## A COMPLETE DESCRIPTION OF CONSTANT VECTOR CURVATURE IN THE 3-DIMENSIONAL SETTING

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ABSTRACT. A relatively new area of interest in differential geometry involves determining if a model space has the properties of constant vector curvature or constant sectional curvature. The natural setting in which to begin studying these properties is in 3-dimensional space. This paper in particular examines these properties in the Lorentzian setting, where the Ricci Operator takes on one of four Jordan-Normal forms. We determine that three of the four forms possess the property of constant vector curvature, and that under an orthonormal basis, only the diagonalizable family has constant sectional curvature, and that is only when the Ricci Operator has precisely one eigenvalue. By examining these families together, we draw some interesting and unifying conclusions that may be useful for exploring these properties in higher dimensions.

## 1. INTRODUCTION & BACKGROUND

We begin by considering a real-valued, *n*-dimensional vector space, and establish the following definitions.

**Definition 1.1.** An inner product on V is a function that takes each ordered pair (u, v) of elements of V to a number  $\langle u, v \rangle \in \mathbb{R}$  and has the following properties

- (1) Symmetry:  $\langle u, v \rangle = \langle v, u \rangle$ ,
- (2) Additivity:  $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$  for all  $u, v, w \in V$ ,
- (3) Homogeneity:  $\langle \lambda u, v \rangle = \lambda \langle u, v \rangle$  for all  $\lambda \in \mathbb{R}$  and  $u, v \in V$ ,
- (4) Non-Degenerate:  $\langle v, w \rangle \ge 0$  for all  $v \in V$ , and  $\langle v, v \rangle = 0$  if and only if v = 0

We say the inner product is **positive definite** rather than non-degenerate if  $\langle v, v \rangle \geq 0$  for all  $v \in V$ , and  $\langle v, v \rangle = 0$  if and only if v = 0.

**Definition 1.2.** Let V be a real-valued, finite-dimensional vector space. Define  $R: V \times V \times V \times V \to \mathbb{R}$  as a multilinear function that satisfies the following conditions:

(1) 
$$R(x, y, z, w) = -R(y, x, z, w)$$
  
(2)  $R(x, y, z, w) = R(z, w, x, y)$   
(3)  $R(x, y, z, w) + R(z, x, y, w) + R(y, z, x, w) = 0$ 

for all  $x, y, z, w \in V$ . We say R is an algebraic curvature tensor (ACT).

**Definition 1.3.** A model space  $M = (V, \langle \cdot, \cdot \rangle, R)$  consists of a vector space  $V = span\{e_1, \ldots, e_n\}$ , an inner product  $\langle \cdot, \cdot \rangle$  on V which is symmetric, bilinear and non-degenerate, as well as an ACT R.

We wish to examine certain properties of model spaces in the 3-dimensional setting. These properties, along with some methods for studying them, are defined below.

**Definition 1.4.** Let V be a real-valued, finite-dimensional vector space with  $v, w \in V$ , and suppose  $\pi = span\{v, w\}$  is a non-degenerate 2-plane. Then the **sectional curvature** is defined as

$$\kappa(\pi) = \frac{R(v, w, w, v)}{\langle v, v \rangle \langle w, w \rangle - \langle v, w \rangle^2}$$

Note that this definition is independent of the basis chosen.

**Definition 1.5.** A model space  $M = (V, \langle \cdot, \cdot \rangle, R)$  has constant sectional curvature  $\varepsilon$ , denoted  $csc(\varepsilon)$ , if  $\kappa(\pi) = \varepsilon$  for all non-degenerate  $\pi$ .

**Definition 1.6.** A model space  $M = (V, \langle \cdot, \cdot \rangle, R)$  has **constant vector curvature**  $\varepsilon$ , denoted  $cvc(\varepsilon)$ , if for every  $v \in V$ , there exists some  $w \in V$ such that  $\kappa(\pi) = \varepsilon$  and  $\pi$  is non-degenerate.

**Definition 1.7.** Let  $M = (V, \langle \cdot, \cdot \rangle, R)$  be a model space with  $\dim V = n$ and a basis  $\{e_1, \ldots, e_n\}$ , which is not necessarily orthonormal. Define  $[g_{ij}] = \langle \cdot, \cdot \rangle$ . The **Ricci Tensor**  $\rho$  is defined by

$$\rho(x,y) = \sum_{i,j}^{n} g^{ij} R_{xijy} = \langle Ae_i, e_j \rangle$$

Where  $[g^{ij}] = [g_{ij}]^{-1}$  and A is the Ricci Operator.

A relatively new and active area of research in Differential Geometry involves studying the properties of constant sectional curvature and constant vector curvature for *n*-dimensional model spaces. In this paper, we study these properties in the case of 3-dimensional space, where the ACT is determined uniquely by a simpler function, the Ricci Tensor. In particular, we consider four different cases of M, as dictated by the four possible Jordan-Normal Forms in the 3-dimensional Lorentzian setting.

For a precise definition of Jordan bases and Jordan forms, see [1]. The four possible forms will be explored later in this paper. Of the four types, Peng and Doktorova have extensively and completely studied the diagonalized form [2] [3]. Their results are included here for completeness. In addition, the Riemannian case has also been completely solved [4]. In that case, it was shown that every model space is  $cvc(\varepsilon)$ . We study the three remaining

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forms in this paper and draw some unifying results that may be useful in studying cvc in higher dimensions.

We proceed by first considering a significant amount of preliminary material that is necessary for examining each of the cases. We then present the results obtained by Peng and Doktorova for the diagonalized case. Afterwards, we move to examining the case of one real eigenvalue, then two real eigenvalues, and finally one real and one complex eigenvalue. In each case, we first recover the metric of that particular Jordan form. Next, we recover the Ricci and curvature tensor entries. Then, we eliminate all possible values of  $\varepsilon$  with the exception of one particular value. Finally, we utilize some preliminary material to either isolate the circumstances which allow M to have  $cvc(\varepsilon)$  or demonstrate that it cannot be  $cvc(\varepsilon)$ .

## 2. Preliminaries

There are a handful of results which are used throughout the paper; they are summarized here. Our first lemma we defer to [citation] for a proof.

Definition 2.1. The Jordan-Normal Forms are defined as follows:

$$J(\lambda,k) = \begin{bmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \lambda & 1 & 0 & \\ \vdots & 0 & \ddots & \ddots & \\ 0 & & \lambda & 1 \\ 0 & & \dots & \lambda \end{bmatrix} \qquad J(\tilde{a} + \tilde{b}i,k) = \begin{bmatrix} \tilde{a} & b & 1 & 0 & 0 & \dots \\ -\tilde{b} & \tilde{a} & 0 & 1 & 0 & \dots \\ 0 & 0 & \tilde{a} & \tilde{b} & 1 & 0 \\ 0 & 0 & -\tilde{b} & \tilde{a} & 0 & 1 \\ \vdots & \vdots & & \ddots & \ddots & \ddots \\ 0 & & \dots & & \tilde{a} \end{bmatrix}$$

## Lemma 2.1. Every matrix A is similar to a direct sum of Jordan blocks.

Using this fact, we can derive the four possible Jordan forms in 3-dimensional space. Also note that, by Lemma 2.1, we can examine all possible families of 3-dimensional model spaces by studying their corresponding Jordan forms. These Jordan forms are:

