldn't pass through another link component. If it does, this movement is shown in Figure 44 and we refer to this as a flype move.

After moving the crossing circle in such a flype move, the nerves of the original and flyped links will differ as shown in figure 51. If either tangle is trivial, the movement clearly does not change the nerve and so is not a flype move.



(a) (b)

FIGURE 51. The nerves of L before and after a flype move.

□

We now need to show that a knot circle in an FAL cannot be moved in a way that will yield a different nerve of that link.

Proposition 30. *For a twist reduced link L, flype moves are the only way to get a different nerve for L.*

Proof. Suppose L is a fully augmented link that has multiple different nerves. Then there exist two different nerves for L. Call these N_1 and N_2. By corollary 28, N_1 and N_2 correspond to the same FAS of L. Then N_1 and N_2 differ depending on movements of crossing circles or knot circles. If the differences between N_1 and N_2 rely on moving crossing circles, by lemma 29, these different nerves correspond to flype moves, so our proposition holds. Now consider the case where the difference in nerves does not rely on the movement in crossing circles.

Suppose for contradiction that movements of knot circles in L must account for the two different nerves, N_1 and N_2. There must exist two planes, P_1 and P_2, where P_1 is the plane of projection for the link corresponding to N_1 and P_2 corresponds to N_2. Then, since N_1 and N_2 are nerves for the same link, L, P_1 and P_2 are both planes of projection for L. Then, given any knot circle, K, of L, $P_1 \cap N(K) = 2\ell$ and $P_2 \cap N(K) = 2\ell$. Furthermore, P_1 must intersect the interior of each crossing circle four times (twice on the toroidal neighborhood of each component passing through the crossing circle). This same logic applies for P_2. Then given any crossing circle, C, in L, P_1 must form three unique geodesics on the interior of C. Similarly, P_2 must form three unique geodesics inside C. Then both P_1 and P_2 must intersect C as shown in figure 52.

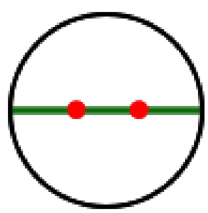


FIGURE 52. The green lines indicates how P_1 and P_2 must form geodesics inside a crossing circle. There are four points of intersection between a projection plane and a crossing disk (two along each crossing circle), and they must be connected through unique, non-intersecting geodesics. Therefore the segments above are the only ways to do this.

Therefore,P_1 and P_2 must form the same geodesics along all crossing circles in L. Then they also must intersect each knot circle neighborhood in the same way, forming the same pair of longitudes on each knot circle. Therefore P_1 and P_2 are the same plane of projection. Thus, P_1 and P_2 correspond to the same nerve, so we arrive at a contradiction. Therefore we may conclude that the only way a fully augmented link can have different nerves is if a crossing circle corresponding to a twist region is moved.

□

We now know that there is only one way an FAL can have multiple nerves, and that is through flype moves. We proceed to show that such differing nerves will decompose into the same prime FALs.

Proposition 31. *Different nerves corresponding to the same FAL have the same decomposition.*

Proof. Consider a link with at least two different nerve representations. By proposition 30, these two different nerve representations are produced by flype moves. Since the crossing disk was previously in a place punctured by two knot strands, we know that this motion yields another thrice punctured sphere. This gives two different representations of the same FAL, given nontrivial fully augmented tangle regions S and T (see figure 44).

After moving the crossing circle, their nerves will differ as shown in figure 51. Decomposing these nerves along the only buckles present, we are left with the nerves shown in figure 53. These correspond to the FALs in figure 54. Since these two decompositions are equivalent, we conclude that decompositions of the same link, regardless of their nerve, is equivalent.

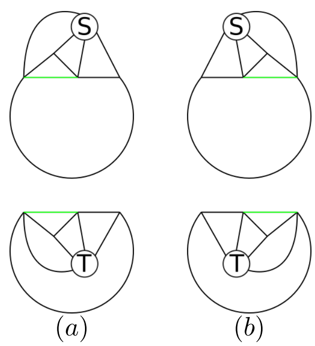


FIGURE 53. Nerves corresponding to the decomposition of figure 51 (a) and (b) along the buckle.

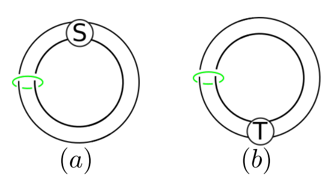


FIGURE 54. Link (a) corresponds to the top row and Link (b) corresponds to the bottom row of figure 53.

□

We now know that all FALs have either one nerve or have multiple nerves that will decompose identically. We now must show that, given one nerve there is only one set of prime FALs that it will decompose into. This relies on the commutativity of belt sum decomposition.

