Constant Vector Curvature for Skew-Adjoint and Self-Adjoint Canonical Algebraic Curvature Tensors

Mack Beveridge
mackbeveridge@lclark.edu

California State University, San Bernardino REU

August 2017

Abstract

This research explores the property of constant vector curvature on model spaces with curvature tensors constructed from skew-adjoint and self-adjoint linear transformations. Constant vector curvature has been studied for general curvature tensors in the past, but only in the 3-dimensional case. For this reason we look at these specific cases to try to generalize results in higher dimensions. We determine the constant vector curvature for all tensors constructed from skew-adjoint linear transformations, some cases of tensors constructed from self-adjoint linear transformations, and we find some general results about constant vector curvature.

1 Introduction

We study a curvature condition called constant vector curvature. In the past, constant vector curvature has only been studied in the three dimensional case. In order to learn more about constant vector curvature in higher dimensions we consider two specific kinds of curvature tensors, those built from skew-adjoint and self-adjoint linear transformations.

Definition 1.1. *Let V be a finite-dimensional real vector space. An **algebraic curvature tensor** is a multilinear function $R : V \times V \times V \times V \to \mathbb{R}$ that satisfies the following conditions*

$$R(x, y, z, w) = -R(y, x, z, w) = R(z, w, x, y) and,$$

$$R(x, y, z, w) + R(z, x, y, w) + R(y, z, x, w) = 0.$$

The kernel of an algebraic curvature tensor is very important in determining what its constant vector curvature is. It will play a central role in Theorem 2.1.

Definition 1.2. *Let R be an algebraic curvature tensor. We define the **kernel of** R as follows:*

$$kerR = \{v \in V : R(x, y, z, w) = 0 \ for \ all \ y, z, w \in V\}$$

We study a specific type of algebraic curvature tensor called a canonical algebraic curvature tensor, which we define here.

Definition 1.3. *Let $\langle \cdot, \cdot \rangle$ be an inner product and let A and J be linear transformations. Suppose J is skew-adjoint with respect to $\langle \cdot, \cdot \rangle$ and A is self-adjoint with respect to $\langle \cdot, \cdot \rangle$. The **canonical algebraic curvature tensor** constructed from J is*

$$R_J(x, y, z, w) = \langle Jx, z\rangle\langle Jy, z\rangle - \langle Jx, z\rangle\langle Jy, w\rangle - 2\langle Jx, y\rangle\langle Jz, w\rangle.$$

*And the **canonical algebraic curvature tensor** constructed from A is*

$$R_A(x, y, z, w) = \langle Ax, z\rangle\langle Ay, z\rangle - \langle Ax, z\rangle\langle Ay, w\rangle.$$

R_J and R_A are algebraic curvature tensors with these builds because J is skew-adjoint and A is self-adjoint [2]. One reason that these tensors are important to study is that $span\{R_J\} = span\{R_A\} = A(V)$, where $A(V)$ is the set of all algebraic curvature tensors on V.

Definition 1.4. *Let $\langle \cdot, \cdot \rangle$ be an inner product on vector space V and R be an algebraic curvature tensor. Then $M = (V, \langle \cdot, \cdot \rangle, R)$ is a **model space**.*

In this paper all inner products are assumed to be positive-definite. We can now introduce a property that is central to this research: sectional curvature.

Definition 1.5. *Let $M = (V, \langle \cdot, \cdot \rangle, R)$ be a model space, and let $v, w \in V$ be such that they span a 2-plane in V. The **sectional curvature** of the two-plane spanned by v and w is*

$$k(v, w) = \frac{R(v, w, w, v)}{R_{\langle \cdot, \cdot \rangle}(v, w, w, v)},$$

where $R_{\langle \cdot, \cdot \rangle}(x, y, z, w) = \langle x, w\rangle\langle y, z\rangle - \langle x, z\rangle\langle y, w\rangle$.

In a 2-dimensional model space, there is only one possible 2-plane so every two dimensional model space has constant sectional curvature. For this reason, in this paper, we assume that every model space is of dimension 3 or greater. Furthermore, since sectional curvature is a function of 2-planes, two pairs of vectors that span the same 2-plane will produce the same sectional curvature. For any vector $v \in V$, if there exists a $w \in V$ such that $k(v, w) = \varepsilon$ then there will be a $u \in v^{\perp}$ with $span\{v, w\} = span\{v, u\}$ for which $k(v, u) = \varepsilon$.