$\begin{bmatrix} \lambda_1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} \lambda & 1 & 0 \end{bmatrix}$
$\begin{bmatrix} 0 & \lambda_2 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & \lambda & 1 \end{bmatrix}$
$\begin{bmatrix} 0 & 0 & \lambda_3 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & \lambda \end{bmatrix}$
Type  I	$Type \ II$
$\begin{bmatrix} \lambda_1 & 1 & 0 \end{bmatrix}$	$\begin{bmatrix} \tilde{a} & \tilde{b} & 0 \end{bmatrix}$
$\begin{bmatrix} 0 & \lambda_1 & 0 \end{bmatrix}$	$\begin{vmatrix} -\tilde{b} & \tilde{a} & 0 \end{vmatrix}$
$\begin{bmatrix} 0 & 0 & \lambda_2 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & \lambda_2 \end{bmatrix}$
Type III	Type   IV

Much of what we do when determining whether a family of model spaces have constant vector curvature involves sectional curvature. To that end, the next two lemmas are key tools that we can utilize. **Lemma 2.2.** Let  $v, w \in V$ , a real-valued, finite-dimensional vector space. v, w span a non-degenerate 2-plane if and only if  $\langle v, v \rangle \langle w, w \rangle - \langle v, w \rangle^2 \neq 0$ .

Proof.  $(\Rightarrow)$ :

Suppose v, w span a non-degenerate 2-plane. Then there exists some orthonormal basis  $\{e_1, e_2\} \in V$  such that we can express v, w as

$$v = ae_1 + be_2$$
$$w = ce_1 + de_2$$

Notice that this can be expressed as the matrix equation

(1) 
$$\begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

where

(2) 
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Notice that  $det(A) \neq 0$  since the row vectors of A are linearly independent (v & w cannot be multiples of one another or they would not span a 2-plane).Now consider the denominator  $\langle v, v \rangle \langle w, w \rangle - \langle v, w \rangle^2$  for v, w defined as above. We obtain

$$\begin{aligned} \langle v, v \rangle &= a^2 \langle e_1, e_1 \rangle + b^2 \langle e_2, e_2 \rangle \\ \langle w, w \rangle &= c^2 \langle e_1, e_1 \rangle + d^2 \langle e_2, e_2 \rangle \\ \langle v, w \rangle^2 &= [ac \langle e_1, e_1 \rangle + bd \langle e_2, e_2 \rangle]^2 \end{aligned}$$

After expanding and collecting terms,

$$[a^2d^2 - 2abcd + b^2c^2]\langle e_1, e_1 \rangle \langle e_2, e_2 \rangle$$
$$\implies \pm 1(ad - bc)^2$$

since  $\langle e_i, e_i \rangle = \pm 1$ . Notice that this is simply  $\pm (det(A))^2$ , which we previously stated is nonzero. Thus

$$\langle v, v \rangle \langle w, w \rangle - \langle v, w \rangle^2 \neq 0$$

as desired.

 $(\Leftarrow)$ : Suppose now that

$$\langle v, v \rangle \langle w, w \rangle - \langle v, w \rangle^2 \neq 0$$

Suppose also, for the sake of contradiction, that there exists some nonzero  $u \in span\{v, w\}$  such that u is perpendicular to itself and every other vector. That is,

$$\langle u, x \rangle = 0$$
 for all  $x \in V$ 

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Now we can write u as a linear combination of v and w:

$$u = av + bw$$

Consider the inner products of u with each of v and w:

$$0 = \langle u, v \rangle = a \langle v, v \rangle + b \langle v, w \rangle$$
$$0 = \langle u, w \rangle = a \langle v, w \rangle + b \langle w, w \rangle$$
$$(3) \qquad \Longrightarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \langle v, v \rangle & \langle v, w \rangle \\ \langle v, w \rangle & \langle w, w \rangle \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

Define

(4) 
$$A = \begin{bmatrix} \langle v, v \rangle & \langle v, w \rangle \\ \langle v, w \rangle & \langle w, w \rangle \end{bmatrix}$$

Notice that  $det(A) \neq 0$  by assumption, so there exists only one solution to (3); in particular, the only solution is the trivial solution. But this contradicts the assumption of the existence of a nonzero u as defined above. Thus v, w must span a non-degenerate 2-plane.

**Lemma 2.3.** If  $v = ae_1 + be_2 + ce_3$  and  $w = xe_1 + ye_2 + ze_3$  and  $\kappa(\pi)$  is defined as above, then the numerator of  $\kappa(\pi)$  is the sum of:

(1)  $R_{1221}(ay - bx)^2$ , (2)  $R_{1331}(az - cx)^2$ , (3)  $R_{2332}(bz - cy)^2$ , (4)  $2R_{1223}(acy^2 - abyz - bcxy + b^2xz)$ , (5)  $2R_{2113}(a^2yz - acxy - abxz + bcx^2)$ , (6)  $2R_{1332}(abz^2 - acyz - bcxz + c^2xy)$ .

Proof. 1 of 6

For reference, the numerator we are working with follows:

$$R(ae_1 + be_2 + ce_3, xe_1 + ye_2 + ze_3, xe_1 + ye_2 + ze_3, ae_1 + be_2 + ce_3)$$

There are 4 possible ways to obtain  $R_{1221}$ . One of these is detailed in the diagram below.

We proceed in this manner for every other case.

(1)  $R_{1221} \implies R_{1221}(a^2y^2)$  (3)  $R_{2112} \implies R_{1221}(b^2x^2)$ (2)  $R_{1212} \implies -R_{1221}(abxy)$  (4)  $R_{2121} \implies -R_{1221}(abxy)$  Collecting these terms gives us

$$R_{1221}(a^2y^2 - 2abxy + b^2x^2)$$
$$\implies R_{1221}(ay - bx)^2$$

as desired.

Proof. 2 of 6

We now consider all of the ways to get  $R_{1331}$ :

(1) 
$$R_{1331} \implies R_{1331}(a^2z^2)$$
 (3)  $R_{3113} \implies R_{1331}(c^2x^2)$   
(2)  $R_{1313} \implies -R_{1331}(acxz)$  (4)  $R_{3131} \implies -R_{1331}(acxz)$ 

Collecting these terms gives us

$$R_{1331}(a^2z^2 - 2acxz + c^2x^2) \implies R_{1331}(az - cx)^2$$

as desired.

Proof. 3 of 6

Next consider all of the ways to get  $R_{2332}$ :

(1) 
$$R_{2332} \implies R_{2332}(c^2y^2)$$
 (3)  $R_{3223} \implies R_{2332}(b^2z^2)$   
(2)  $R_{2323} \implies -R_{2332}(bcyz)$  (4)  $R_{3232} \implies -R_{2332}(bcyz)$ 

Collecting these terms gives us

$$R_{2332}(c^2y^2 - 2bcyz + b^2z^2)$$
$$\implies R_{2332}(cy - bz)^2$$

as desired.

Proof. 4 of 6

Next consider all of the ways to get  $R_{1223}$ :

Collecting these terms gives us

$$R_{1223}(2acy^2 - 2abyz - 2bcxy + b^2xz)$$
  
$$\implies 2R_{1223}(acy^2 - abyz - bcxy + b^2xz)$$

as desired.

Proof. 5 of 6

Next consider all of the ways to get  $R_{2113}$ :

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Collecting these terms gives us

$$R_{2113}(2a^2yz - 2acxy - 2acxz + 2bcx^2)$$
  
$$\implies 2R_{2113}(a^2yz - acxy - abxz + bcx^2)$$

as desired.