Theorem 32. *Belt sum decomposition is commutative.*

Proof. To show belt sum decomposition is commutative, we must show that if we have a nerve such as the one in Figure 55 with arbitrary tangles S, T, and U that do not contain any part of a buckle, decomposing by the top buckle and then the bottom one is equivalent to decomposing by the bottom buckle followed by the top one.

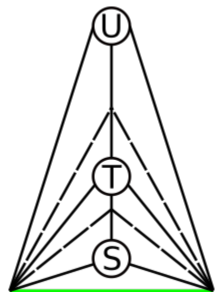


FIGURE 55. A generic nerve of an FAL that is made up of three prime FALs. The dotted edges denote unpainted edges of the two buckles.

These decompositions are shown in Figure 56. In each case, we end up with the same three prime FALs.

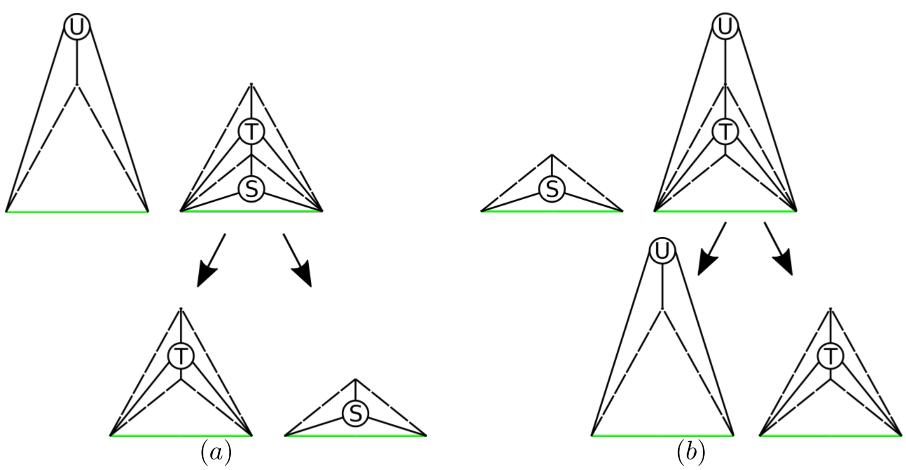


FIGURE 56. Two different orders in which to decompose the nerve shown in Figure 55.

□

Thus, given a set of buckles to decompose along, the order in which we decompose along them doesn't matter.

Theorem 33. *The decomposition of Theorem 23 is unique. That is, for each FAL, there is exactly one decomposition into prime FALs.*

Proof. By Proposition 31, we know that if a given FAL has multiple possible nerves, they all share a prime decomposition. Now we must show that no FAL can be decomposed into two different sets of prime FALs. This is a result of how we decompose the nerve along buckles, as shown in Figures 37 and 38.

Each step of the decomposition removes one buckle from the nerve as the three-cycle we decompose along becomes trivial in both resulting nerves. Imagine if a second buckle could be removed in the same decomposition step. Then the decomposition must split up its one-colored three-cycle. This second buckle would have to consist of at least one edge on each side of the first buckle, and so it would contain at least one vertex on each side. Its third vertex would have to be shared with the first buckle. The third edge would have to connect a vertex from each side of the first buckle to each other, passing through the first buckle a second time, creating a fourth vertex. Thus, such a buckle could not exist.

As each buckle corresponds to one decomposition step, an FAL with n buckles can be decomposed into $n+1$ prime FALs. In any decomposition, we will have a decomposition step for each buckle in the original nerve. Then, because this decomposition is commutative, there is only one resulting set of prime FALs for a given nerve. □

6. Octahedral FALs

In this section, we apply our previous results to a subclass of fully augmented links called octahedral FALs. We define *octahedral FALs* as FALs whose polyhedral decomposition is obtained by the gluing of regular ideal ocahedra. We prove that all members of this class decompose into a well-defined family of prime octahedral FALs. Moreover, Adams proved in [1] that the sum of the volumes of belted sum decompositions is equivalent to the volume of the belted sum. Hence, we prove that given an otherwise difficult volume computation for a complicated nerve, we may reduce it to the sum of easy volume computations.

To proceed, we need the following result from Purcell, proven in 2011. Recall that a *central subdivision* is the subdivision obtained by adding a vertex to any face of a triangulation and adding 3 edges, each connecting the new vertex to one of the three vertices along the boundary of the face.