For this reason we can always assume, without loss of generality, that v and w are orthogonal. This amounts to finding an orthonormal basis for the 2-plane spanned by v and w. We use this fact to simplify the calculations of sectional curvature.

We can use sectional curvature to define curvature properties on model spaces.

Definition 1.6. *A model space M is said to to have* ***constant sectional curvature*** *ε, denoted $csc(\varepsilon)$, if for all $v, w \in V$ such that v, w span a 2-plane, $k(v, w) = \varepsilon$.*

Constant sectional curvature is a very strong and somewhat uncommon property of model spaces, so we study a slightly weaker, but more common property: constant vector curvature.

Definition 1.7. *A model space M is said to to have* ***constant vector curvature*** *ε, denoted $cvc(\varepsilon)$, if for all $v \neq 0 \in V$ there exists a $w \in V$ such that $k(v, w) = \varepsilon$.*

Note that constant sectional curvature always implies constant vector curvature. We can also define a slightly stronger property extremal constant vector curvature.

Definition 1.8. *A model space M is said to to have* ***extremal constant vector curvature*** *ε, denoted $ecvc(\varepsilon)$, if M has $cvc(\varepsilon)$ and ε is a bound on the possible sectional curvature values for M.*

In Section 2 we present general results about constant vector curvature that can be applied to all model spaces. In Section 3 we determine all the sectional curvature and *cvc* values for all canonical algebraic curvature tensors built from skew-adjoint linear transformations. Additionally we provide a method for constructing a model space with a specific *cvc* value. In Section 4 we determine the *cvc* values for some canonical algebraic curvature tensors built from self-adjoint linear transformations, and put some restrictions on the possible *cvc* values for others.

2 General Results

In this section we present general results about the constant vector curvature of model spaces. We determine the *cvc* values of any model space that has a curvature tensor with a non-trivial kernel. We also provide some methods for constructing a model space with a desired *cvc* value through linear combinations of curvature tensors.

Theorem 2.1. *Let $M = (V, \langle \cdot, \cdot \rangle, R)$ be a model space such that $ker(R) \neq 0$. Then M is $cvc(0)$ and only $cvc(0)$.*

Proof. If $ker(R) = V$, then $R = 0$ so M has $csc(0)$, and therefore has $cvc(0)$. If $ker(R) \neq V$ let $v \in V$ be a non-zero vector. If $v \in ker(R)$ then for any choice of w, $k(v, w) = 0$. So 0 is the only possible *cvc* value. In this case let $w \in ker(R)^{\perp}$ so v and w are linearly independent. Then M has $cvc(0)$.

Now if $v \notin ker(R)$, let $w \in ker(R)$. Then M has $cvc(0)$. □

Furthermore, it is known that if T is a self-adjoint or skew-adjoint linear transformation $ker(T) = ker(R_T)$ unless T has rank 1 [2]. So, by Theorem 2.1, all canonical algebraic curvature tensors built from linear transformations with a non-zero kernel have only $cvc(0)$. For this reason, we assume that all canonical algebraic curvature tensors and linear transformations in this paper have a zero kernel.

Theorem 2.2. *If $M = (V, \langle \cdot, \cdot \rangle, R)$ is a model space with $cvc(\varepsilon)$, and c is a real number, then $M = (V, \langle \cdot, \cdot \rangle, cR)$ has $cvc(c\varepsilon)$.*

Proof. Let $v \in V$ be a non-zero vector. Since M is $cvc(\varepsilon)$ there exists w such that,

$$k(v, w) = \frac{R(v, w, w, v)}{R_{\langle \cdot, \cdot \rangle}(v, w, w, v)} = \varepsilon.$$

By replacing R with cR we get $k(v, w) = c\varepsilon$. So M has $cvc(c\varepsilon)$. □

We also know the *cvc* values for the sums of certain algebraic curvature tensors.