Proof. 6 of 6

Finally, consider all of the ways to get  $R_{1332}$ :

(1) $R_{1332} \implies R_{1332}(abz^2)$	(5) $R_{2313} \implies -R_{1332}(bcxz)$
(2) $R_{2331} \implies R_{1332}(abz^2)$	(6) $R_{3132} \implies -R_{1332}(bcxz)$
(3) $R_{1323} \implies -R_{1332}(acyz)$	(7) $R_{3123} \implies R_{1332}(c^2 xy)$
(4) $R_{3231} \implies -R_{1332}(acyz)$	(8) $R_{3213} \implies R_{1332}(c^2 xy)$

Collecting these terms gives us

$$R_{1332}(2abz^2 - 2acyz - 2bcxz + 2c^2xy)$$
  
$$\implies 2R_{1332}(abz^2 - acyz - bcxz + c^2xy)$$

as desired.

The last result we need involves the concept of generalized eigenspaces. All of the material that follows in this section is adapted from [1].

**Definition 2.2.** Suppose  $A \in \mathcal{L}(V)$  and  $\lambda \in \mathbf{F}$ , where  $\mathbf{F} = \mathbb{R}$  or  $\mathbb{C}$ . The generalized eigenspace of A corresponding to  $\lambda$ , denoted  $G(\lambda, A)$ , is defined by

 $G(\lambda, A) = null(A - \lambda I)^{dimV}$ 

**Definition 2.3.** An operator on a complex inner product space is normal if it commutes with its adjoint.

**Lemma 2.4.** Suppose  $A \in \mathcal{L}(V)$  is normal. Then eigenvectors of A corresponding to distinct eigenvalues are orthogonal.

**Lemma 2.5.** Suppose  $A \in \mathcal{L}(V)$ . Let  $n = \dim V$ . Then

$$V = nullA^n \oplus rangeA^n$$

**Lemma 2.6.** Suppose  $A \in \mathcal{L}(V)$  and  $p \in \mathcal{P}(F)$ . Then null p(A) and range p(A) are invariant under A.

**Lemma 2.7.** Suppose V is a complex vector space and  $A \in \mathcal{L}(V)$ . Let  $\lambda_1, \ldots, \lambda_m$  be the distinct eigenvalues of A, with multiplicities  $d_1, \ldots, d_m$ . Then there is a basis of V with respect to which A has a block diagonal matrix of the form

$$\begin{bmatrix} \mathcal{M}_1 & 0 \\ & \ddots & \\ 0 & \mathcal{M}_m \end{bmatrix}$$

where each  $\mathcal{M}_i$  is a  $d_i$ -by- $d_i$  upper-triangular matrix of the form

(5) 
$$\mathcal{M}_j = \begin{bmatrix} \lambda_j & * \\ & \ddots \\ 0 & & \lambda_j \end{bmatrix}$$

**Lemma 2.8.** Let  $A \in \mathcal{L}(V)$ . Suppose  $\lambda_1, \ldots, \lambda_m$  are distinct eigenvalues of T and  $v_1, \ldots, v_m$  are corresponding generalized eigenvectors. Then  $v_1, \ldots, v_m$  is linearly independent.

**Theorem 2.1.** Let V be an arbitrary complex vector space and let A be a linear operator on V. Then the following assertions are true:

- (1) V decomposes as the direct sum of its generalized eigenspaces.
- (2) If  $\varphi_{ij}$  is a non-degenerate inner product on V, and A is self-adjoint with respect to  $\varphi_{ij}$ , then the generalized eigenspaces of A are orthogonal.

*Proof.* In this proof, we allow  $\lambda$  to be either real or complex. We also let V have m eigenvalues.

We begin by showing (1). Let  $n = \dim V$ . We prove by induction on n. The result clearly holds for n = 1. We proceed by assuming n > 1, and that the result holds for all vector spaces of a smaller dimension.

Since V is a complex vector space, it has at least one eigenvalue. Define  $G(\lambda_i, A)$  as the generalized eigenspace for each distinct  $\lambda_i$  of A. We use Lemma 2.5 with  $A - \lambda_1 I$  to obtain

(\*) 
$$V = G(\lambda_1, A) \oplus U$$

where  $U = range(A - \lambda_1 I)^n$ . Now since U is invariant under A (simply use  $p(z) = (z - \lambda I)^n$  with Lemma 2.6) and  $G(\lambda_1, A) \neq \{0\}$ , we have dim U < n. Thus our induction hypothesis applies to  $A|_U$ . Furthermore, each eigenvalue of  $A|_U$  is in  $\{\lambda_2, \ldots, \lambda_m\}$ . By the induction hypothesis,

$$U = G(\lambda_2, A|_U) \oplus \ldots \oplus G(\lambda_m, A|_U)$$

. It remains only to show that  $G(\lambda_k, A|_U) = G(\lambda_k, A)$  for k = 2, ..., m.

Fix  $k \in \{2, \ldots, m\}$ , so that  $G(\lambda_k, A|_U) \subset G(\lambda_k, A)$  is clear. Now suppose  $v \in G(\lambda_k, A)$ . By (\*), we have

$$v = v_1 + u$$

where  $v_1 \in G(\lambda_1, A)$  and  $u \in U$ . We have

$$u = v_2 + \ldots + v_m$$

where each  $v_j \in G(\lambda_j, A|_U) \subset G(\lambda_j, A)$ . Thus

$$v = v_1 + v_2 + \ldots + v_m$$

Now by Lemma 2.8, we have  $v_j = 0$  except, possibly, when j = k. Since  $k \neq 1$ , we know at least that  $v_1 = 0$  and so  $v = u \in U$ . That is,  $v \in U \implies v \in G(\lambda_k, A|_U)$ . So we have shown that  $G(\lambda_k, A|_U) = G(\lambda_k, A)$  for  $k = 2, \ldots, m$ . Thus we have

$$V = G(\lambda_1, A) \oplus \ldots \oplus G(\lambda_m, A)$$

as desired.

We now need to show that the generalized eigenspaces are perpendicular. By Definition 2.2, A is normal; then by Lemma 2.4, its eigenvectors are orthogonal. Now we defined each generalized eigenspace as corresponding to a distinct eigenvalue, meaning that  $\lambda_j \in G(\lambda_j, A)$  and, in particular, the *jth* eigenvector resides only in the *jth* eigenspace. Let  $w_j \in G(\lambda_j, A)$  denote the *jth* eigenvector corresponding to  $\lambda_j$ , and let  $v_j \in G(\lambda_j, A)$  be a vector other than the eigenvector. Note that

$$v_i = aw_i + br$$

where r is a unit vector and  $a, b \in \mathbf{F}$ . r cannot have entries below or above the rows corresponding to  $\lambda_i$  by Lemma 2.7. Clearly, then,

$$v_i \perp v_j$$
  
 $\implies G(\lambda_i, A) \perp G(\lambda_j, A)$ 

for  $i \neq j$  since any vector from one space is perpendicular to any vector from another space.

**Corollary 2.1.** The eigenspace corresponding to any Jordan block is perpendicular to the eigenspace corresponding to any other Jordan block.

*Proof.* Let  $G(\lambda_i, A)$  be the eigenspace corresponding to the Jordan block  $J(\lambda_i, k)$ , where  $\lambda_i \in \mathbb{C}$  or  $\mathbb{R}$  and k is the size of the Jordan block. We may construct V in the following way (by Lemma 2.1):

$$V = \bigoplus_{i=1}^{n} J(\lambda_i, k)$$

where  $k \in \{1, \ldots, dimV\}$ . The result now follows immediately from Theorem 2.2.