Proposition 34. *(Purcell) For fully augmented links with polyhedral decomposition, these polyhedra are obtained by gluing regular ideal octahedra if and only if the nerve of the circle packing is obtained by central subdivision of K_4.*

This is a strong statement. One thing it tells us is that given the nerve of an octahedral FAL, it must have at least one vertex of degree three corresponding to a central subdivison. Moreover, it is known that every well-painting of K_4 corresponds to a nerve of the Borromean rings. So, trivially, we know that the Borromean rings are prime octahedral FALs. It is true that if we centrally subdivide any face of a well-painted K_4, we may decompose the result into two Borromean rings. However, we must still check for hidden prime octahedral FALs. Given that all octahedral FALs are obtained via central subdivisions of K_4, instead of assuming a well-painted K_4 subgraph, and central subdivisions that correspond to more Borromean rings, suppose the central subdivisions are what produce a well-painted nerve. That is, suppose there exists a painted subgraph within a well-painted nerve that corresponds to K_3, K_4, or some central subdivision of K_4. Consider the constructions in the following figure 57.

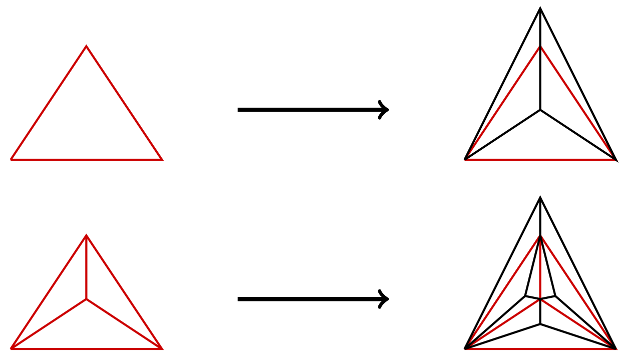


FIGURE 57. **TOP** Starting with a painted K_3, we perform central subdivisions of each face such that we are left with a well painted nerve that corresponds to central subdivisions of K_4. **BOTTEM** If the colored subgraph is not K_3, then it must follow from some number of central subdivisions of K_4 in order for it to be octahedral.

We define this construction in better terms as follows.

Definition 35. The *Addams Family* is a set of fully augmented links whose members' nerves meet the following criteria.

1. The painted subgraph must be obtained by central subdivision of K_3.
2. Centrally subdividing every face of this subgraph using uncolored edges produces the full nerve.

Note that we describe members of the Addams family as having no more central subdivisions than the ones needed to obtain a well-painted nerve by the construction. Else, a buckle is produced. This is easy to observe: in an Addams family member, we may only choose a once-painted triangle to centrally subdivide. If we are to subdivide here, then the boundary of the once-painted triangle becomes a buckle. Now, we prove that our construction produces only prime octahedral FALs.

Proposition 36. *Every member of the Addams family is prime.*

Proof. In the painted subgraph of an Addams Family member's nerve, each face is bounded by a thrice-painted 3-cycle. Consider one of these 3-cycles and, for sake of contradiction, suppose that one of these painted edges is the painted edge of a buckle. Call this edge e. The endpoints of e must both be adjacent to a vertex v. However, in order to be a buckle, the edges between the endpoints of e and v must be unpainted. By the construction of an Addams family member, this is only the case if v is gained during the central subdivisions. Furthermore, all vertices gained this way are of degree three, since all vertices not in the painted subgraph are obtained by a central subdivision. However, for this 3-cycle to be a buckle, it must be nontrivial. Without loss of generality, suppose the third edge adjacent to v goes to the "outside" of the buckle. Then the "inside" of the buckle has only two vertices (the endpoints of e) to form a triangulation. This is impossible with just central subdivisions. Hence a buckle is impossible within a member of the Addams family. $\square$

Now, we prove that all prime octahedral FALs are now found.

Proposition 37. *All prime octahedral fully augmented links are either the Borromean Rings or members of the Addams Family.*

Proof. Let $\mathcal{F}$ be an octahedral FAL containing no buckle, and let N be the nerve of $\mathcal{F}$. Notice that all 3-cycles in N are either trivial or fully painted. We cannot have both, since a trivial fully painted 3-cycle would not be well-painted. Furthermore, all 3-cycles must fall into either of these cases since we cannot have twice painted 3-cycles by Theorem 1.

Case 1: Suppose N contains no thrice painted 3-cycles. Then N contains only once painted 3-cycles that are trivial. However, if we subdivide K_4 even once, and paint it such that there are no thrice painted 3-cycles, we are left with a buckle. Hence, the only prime FAL of this form is K_4 with two painted edges. This is the nerve of the Borromean Rings.

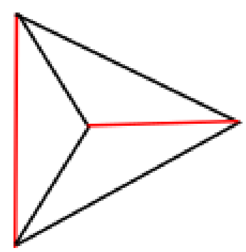


FIGURE 58. One of the possible nerves of the Borromean rings.