Theorem 2.3. *If $M_1 = (V, \langle \cdot, \cdot \rangle, R_1)$ is a model space with $cvc(\varepsilon)$ and $M_2 = (V, \langle \cdot, \cdot \rangle, R_2)$ is a model space with $csc(\delta)$ then $M = (V, \langle \cdot, \cdot \rangle, R_1 + R_2)$ is a model space with $cvc(\varepsilon + \delta)$.*

Proof. Let $v \in V$ be a non-zero vector. Since M_1 has $cvc(\varepsilon)$, there exists w such that $\frac{R_1(v,w,w,v)}{R_{\langle \cdot, \cdot \rangle}(v,w,w,v)} = \varepsilon$. Furthermore, since M_2 has $csc(\delta)$, $\frac{R_1(v,w,w,v)}{R_{\langle \cdot, \cdot \rangle}(v,w,w,v)} = \delta$. This gives us

$$k(v, w) = \frac{R_1(v, w, w, v) + R_2(v, w, w, v)}{R_{\langle \cdot, \cdot \rangle}(v, w, w, v)} = \frac{R_1(v, w, w, v)}{R_{\langle \cdot, \cdot \rangle}(v, w, w, v)} + \frac{R_2(v, w, w, v)}{R_{\langle \cdot, \cdot \rangle}(v, w, w, v)} = \varepsilon + \delta.$$

So M has $cvc(\varepsilon + \delta)$. □

These two theorems are useful for constructing a model space with a desired *cvc* value in Corollary 3.5.2. Using them we can determine the *cvc* values for some linear combinations of algebraic curvature tensors. In particular, we use Theorem 2.3 to demonstrate that for every closed interval $[a, b]$ over the real numbers, there is a model space which has $cvc(\varepsilon)$ for all $\varepsilon \in [a, b]$.

3 Curvature Tensors of Skew-Adjoint Linear Transformations

In this section we determine the sectional curvature values and the constant vector curvature values of all canonical algebraic curvature tensors constructed from skew-adjoint linear transformations. First we present a useful theorem regarding skew-adjoint linear transformations. Second, we present some lemmas that are useful in the later proofs. Then we present the main result, that all canonical algebraic curvature tensors constructed from skew-adjoint linear transformations have an interval of *cvc* values. Finally we provide a method for constructing a model space with specific *cvc* values.

Theorem 3.1. *If J is a skew-adjoint linear transformation on a finite dimensional vector space, then there exists an orthonormal basis $\{x_1, y_1, ...x_k, y_k, z_1, ...z_p\}$ such that $Ker(J) = span\{z_1, ...z_p\}$, $Jx_i = \lambda_i y_i$, and $Jy_i = -\lambda_i x_i$. Furthermore, by replacing x_i with y_i we can assume that $\lambda_i > 0$. [5]*

Note that this implies that v and Jv are orthogonal, a critical fact in the later proofs.

Lemma 3.2. *If $v, w \in V$ are orthogonal unit vectors and $M = (V, \langle\cdot,\cdot\rangle, R_J)$ is a model space where J is a skew-adjoint linear operator, then $k(v, w) = 3\langle Jv, w\rangle^2$.*

Proof. Since J is skew-adjoint, v and Jv are orthogonal, so $\langle Jv, v\rangle = 0$ for all $v \in V$. So $R(v, w, w, v) = -\langle Jv, w\rangle\langle Jw, v\rangle - 2\langle Jv, w\rangle\langle Jw, v\rangle = 3\langle Jv, w\rangle^2$. Since v and w are unit and orthogonal, the denominator of $k(v, w)$ is 1. □

Because we can always assume, without loss of generality, that v and w are orthogonal, this lemma is useful for simplifying calculations of sectional curvatures. The following Lemma is very useful in the proofs of the next two theorems.

Lemma 3.3. *Let J be a skew-adjoint linear operator and let v be a unit vector. Then $min(\lambda_i) \le ||Jv|| \le max(\lambda_i)$ where λ_i is as in Theorem 3.1.*

Proof. Let $v = \Sigma_{i=1}^n v_i e_i$

$$||Jv||^2 = \lambda_1 v_2^2 + \lambda_1 v_1^2 + ... + \lambda_k v_{n-1}^2 + \lambda_k v_n^2.$$

By replacing λ_i with $max(\lambda_i)$ we get

$$||Jv||^2 \le max(\lambda_i^2)(v_1^2 + v_2^2 + ... + v_n^2) = max(\lambda_i^2).$$

Similarly, by replacing λ_i with $min\lambda_i$ we get $||Jv||^2 \ge min(\lambda_i^2)$. So $min(\lambda_i) \le ||Jv|| \le max(\lambda_i)$.