It's worth noting that this result holds for real vector spaces through the process of complexification. For a detailed discussion of complexification, see [1]. With these results established, we can begin to examine which models spaces are  $cvc(\varepsilon)$  and which are not.

## 3. Three Real Eigenvalues

We begin by presenting the work done by [2] and [3]. Define

(6) 
$$A = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

Since this matrix is diagonalized (and A is self-adjoint), we are guaranteed by the spectral theorem to have an orthonormal basis as follows:

(7) 
$$[\varphi_{ij}] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where the negative in the first slot comes from the Lorentzian setting. We can now determine both the Ricci and curvature tensor entries:

$\rho_{11} = -\lambda_1 = R_{1221} + R_{1331}$	$\rho_{12} = 0 = R_{1332}$
$\rho_{22} = \lambda_2 = -R_{1221} + R_{2332}$	$\rho_{13} = 0 = R_{1223}$
$\rho_{33} = -R1331 + R_{2332}$	$\rho_{23} = 0 = R_{2113}$
$R_{1221} = \frac{\lambda_3 - \lambda_1 - \lambda_2}{2}$	$R_{1223} = 0$
$R_{1331} = \frac{\lambda_2 - \lambda_3 - \lambda_1}{2}$	$R_{1332} = 0$
$R_{2332} = \frac{\lambda_2 + \bar{\lambda_3} - \lambda_1}{2}$	$R_{2113} = 0$
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For the remainder of this section, we let

$$\alpha = R_{1221}, \beta = R_{1331}, \gamma = R_{2332}$$

We also assume  $\alpha > \beta$ . With the curvature entries, we can begin to determine whether this family of model spaces has  $cvc(\varepsilon)$ .

**Lemma 3.1.** Let  $M = (V, \langle \cdot, \cdot \rangle, R)$  be a 3-dimensional Lorentzian model space of Jordan Type I with the derived metric and curvature entries. If M has  $cvc(\varepsilon)$ , and:

(1)  $-\alpha, -\beta \leq \gamma$ , then  $\varepsilon = -\alpha$ . (2)  $\gamma \leq -\alpha, -\beta$ , then  $\varepsilon = -\beta$ .

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(3)  $-\alpha < \gamma < \beta$ , then  $\varepsilon \in \{-\alpha, \gamma, -\beta\}$ . (4)  $\alpha = \beta$ , then  $\varepsilon = -\alpha = -\beta$ .

For a proof of this lemma, see the original statement in [3] or [2].

**Theorem 3.1.** Let  $M = (V, \langle \cdot, \cdot \rangle, R)$  be a 3-dimensional Lorentzian model space of Jordan Type I with the derived metric and curvature entries. Then the following hold:

- (1) If  $\gamma > -\alpha, -\beta$  then M has  $cvc(-\alpha)$ .
- (2) If  $\gamma < -\alpha, -\beta$  then M has  $cvc(-\beta)$ .
- (3) If  $-\alpha \leq \gamma < -\beta$  or  $-\alpha < \gamma \leq -\beta$  then M does not have  $cvc(\varepsilon)$  for any  $\varepsilon \in \mathbb{R}$ .
- (4) If  $-\alpha = -\beta \neq \gamma$ , then M has  $cvc(-\alpha)$ .
- (5) If  $-\alpha = -\beta = \gamma$ , then M has  $\csc(\frac{\lambda}{2})$ .

For a proof of this theorem, see the original statement in [2]. What we have shown is that depending on the ordering of the eigenvalues of A, M either has  $cvc(\varepsilon)$ ,  $csc(\varepsilon)$ , or neither. Going forward, we'll see similar results appear in the other forms.

## 4. One Real Eigenvalue

We turn our attention to the model spaces of type II; for quick reference, define

(8) 
$$A = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$$

Our first order of business will be to recover the metric of this form so that we can use it, in conjunction with the matrix, to evaluate the Ricci tensor entries. Note that we allow  $\varphi_{ij} = \langle e_i, e_j \rangle$ , where  $\{e_i, e_j\}$  come from whatever basis we are working with. We first compare the inner products  $\langle Ae_1, e_2 \rangle = \langle Ae_2, e_1 \rangle$ ,  $\langle Ae_1, e_3 \rangle = \langle Ae_3, e_1 \rangle$  and  $\langle Ae_2, e_3 \rangle = \langle Ae_3, e_2 \rangle$ . As an example, consider:

$$\begin{array}{lll} \langle Ae_1, e_2 \rangle & = & \lambda \varphi_{12} \\ \langle Ae_2, e_1 \rangle & = & \varphi_{11} + \lambda \varphi_{12} \end{array}$$

Then we conclude that  $\varphi_{11} = 0$ . Repeating this process, we obtain:

(9) 
$$\varphi_{ij} = \begin{bmatrix} 0 & 0 & \varphi_{22} \\ 0 & \varphi_{22} & \varphi_{23} \\ \varphi_{22} & \varphi_{23} & \varphi_{33} \end{bmatrix}$$

We let  $a = \varphi_{22}$  and employ the change of basis:

$$f_1 = \frac{1}{\sqrt{a}}e_1,$$
  

$$f_2 = \frac{1}{\sqrt{a}}e_2,$$
  

$$f_3 = \frac{1}{\sqrt{a}}e_3.$$

Note that a > 0 since if a < 0 then we exchange the metric for its negative, and if a = 0 the metric is degenerate, which contradicts our choice of the inner product. This has the effect of putting 1's on the off-diagonal of  $\varphi_{ij}$ . Now make one last change of basis:

$$g_1 = f_1, g_2 = xf_1 + f_2, g_3 = yf_1 + xf_2 + f_3.$$

We wish to have  $\varphi_{23} = \varphi_{33} = 0$ . Solving for x and y gives

$$x = \frac{-b}{2} \qquad y = \frac{3}{4}b^2 - c$$

It can be shown that these change of bases preserve the matrix A. Thus we end up with

(10) 
$$\varphi_{ij} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Utilizing this metric with the matrix A, we calculate the Ricci tensor entries to be

$$\begin{array}{ll} \rho_{11} = 0 = R_{1221} & \rho_{22} = \lambda = 2R_{1223} \\ \rho_{12} = 0 = R_{2113} & \rho_{23} = -1 = R_{1332} \\ \rho_{13} = \lambda = R_{1313} + R_{1223} & \rho_{33} = 0 = R_{2332} \end{array}$$

Thus we have

With the curvature entries in hand, we can now examine whether this family of model spaces has  $cvc(\varepsilon)$  for some  $\varepsilon$ .

**Lemma 4.1.** Let  $M = (V, \langle \cdot, \cdot \rangle, R)$  be a 3-dimensional Lorentzian model space of Jordan Type II with the derived metric and curvature entries. If M has  $cvc(\varepsilon)$ , then  $\varepsilon = \frac{\lambda}{2}$ .

*Proof.* Let  $w = xe_1 + ye_2 + ze_3$  for some  $x, y, z \in \mathbb{R}$ , with not more than one of them 0. Suppose  $v = e_1$ . Then from Definition 1.4, we have

$$\kappa(\pi) = \frac{z^2(-\lambda/2)}{-z^2} = \frac{\lambda}{2} = \varepsilon$$

Notice that we are free to choose  $z \neq 0$  since we can stipulate x = 0 without affecting this equation.