Case 2: Now suppose that N contains a thrice painted 3-cycle. This has to be well-painted, hence there must have been at least 2 central subdivisions (the "inner" face and the" outer face").

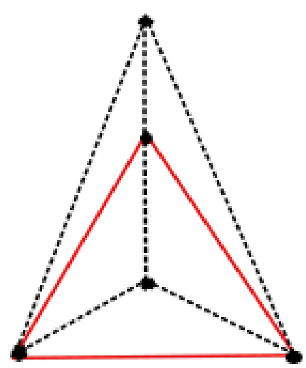


FIGURE 59. For a nontrivial 3-colored 3-cycle, there must be at least two central subdivisions present to form a well-painted nerve.

Consider, without loss of generality, the central subdivision preformed on the outside face. Notice that the edges added during this central subdivision must be either all painted or fully unpainted, since adding one or two painted edges results in a twice painted 3-cycle, which, again, contradicts Theorem 1.

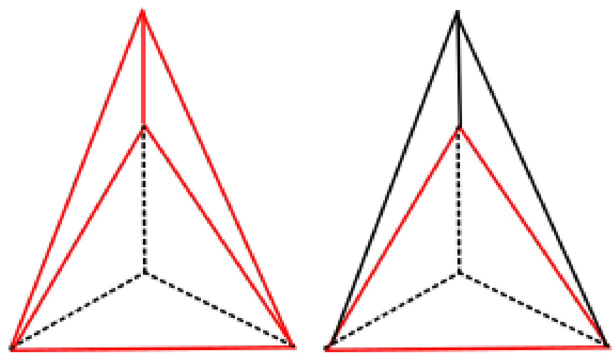


FIGURE 60. **Left**: a 3-colored central subdivision. **Right**: an uncolored central subdivision.

Suppose the central subdivision results in the addition of three thrice painted edges. Then we have new thrice painted 3-cycles, so this process repeats, as the edges must be well-painted.

Suppose the central subdivision results in the addition of 3 unpainted edges. Then the result is three 3-cycles with one painted edge each. So if we subdivide any of these three triangles, we would get a buckle. This yields the FAL composite. Thus none of these faces can be subdivided. Hence when all faces of N are well-painted, the process ceases, and we are left with a prime FAL. This FAL is a member of the Addams Family.

Case 1 yields the Borromean rings and Case 2 yields members of the Addams family. These, by construction, are the only cases possible, so we conclude that they are the only prime octahedral FALs.

□

Having both this proposition and our main existence theorem, the following result is clear.

Theorem 38. *L is an octahedral FAL if and only if it has a prime decomposition into only Borromean Rings and/or members of the Addams Family.*

Proof. Suppose L is an octahedral FAL. Then it is either a prime FAL or it is composite. If L is prime, by Proposition 37, it is necessarily either the Borromean rings or a member of the Addams Family. Suppose L is composite. Then there exist buckles in the nerve N of L. L is octahedral, so N is obtained via central subdivisions of K_4. Hence, every region enclosed by a buckle in the nerve of L must be a central subdivision of K_4. Thus, L can be decomposed into a number of prime FALs, all of which are octahedral. By Proposition 37, each of these FALs must be either the Borromean Rings or members of the Addams Family.

Suppose that the link L has a prime decomposition into only Borromean Rings and/or members of the Addams Family. Then L is made up of the belted sum of octahedral FALs and is itself octahedral. □

So, given any complex octahedral FAL who's complement's volume is unknown, one may now decompose it into a finite number of smaller octahedral FALs whose structure is known. Taking the sum of the volumes of these complements yields that of the initial FAL.

Open questions

(1) Can any of our results be generalized to links that aren't fully augmented? For example, consider the belt-sum decomposition along the thrice punctures spheres in the Borromean rings that yields two Whitehead links. More generally, what more can be said of the other 5 thrice punctured spheres found in FALs and their unions?
(2) What prime FALs exist? Is there a way to categorize all of them? For example, it is easy to prove that for $n \geq 3$, all $2n$-component chains are prime, where the 6-component chain is also a prime octahedral FAL. Furthermore, if there were an infinite number of these families of prime FALs, can we prove that there exists a prime FAL that doesn't belong to at least one of these families? Are the Borromean Rings an example of this?
(3) Is there any relationship between the volumes of prime FALs of the same family? For example, is there some way to relate the volumes of members of the Addams Family? This would make complex volume computations even more simple.
(4) While the decomposition of FALs is unique, we have also shown that, given a set of prime FALs, different sequences of belt-suming will result in potentially different FALs. Consider the different choices one can belt sum a set of FALs. What can be said of the paths of choices that lead to different FALs? What can be said of distinct paths of choices that lead to equivalent FALs?
(5) What can be said of belt-summing with regards to algebraic FALs?

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