□

Notice that $min(\lambda_i)$ and $max(\lambda_i)$ are attained when $v = x_i$ where x_i corresponds to $min(\lambda_i)$ or $max(\lambda_i)$. Since $||Jv||$ is a continuous function, by the Intermediate Value Theorem, $||Jv||$ attains every value in $[min(\lambda_i), max(\lambda_i)]$.

Now we can determine all possible sectional curvature and constant vector curvature values of an algebraic curvature tensor built from a skew-adjoint linear transformation.

Theorem 3.4. *Let* $M = (V, \langle\cdot,\cdot\rangle, R_J)$ *be a model space where* J *is a skew-adjoint linear transformation with a zero kernel. The set of all possible sectional curvatures is* $[0, 3max(\lambda_i^2)]$ *where* λ_i *is as in Theorem 3.1.*

Proof. Without loss of generality assume that the λ_i's are arranged such that they are in decreasing order, so $max(\lambda_i) = \lambda_1$. From Lemma 3.2 we can see that the sectional curvature will always be positive. Since J is skew-adjoint we have

$$0 \leq \langle Jv, w\rangle^2 = -\langle Jv, w\rangle\langle Jw, v\rangle.$$

This implies that either $\langle Jv, w\rangle$ or $\langle Jw, v\rangle$ is negative. Without loss of generality assume that $\langle Jw, v\rangle$ is negative. Then, by the Cauchy-Schwarz Inequality and Lemma 3.3,

$$0 \leq \langle Jv, w\rangle\langle -Jw, v\rangle \leq ||Jv|| \cdot ||Jw|| \leq max(\lambda_i^2).$$

Now let $\varepsilon \in [0, 3\lambda_1^2]$ and let $\delta = \sqrt{\frac{\varepsilon}{3\lambda_1^2}}$. Let $v = x_1$ and let $w = \delta y_1 + \sqrt{1-\delta^2}x_2$. Such an x_2 exists because we assume three or more dimensions. Since $0 \leq \delta \leq 1$, w is a unit vector, and v and w are orthogonal, so, by Lemma 3.2, $k(v, w) = \varepsilon$. □

Theorem 3.5. *Let* $M = (V, \langle\cdot,\cdot\rangle, R_J)$ *be a model space where* J *is a skew adjoint linear transformation with a zero kernel, then* M *has* $cvc(\varepsilon)$ *if and only if* $\varepsilon \in [0, 3min(\lambda_i^2)]$.

Proof. Let $v \in V$ be a unit vector and let $\varepsilon \in [0, 3min\lambda_i^2]$. We will construct an orthonormal basis $F = \{f_1, f_2, ..., f_n\}$ as follows.

$$f_1 = v, \quad f_2 = \frac{Jv}{||Jv||}, \text{ and } f_i \in span\{f_1, f_2\}^\perp \text{ for all } i \geq 3.$$

Note that $||Jv|| \neq 0$ as J has a trivial kernel and $v \neq 0$. This is an orthonormal basis becuase $Jv \perp v$.

Let $\delta = \sqrt{\frac{\varepsilon}{3||Jv||^2}}$, and $w = \delta f_2 + \sqrt{1-\delta^2}f_3$. Note that $0 \leq \delta \leq 1$ so v and w are orthogonal unit vectors, and, by Lemma 3.2, $k(v, w) = 3||Jv||^2\delta^2 = \varepsilon$.

Now let M be $cvc(\varepsilon)$. Let v be a unit vector such that $||Jv|| = min|\lambda_i|$. Then, by Cauchy-Schwarz, $k(v, w) = 3\langle Jv, w\rangle^2 \leq 3min\lambda_i^2$. We know that $k(v, w)$ is always positive, so $0 \leq \varepsilon \leq 3min\lambda_i^2$. □

This result is particularly interesting because we have an interval of *cvc* values. Previous work, in the three dimensional case, has only found model spaces with a single *cvc* value [3].

We can also make a slightly stronger statement about *ecvc* for canonical algebraic curvature tensors constructed from skew-adjoint linear transformations.