This lemma only establishes what the value of  $\varepsilon$  must be for M to have  $cvc(\varepsilon)$ . It does not guarantee that there exists a w for every possible v that will result in this  $\varepsilon$ . As it turns out, there is a particular group of vectors that cause M to fail to be  $cvc(\frac{\lambda}{2})$ . In order to see why this is the case, we first establish this lemma.

**Lemma 4.2.** If a model space that is Type II Jordan form has  $cvc(\frac{\lambda}{2})$ , then the following equation must be satisfied:

$$abz^2 - acyz - bcxz + c^2xy = 0$$

for  $v = ae_1 + be_2 + ce_3$  and  $w = xe_1 + ye_2 + ze_3$ .

*Proof.* Let v, w be defined as above. We start with the numerator, using our curvature entries and Lemma 2.3:

$$-\frac{\lambda}{2}(az-cx)^2 + 2(\frac{\lambda}{2})(acy^2 - abyz - bcxy + b^2xz) + 2(-1)(abz^2 - acyz - bcxz + c^2xy)$$
$$\implies -\frac{\lambda}{2}(az-cx)^2 + \lambda(acy^2 - abyz - bcxy + b^2xz) - 2(abz^2 - acyz - bcxz + c^2xy)$$

We now calculate the denominator:

Which, plugging in to the denominator of Definition 1.4, gives us:

$$\begin{aligned} \langle v,v\rangle\langle w,w\rangle &= 4acxz+2acy^2+2b^2xz+b^2y^2,\\ -\langle v,w\rangle^2 &= -a^2z^2-c^2x^2-b^2y^2-2acxz-2abyz-2bcxy. \end{aligned}$$

Thus we have

$$-(az - cx)^2 + 2(acy^2 - abyz - bcxy + b^2xz).$$

Combining the numerator and denominator into Definition 1.4, and setting  $\kappa(\pi) = \frac{\lambda}{2}$  from Lemma 4.1, we have:

$$\frac{-\frac{\lambda}{2}(az - cx)^2 + \lambda(acy^2 - abyz - bcxy + b^2xz) - 2(abz^2 - acyz - bcxz + c^2xy)}{-(az - cx)^2 + 2(acy^2 - abyz - bcxy + b^2xz)} = \frac{\lambda}{2}$$

The right-hand side of this equation will be

$$-\frac{\lambda}{2}(az-cx)^2 + \lambda(acy^2 - abyz - bcxy + b^2xz)$$

Cancelling terms and simplifying, we arrive at the desired result:

$$abz^2 - acyz - bcxz + c^2xy = 0$$

**Theorem 4.1.** Let  $M = (V, \langle \cdot, \cdot \rangle, R)$  be a 3-dimensional Lorentzian model space of Jordan Type II. Such a model space cannot be  $cvc(\varepsilon)$  for any  $\varepsilon$ .

*Proof.* Let v and w be defined as in Lemma 4.2. Suppose M is  $cvc(\varepsilon)$ . Then, from Lemma 4.1, we know that  $\varepsilon = \frac{\lambda}{2}$ . Suppose we are given  $v = ae_1 + be_2$ . From Lemmas 2.3 and 4.2, we obtain

$$abz^2 = 0.$$

Suppose it were the case that  $a \neq 0$  and  $b \neq 0$ , so that z = 0. Now, utilizing Lemma 2.2, we have:

$$\begin{split} \langle v,v\rangle &= b^2,\\ \langle w,w\rangle &= 2xz+y^2,\\ \langle v,w\rangle^2 &= (az+by)^2. \end{split}$$

Which gives us:

$$2b^{2}xz + b^{2}y^{2} - a^{2}z^{2} - 2abyz - b^{2}y^{2} \neq 0$$
  
$$\implies z(2b^{2}x - a^{2}z - 2aby) \neq 0.$$

But we already stated z = 0, which contradicts Lemma 2.2. Then we know that v, w must span a degenerate 2-plane, and so we have found a set of "bad vectors" that prevent the *cvc* condition from holding. It is the case, then, that  $\varepsilon \neq \frac{\lambda}{2}$ , which with Lemma 4.1 implies that M is not  $cvc(\varepsilon)$  for any  $\varepsilon$ .

And so we have now encountered our first example of a family of model spaces that does not have the property of constant vector curvature. The next case will prove to be more interesting by having  $cvc(\varepsilon)$  in some cases but not others.

## 5. Two Real Eigenvalues

We now consider the family of model spaces of Type III Jordan-Normal form. For quick reference, let

(11) 
$$A = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}$$

Once again, we must first recover the metric for this form. Here we make use of Theorem 2.2. Notice that  $\lambda_1$  and  $\lambda_2$  constitute two different

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eigenspaces, so that we know  $G(\lambda_1, A) \perp G(\lambda_2, A)$  and that they span the vector space. This gives us

(12) 
$$[\varphi_{ij}] = \begin{bmatrix} \varphi_{11} & \varphi_{12} & 0\\ \varphi_{12} & \varphi_{22} & 0\\ 0 & 0 & \varphi_{33} \end{bmatrix}$$

We now employ the following change of basis to A:

$$f_1 = e_1, f_2 = e_2 + xe_1, f_3 = e_3.$$

This provides  $\varphi_{22} = 0$ . The next change of basis provides  $\varphi_{11} = 0$ :

$$f_1 = e_1,$$
  
 $f_2 = e_2,$   
 $f_3 = e_3 + xe_1.$ 

Note that these change of bases do not change A. Upon scaling  $\varphi_{12} = \varphi_{33} = 1$ , we obtain

(13) 
$$[\varphi_{ij}] = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

With this metric in hand, we can calculate the Ricci Tensor as well as the corresponding curvature entries. From Definition 1.8, we determine that the Ricci Tensor is expressed as

$$\rho(x,y) = \sum_{i,j}^{3} R_{x12y} + R_{x21y} + R_{x33y}.$$

From this we obtain the following:

$$\begin{aligned}
\rho_{11} &= 0 = R_{1331} & \rho_{22} = 1 = R_{2332} \\
\rho_{12} &= \lambda_1 = -R_{1221} + R_{1332} & \rho_{23} = 0 = R_{1223} \\
\rho_{13} &= 0 = R_{2113} & \rho_{33} = \lambda_2 = 2R_{1332}
\end{aligned}$$

Now let  $\alpha = \frac{\lambda_2}{2} - \lambda_1$  and  $\beta = \frac{\lambda_2}{2}$ . Then we have

$$\begin{array}{ll} R_{1221} = \alpha & & R_{2332} = 1 & & R_{1332} = \beta \\ R_{1331} = 0 & & R_{1223} = 0 & & R_{2113} = 0 \end{array}$$

With these curvature entries in hand, we can determine what the value of  $\varepsilon$  should be if M is  $cvc(\varepsilon)$ .

**Lemma 5.1.** Let  $M = (V, \langle \cdot, \cdot \rangle, R)$  be a 3-dimensional Lorentzian model space of Type III with the derived metric and curvature entries. If M has  $cvc(\varepsilon)$ , then  $\varepsilon = -\alpha$ .

*Proof.* Suppose M has  $cvc(\varepsilon)$ . Let  $w = xe_1 + ye_2 + ze_3$  for some  $x, y, z \in \mathbb{R}$ , with not more than one of them being 0.