Corollary 3.5.1. *If $M = (V, \langle \cdot, \cdot \rangle, R_J)$ is a model space where J is a skew adjoint linear operator, then M is ecvc(0). If all λ_i of J are equal then M is ecvc(λ^2).*

Proof. This follows from Theorems 3.4 and 3.5 as 0 is a lower bound on all sectional curvature values and always a *cvc* value for M. If all λ_i of J are equal then $min\lambda_i^2 = max\lambda_i^2 = \lambda^2$, so the set of sectional curvature values is equal to the set of *cvc* values. $\square$

Now we provide a method for constructing a model space with a desired interval of *cvc* values.

Corollary 3.5.2. *Let $[a, b]$ be a closed interval over the real numbers. The model space $M = (V, \langle \cdot, \cdot \rangle, aR_{\langle \cdot, \cdot \rangle} + \frac{b}{3} R_J)$ where J is the skew-adjoint linear transformation where $\lambda_i = 1$ has cvc(ε) if and only if $\varepsilon \in [a, b]$.*

Proof. This results follows from Theorems 3.5, 2.2, and 2.3. It is easy to see that any model space with $R_{\langle \cdot, \cdot \rangle}$ as its curvature tensor is *csc*(1). So M will be *cvc*(ε) for all $\varepsilon \in [a, b]$. $\square$

4 Curvature Tensors of Self-Adjoint Linear Transformations

In this section we determine the *cvc* values for all canonical algebraic curvature tensors built from self-adjoint linear transformations with 3 or fewer eigenvalues. We begin by noting some significant restrictions on the possible *cvc* values. Then we determine the *cvc* values when A has one, two, or three eigenvalues.

First note that similarily to the skew adjoint case, we can assume that v, and w are orthogonal unit vectors. This gives us $k(v, w) = R_A(v, w, w, v)$. Since *cvc* is a property that must hold for all $v \in V$ we can put some restrictions on the possible *cvc* values by looking at what sectional curvature values we can get from a specific v.

Lemma 4.1. *Let $M = (V, \langle \cdot, \cdot \rangle, R_A)$ be a model space where A is a self-adjoint linear transformation with a zero-kernel. We can restrict the possible cvc values of M in the following cases.*

1. *If all the eigenvalues of A have the same sign then M can only have cvc$(\lambda_1 \lambda_k)$ where λ_k is the largest eigenvalue.*

2. *If A has only one negative eigenvalue and at least three distinct eigenvalues then M can only have cvc$(\lambda_1 \lambda_2)$.*

3. *If A has only one positive eigenvalue and at least three distinct eigenvalues then M can only have cvc$(\lambda_k \lambda_{k-1})$.*

Proof.

1. By the Spectral Theorem [1], there exists an orthonormal basis $E = \{e_1, e_2, ..., e_n\}$ such that $Ae_i = \lambda_i e_i$. Let $v = e_1$. We can assume that w is orthongonal to v so it must be a linear combination of the other eigenvectors. So $k(v, w) = ||Av|| \cdot ||Aw||$. Since $\lambda_2 \leq ||Aw|| \leq \lambda_k$ our possible *cvc* values are $[\lambda_1\lambda_2, \lambda_1\lambda_k]$. Now let $v = e_k$. Using the same process we get that all *cvc* values must be in $[\lambda_1\lambda_2, \lambda_k - 1\lambda_k]$. Therefore, the only possible *cvc* value is $\lambda_1\lambda_k$
2. Let $v = e_1$. Then the set of possible *cvc* values is $[\lambda_1\lambda_k, \lambda_1\lambda_2]$. Now let $v = e_2$. Then all *cvc* values must be in $[\lambda_1\lambda_2, \lambda_2\lambda_4]$. Therefore, the only possible *cvc* value is $\lambda_1\lambda_2$.
3. It is known that $R_A = R_{-A}$, so, by 2, we have $cvc(\lambda_2\lambda_3)$.

□

Now we only need to show that our model spaces have a single *cvc* value.