Suppose  $v = e_1$ . Then from Definition 1.4 we have

$$\frac{\alpha y^2}{-y^2} = -\alpha = \varepsilon.$$

Note that we can allow  $y \neq 0$  by stipulating x = 0.

We now need to see if, for every v, there is guaranteed an appropriate w. We first impose the restriction that  $\lambda_1 = \lambda_2$ .

**Lemma 5.2.** Let  $M = (V, \langle \cdot, \cdot \rangle, R)$  be a 3-dimensional Lorentzian model space of Type III with the derived metric and curvature entries. M does not have  $cvc(\varepsilon)$  when  $\lambda_1 = \lambda_2$ .

*Proof.* Let  $w = xe_1 + ye_2 + ze_3$  for some  $x, y, z \in \mathbb{R}$ , with not more than one of them being 0. Suppose also for the sake of contradiction that M has  $cvc(\varepsilon)$  for  $\lambda_1 = \lambda_2$ , and consider  $v = e_3$ . Then we have

$$\kappa(\pi) = \frac{2xy\beta + y^2}{2xy + z^2 - z^2}$$
$$= \beta + \frac{y}{2x}.$$

Note that  $\beta = \alpha + \lambda_1$ . We wish to have  $\kappa(\pi) = \varepsilon = -\alpha$ . So we have

$$\alpha + \lambda_1 + \frac{y}{2x} = -\alpha$$
$$\implies \frac{y}{2x} = -2\alpha - \lambda_1$$

Now substituing  $\alpha = \frac{\lambda_2}{2} - \lambda_1$ , we have

$$\lambda_1 - \lambda_2 = \frac{y}{2x}.$$

But  $\lambda_1 = \lambda_2 \implies y = 0$ , which we have seen results in a degenerate 2-plane by Lemma 2.2. This contradicts our assumption of  $\lambda_1 = \lambda_2$ . Thus, we have  $\kappa(\pi) = \varepsilon = -\alpha$  if  $\lambda_1 \neq \lambda_2$ .

Before considering further restrictions on M, we establish this lemma.

**Lemma 5.3.** Let  $M = (V, \langle \cdot, \cdot \rangle, R)$  be a 3-dimensional Lorentzian model space of Type III with the derived metric and curvature entries. If M has  $cvc(-\alpha)$ , then the following equation must be satisfied:

$$(cy - bz)^{2} + 2(\alpha + \beta)(abz^{2} - acyz - bcxz + c^{2}xy) = 0.$$

*Proof.* We begin with the numerator of the sectional curvature. From Lemma 2.3, we have

$$\alpha(ay - bx)^{2} + (1)(cy - bz)^{2} + 2(\beta)(abz^{2} - acyz - bcxz + c^{2}xy).$$

Now for the denominator:

$$\begin{aligned} \langle v, v \rangle &= 2ab + c^2, \\ \langle w, w \rangle &= 2xy + z^2, \\ \langle v, w \rangle^2 &= (ay + bx + cz)^2. \end{aligned}$$

Expanding and collecting terms, we get

$$-(ay - bx)^2 + 2(abz^2 - acyz - bcxz + c^2xy).$$

Now utilizing Definition 1.4 with  $\kappa(\pi) = -\alpha$ , we have

$$\frac{\alpha(ay - bx)^2 + (cy - bz)^2 + 2\beta(abz^2 - acyz - bcxz + c^2xy)}{-(ay - bx)^2 + 2(abz^2 - acyz - bcxz + c^2xy)} = -\alpha.$$

The right-hand side of this equation will be

$$\alpha(ay - bx)^2 - 2\alpha(abz_a^2 cyz - bcxz + c^2 xy).$$

Bringing everything to the left-hand side gives us the desired result:

$$(cy - bz)^{2} + 2(\alpha + \beta)(abz^{2} - acyz - bcxz + c^{2}xy) = 0.$$

**Lemma 5.4.** Let  $M = (V, \langle \cdot, \cdot \rangle, R)$  be a 3-dimensional Lorentzian model space of Type III with the derived metric and curvature entries. M does not have  $cvc(\varepsilon)$  when  $\lambda_1 < \lambda_2$ .

*Proof.* Suppose  $v = ae_1 + ce_3$  and  $w = xe_1 + ye_2 + ze_3$ , and note that  $\alpha + \beta = \lambda_2 - \lambda_1 \neq 0$ . We scale c down to 1 for simplicity in this case. From Lemma 5.1, we know that if M attains  $cvc(\varepsilon)$ , then it must be the case that  $\varepsilon = -\alpha$  for  $\alpha$  defined as above. Then from Lemmas 2.3 and 5.2, we have

$$y^2 + 2(\alpha + \beta)(xy - ayz) = 0.$$

If y = 0, then v, w span a degenerate 2-plane, so it must be the case that  $y \neq 0$ . Then the above equation simplifies to

$$y + 2(\alpha + \beta)(x - az) = 0$$

(\*) 
$$\implies 2(x-az) = \frac{-y}{\alpha+\beta}.$$

We now need to check if v, w span a non-degenerate 2-plane. Using Lemma 2.2 and v, w as defined above, we have

$$\langle v, v \rangle \langle w, w \rangle - \langle v, w \rangle^2 = 2xy + z^2 - (ay + z)^2$$
  
=  $2xy - a^2y^2 - ayz$ 

$$= y[-a^2y - 2(x - az)].$$

Now substituting in (\*) and factoring a y gives us

$$y^2[-a^2 + \frac{1}{\alpha + \beta}].$$

Choose  $a^2 = \frac{1}{\alpha + \beta}$  where  $\alpha + \beta > 0$ . Then we have

$$y^{2}\left[\frac{-1}{\alpha+\beta} + \frac{1}{\alpha+\beta}\right] = 0,$$

which demonstrates that v, w actually span a degenerate 2-plane. Since we assumed  $\alpha + \beta > 0$ , we now know that  $\lambda_1 < \lambda_2$  produces degenerate 2-planes. That is, for the given v, there exists no  $w \in V$  for which  $\kappa(\pi) = -\alpha$ .

**Theorem 5.1.** Let  $M = (V, \langle \cdot, \cdot \rangle, R)$  be a 3-dimensional Lorentzian model space of Type III with the derived metric and curvature entries. If  $\lambda_1 > \lambda_2$ , then M has  $cvc(-\alpha)$ .

*Proof.* Suppose  $\lambda_1 > \lambda_2$ , and let  $w = xe_1 + ye_2 + ze_3$  for some  $x, y, z \in \mathbb{R}$ . We now consider the remaining forms of v.

Case 1:

Suppose  $v = e_2$ . Then we have

$$\kappa(\pi) = \frac{\alpha x^2 + z^2}{-x^2}$$
$$= -\alpha - \frac{z^2}{x^2}$$
$$= -\alpha = \varepsilon$$

when z = 0 and  $x \neq 0$ , which is allowable under the restrictions on w. Case 2:

Suppose  $v = ae_1 + be_2$ . From Lemma 5.2, we have

$$-b^2z^2 + 2abz^2(\alpha + \beta) = 0$$

We can let z = 0 since  $e_1, e_2$  are a hyperbolic pair and so span a nondegenerate 2-plane. To prove this, we utilize Lemma 2.2 with  $w = xe_1 + ye_2$ . We get:

$$(2ab)(2xy) - (ay + bx)^2) \neq 0$$
  
$$\implies 4abxy - a^2y^2 - 2abxy - b^2x^2 \neq 0$$
  
$$\implies -(ay - bx)^2 \neq 0.$$

This is only 0 when ay = bx, but we can choose x and y for any given nonzero a and b such that this does not occur.

Case 3:

Suppose 
$$v = be_2 + ce_3$$
, with  $b \neq 0$  and  $c \neq 0$ . From Lemma 5.2, we have  
 $(cy - bz)^2 + 2(\alpha + \beta)(-bcxz + c^2xy) = 0$ 

$$\implies (cy - bz)^2 + 2x(\alpha + \beta)(c^2y - bcz) = 0$$

Setting  $y = \frac{bz}{c}$  with  $x \neq 0$ , we find  $w = xe_1 + \frac{bz}{c}e_2$  as a non-degenerate pairing. From Lemma 2.2, we have

$$c^{2}\left(\frac{2xbz}{c}\right) - (bx)^{2} \neq 0$$
$$\implies 2bcxz - b^{2}x^{2} \neq 0.$$

Since  $b, x \neq 0$ , we simply choose an appropriate z so that  $2cz \neq bx$ . Case 4:

Suppose  $v = ae_1 + be_2 + ce_3$  with  $a, b, c \neq 0$ . For simplicity, we scale c down to 1. From Lemma 5.2, we have

$$(y-bz)^2 + 2(\alpha+\beta)(abz^2 - ayz - bxz + xy) = 0$$

$$\implies z^2[b^2 + 2ab(\alpha + \beta)] + z[-2by - 2ay(\alpha + \beta) - 2bx(\alpha + \beta)] + [y^2 + 2xy(\alpha + \beta)] = 0$$

when factoring for z. Using the quadratic formula, we arrive at

$$z = \frac{by + ay(\alpha + \beta) + bx(\alpha + \beta) \pm (\alpha + \beta)(ay - bx)}{b^2 + 2ab(\alpha + \beta)}.$$

Then the positive and negative expressions for z are given by

(14) 
$$z = \frac{y}{b} \qquad z = \frac{y + 2x(\alpha + \beta)}{b + 2a(\alpha + \beta)}$$

In either case, we can find a w that pairs with this v such that v, w span a non-degenerate 2-plane. However, we need only consider  $z = \frac{y}{b}$  since  $b \neq 0$ . Suppose  $w = xe_1 + ye_2 + \frac{y}{b}e_3$ . Then from Lemma 2.2 we have:

$$(2ab+1)(2xy + \frac{y^2}{b^2}) - (ay + bx + \frac{y}{b})^2 \neq 0$$
  
$$\implies 4abxy + \frac{2aby^2}{b} + 2xy + \frac{y^2}{b^2} - a^2y^2 - b^2x^2 - \frac{y^2}{b^2} - \frac{2ay^2}{b} - 2abxy - 2xy \neq 0$$
  
$$\implies -(ay - bx)^2 \neq 0.$$

As in Case 1, this is a non-degenerate pairing.

Having exhausted all possible vectors v, we have shown that for  $\lambda_1 > \lambda_2$ , M has  $cvc(\varepsilon)$  when  $\varepsilon = -\alpha$ .

This case has some similarities with the diagonalized case, in that the ordering of the eigenvalues is important in determining whether M has  $cvc(\varepsilon)$ . We will see in the next case that introducing a complex Jordan block eliminates the need for any restrictions on the eigenvalues.

## 6. One Real and One Complex Eigenvalue

The final case for our consideration is a Type IV model space. For quick reference, define

(15) 
$$A = \begin{bmatrix} \tilde{a} & \tilde{b} & 0\\ -\tilde{b} & \tilde{a} & 0\\ 0 & 0 & \lambda_2 \end{bmatrix}$$

We may assume without loss of generality that  $\tilde{b} > 0$ .

In this form we needn't concern ourselves with whether  $\lambda_1 = \lambda_2$ , because that simply is not possible. Therefore we can immediately apply Theorem 2.2, and so  $G(\lambda_1, A) \perp G(\lambda_2, A)$  and these eigenspaces span the vector space. Let  $\varphi_{ij} = \langle e_i, e_j \rangle$ , where  $\{e_i, e_j\}$  comes from the basis we are working with; then we have

(16) 
$$\varphi_{ij} = \begin{bmatrix} \varphi_{11} & \varphi_{12} & 0\\ \varphi_{12} & \varphi_{22} & 0\\ 0 & 0 & \varphi_{33} \end{bmatrix}.$$

By comparing the inner products  $\langle Ae_1, e_2 \rangle = \langle Ae_2, e_1 \rangle$  we obtain  $\varphi_{11} = -\varphi_{22}$ . Now employ the following change of basis:

$$f_1 = xe_1 + ye_2 + e_3$$
  

$$f_2 = ye_1 - xe_2 + e_3$$
  

$$f_3 = e_3$$

Note that this change of basis preserves the form of A. We wish to have  $\varphi_{11} = 0$ , so we examine

$$\langle f_1, f_1 \rangle = 2xy\varphi_{12} + (x^2 - y^2)\varphi_{11} = 0$$

and set x = 1. Solving for y then gives us

$$y = \frac{\varphi_{12}}{\varphi_{11}} + \sqrt{1 + \frac{\varphi_{12}^2}{\varphi_{11}^2}}$$

Notice that y = 0 is not possible here (since that implies 1 = 0), and that if  $\varphi_{11} = 0$  then we would not be making this change of basis in the first place. So we have found a suitable x and y that ensure  $\varphi_{11} = -\varphi_{22} = 0$ . Now our metric looks like the following:

(17) 
$$\varphi_{ij} = \begin{bmatrix} 0 & \varphi_{12} & 0 \\ \varphi_{12} & 0 & 0 \\ 0 & 0 & \varphi_{33} \end{bmatrix}$$

Naturally, a change of basis will scale the remaining entries to 1's (if  $\varphi_{33} = -1$ , exchange  $\varphi$  for  $-\varphi$ . With our metric in hand, we can recover the Ricci and Curvature entries.

 $\begin{array}{ll} \rho_{11} = -\tilde{b} = R_{1331} & \rho_{22} = \tilde{b} = R_{2332} \\ \rho_{12} = \tilde{a} = -R_{1221} + R_{1332} & \rho_{23} = 0 = R_{1223} \\ \rho_{13} = 0 = R_{2113} & \rho_{33} = \lambda_2 = 2R_{1332} \end{array}$ 

Let  $\alpha = \frac{\lambda_2}{2} - \tilde{a}$  and  $\beta = \frac{\lambda_2}{2}$ . Then we have

 $\begin{array}{ll} R_{1221} = \alpha & R_{2332} = \tilde{b} & R_{1332} = \beta \\ R_{1331} = -\tilde{b} & R_{1223} = 0 & R_{2113} = 0 \end{array}$ 

We continue as usual by checking to see if M can be  $cvc(\varepsilon)$  for some  $\varepsilon$ .

**Lemma 6.1.** Let  $M = (V, \langle \cdot, \cdot \rangle, R)$  be a 3-dimensional Lorentzian model space of Jordan type IV with the derived metric and curvature entires. If M has  $cvc(\varepsilon)$ , then  $\varepsilon = -\alpha$ .

*Proof.* We consider this proof by cases. Let  $w = xe_1 + ye_2 + ze_3$  for some  $x, y, z \in \mathbb{R}$ , with not more than one of them zero.