Theorem 4.2. *Let $M = (V, \langle\cdot,\cdot\rangle, R_A)$ be a model space where A is a self-adjoint linear transformation with only one eigenvalue, λ, and a zero kernel. Then M only has $cvc(\lambda^2)$*

Proof. Let $v, w \in V$. $R_A(v, w, w, v) = \lambda^2 R_{\langle\cdot,\cdot\rangle}$ so $k(v, w) = \lambda^2$. □

Theorem 4.3. *Let $M = (V, \langle\cdot,\cdot\rangle, R_A)$ be a model space where A is a self-adjoint linear transformation with two eigenvalues then M is $cvc(\lambda_1\lambda_2)$ and if the eigenvalues have the same sign, it is only $cvc(\lambda_1\lambda_2)$.*

Proof. Let $v \in V$ be a unit vector. v can be orthogonally decomposed into v_1 and v_2 where v_i is in the eigenspaces for λ_i. Let $w = -||v_2||v_1 + ||v_1||v_2$. Then $k(v, w) = \lambda_1\lambda_2(||v_1||^2 + ||v_2||^2)^2 = \lambda_1\lambda_2$. □

Theorem 4.4. *Let $M = (V, \langle\cdot,\cdot\rangle, R_A)$ be a model space where A is a self-adjoint linear transformation with three eigenvalues. If all eigenvalue have the same sign then M is only $cvc(\lambda_1\lambda_3)$ If only one eigenvalue is negative then M is $cvc(\lambda_1\lambda_2)$. If two eigenvalues are negative then M is $cvc(\lambda_2\lambda_3)$*

Proof. Let $v \in V$. Note that v can be orthogonally decomposed into a linear combinations v_1, v_2 and v_3 where v_i is a unit vector in the eigenspace corresponding to λ_i. So $v = av_1 + bv_2 + cv_3$. Since $R_A = R_{-A}$ we have two cases, all eigenvalues are positive, or there is only one negative eigenvalue. If all the eigenvalues are positive, let $\delta = \lambda_1\lambda_2, \varepsilon = \lambda_1\lambda_3$, and $\tau = \lambda_2\lambda_3$. If there is one negative eigenvalue let $\delta = \lambda_1\lambda_3, \varepsilon = \lambda_1\lambda_2$, and $\tau = \lambda_2\lambda_3$.

Now let $w = -\sqrt{\frac{(\delta-\varepsilon)(\varepsilon-\tau)}{\varepsilon-\delta}}v_1 + v_3$. Thompson [4] did the necessary algebra to show that $k(v, w) = \varepsilon$. □

These results are all consistent with previous research that only found model spaces with a single *cvc* value in three dimensions. These rsults also highlight an important difference between canonical algebraic curvature tensors built from skew-adjoint transformations and those built from self-adjoint transformations. In the skew-adjoint case we always get an interval of *cvc* values while in the self-adjoint case we seem to only get a single value. It is known that the span of curvature tensors built from skew-adjoint linear transformations is equal to the span of those built from self-adjoint transformations. This means that a curvature tensor with a single *cvc* value can be built by summing those with intervals of *cvc* values.

5 Open Questions

- What are the *cvc* value for curvature tensors constructed from self-adjoint linear transformations with four or more eigenvalues? The pattern suggests that they are only $cvc(\lambda_1\lambda_k)$ if all eigenvalues have the same sign and $cvc(\lambda_l\lambda_{l+1})$ where λ_l is the greatest negative eigenvalue, if the eigenvalues have differing signs.
- What is the relation between the *cvc* values of two model spaces with skew-adjoint curvature tensors to the model space whose curvature tensor is their sum?

6 Acknowledgements

Thanks to Dr. Corey Dunn for advising this research, and California State University, San Bernardino for hosting the REU. This research funded by NSF grant DMS-1461286

References

[1] S. Axler, Linear Algebra Done Right, Springer International Publishing, 2015.

[2] P. Gilkey, The Geometry of Curvature Homogeneous Pseudo-Riemannian Manifolds, Imperial College Press, 2007.

[3] B. Schmidt, J. Wolfson, *Three-manifolds with constant vector curvature*, Indiana University Mathematics Journal, 63 (2014), no. 6, 1757–1783.

[4] A. Thompson, *A look at constant vector curvature on three-dimensional model spaces according to curvature tensor*, CSUSB REU, 2014.

[5] A.C. Aitken, H.W. Turnbull, An Introduction to the Theory of Canonical Matrices, Dover Publications, 2004.