Case 1:

Suppose  $v = e_1$ . We select  $y \neq 0$  for this case. From Definition 1.4, we have

$$\kappa(\pi) = \frac{\alpha y^2 + bz^2}{-y^2} = \varepsilon$$
$$\implies \tilde{b}z^2 - \alpha y^2 = \varepsilon y^2$$
$$\implies \tilde{b}z^2 - y^2(\alpha + \varepsilon) = 0$$

Now since  $\tilde{b}z^2 \ge 0$ , we know that  $y^2(\alpha + \varepsilon) \ge 0$ ; that is,

 $\varepsilon \geq -\alpha.$ 

Case 2:

Suppose  $v = e_2$ . We select  $x \neq 0$  for this case. From Definition 1.4, we have

$$\kappa(\pi) = \frac{\alpha x^2 + bz^2}{-x^2} = \varepsilon$$
$$\implies \alpha x^2 + \tilde{b}z^2 = -\varepsilon x^2$$
$$\implies \tilde{b}z^2 + x^2(\alpha + \varepsilon) = 0$$

Again, since  $\tilde{b}z^2 \ge 0$ , we know that  $x^2(\alpha + \varepsilon) \le 0$  which means

$$\varepsilon \leq -\alpha$$

Combining these cases, we know that  $\varepsilon = -\alpha$  is the only possibility.  $\Box$ 

We have established what the value of  $\varepsilon$  must be, but before we prove that M is always  $cvc(-\alpha)$ , we establish one last lemma.

**Lemma 6.2.** Suppose  $M = (V, \langle \cdot, \cdot \rangle, R)$  is a 3-dimensional Lorentzian model space of Jordan Type IV. If M has  $cvc(-\alpha)$ , then the equation that must be satisfied is:

$$\tilde{b}[(bz - cy)^2 - (az - cx)^2] + 2(\alpha + \beta)(abz^2 - acyz - bcxz + c^2xy) = 0.$$

*Proof.* First we consider the numerator of Definition 1.4. From Lemma 2.3, we have

$$\alpha(ay - bx)^2 - \tilde{b}(az - cx)^2 + \tilde{b}(bz - cy)^2 + 2\beta(abz^2 - acyz - bcxy + c^2xy)$$
  
=  $\alpha(ay - bx)^2 + \tilde{b}[(bz - cy)^2 - (az - cx)^2] + 2\beta(abz^2 - acyz - bcxy + c^2xy).$ 

Now for the denominator, we have:

$$\begin{aligned} \langle v, v \rangle &= 2ab + c^2, \\ \langle w, w \rangle &= 2xy + z^2, \\ \langle v, w \rangle^2 &= (ay + bx + cz)^2. \end{aligned}$$

Expanding and collecting terms, we have:

$$-(ay - bx)^{2} + 2(abz^{2} - acyz - bcxz + c^{2}xy).$$

Now we know that  $\varepsilon = -\alpha$  from Lemma 6.1, so we have:

$$\frac{\alpha(ay - bx)^2 + \tilde{b}[(bz - cy)^2 - (az - cx)^2] + 2\beta(abz^2 - acyz - bcxy + c^2xy)}{-(ay - bx)^2 + 2(abz^2 - acyz - bcxz + c^2xy)} = -\alpha$$

The right-hand side of this equation will be

$$\alpha(ay - bx)^2 - 2\alpha(abz^2 - acyz - bcxz + c^2xy).$$

And, after bringing everything to one side, we obtained the desired result:

$$\tilde{b}[(bz - cy)^2 - (az - cx)^2] + 2(\alpha + \beta)(abz^2 - acyz - bcxz + c^2xy) = 0.$$

**Theorem 6.1.** Suppose  $M = (V, \langle \cdot, \cdot \rangle, R)$  is a 3-dimensional Lorentzian model space of Jordan Type IV. M has  $cvc(\varepsilon)$  for  $\varepsilon = -\alpha$ .

*Proof.* We consider this proof by cases. Let  $w = xe_1 + ye_2 + ze_3$  for some  $x, y, z \in \mathbb{R}$ , with not more than one of them being 0.

Case 1:

Suppose  $v = e_3$ . From Definition 1.4, we have

$$\kappa(\pi) = \frac{-\tilde{b}x^2 + \tilde{b}y^2 + 2\beta xy}{2xy} = \varepsilon.$$

We wish to see if it is possible for  $\varepsilon = -\alpha$  in this case. Make the substitution and multiply the denominator over to obtain:

$$\tilde{b}(y^2 - x^2) + 2xy\beta = -2xy\alpha$$
$$\implies 2xy(\alpha + \beta) = \tilde{b}(x^2 - y^2)$$

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Note that  $\alpha + \beta = \lambda_2 - \tilde{a}$ , which gives us:

$$2xy(\lambda_2 - \tilde{a}) = \tilde{b}(x^2 - y^2)$$
$$\implies x^2 + 2xy(\frac{\tilde{a} - \lambda_2}{\tilde{b}}) = y^2.$$

Let  $p = \frac{\tilde{a} - \lambda_2}{\tilde{b}}$ . Then by completing the square we have

$$(x + py)^2 = y^2(1 + p^2)$$
$$\implies x + py = y\sqrt{1 + p^2}$$
$$\implies x = y(\sqrt{1 + p^2} - p).$$

Note that the coefficient on y cannot be zero since that would imply 1 = 0. This means that, unless x = y = 0 (a ludicrous case), we have found a y that pairs with x to produce a non-degenerate pairing for v. That is,  $\varepsilon = -\alpha$ .

For the remaining cases, we find an appropriate w for the given v by swapping the coefficients of  $e_1$  and  $e_2$ . In two cases, we change one of the signs.

#### Case 2:

Suppose  $v = ae_1 + be_2$ , with a and b nonzero. Then we select  $w = -be_1 + ae_2$ . By Lemma 2.2, we can ensure this is a non-degenerate pairing by examining the denominator of the sectional curvature.

$$-(2ab)^{2} - (a^{2} - b^{2})^{2} = 0$$
  
$$\implies -4a^{2}b^{2} - a^{4} - b^{4} + 2a^{2}b^{2} = 0$$
  
$$\implies -(a^{2} + b^{2})^{2} = 0.$$

Which occurs only if a = b = 0, which contradicts the given v. Thus v, w span a non-degenerate 2-plane.

#### Case 3:

Suppose  $v = ae_1 + ce_3$ , with a and c nonzero. We select  $w = ae_2 + ce_3$ . Then we have

$$c^{4} - (a^{2} + c^{2})^{2} = 0$$
  
$$\implies a^{4} + 2a^{2}c^{2} = 0$$
  
$$\implies a^{2}(a^{2} + 2c^{2}) = 0.$$

Which is true only if either a = 0 or a = c = 0; both situations contradict the given v. Then by Lemma 2.2, v, w span a non-degenerate 2-plane.

## Case 4:

Suppose  $v = be_2 + ce_3$ , with b and c nonzero. We select  $w = be_1 + ce_3$ . Then we have

$$c^4 - (b^2 + c^2)^2 = 0$$

and this proof proceeds exactly as in the previous case.

Case 5:

Suppose  $v = ae_1 + be_2 + ce_3$ . For simplicity, we scale c down to 1. We select a w by comparing  $a^2$  and  $b^2$ . If  $a^2 \ge b^2$ , we select  $w = -be_1 + ae_2 + e_3$ . Then we have

$$(2ab+1)(-2ab+1) - (a^2 - b^2 + 1)^2 = 0$$
  
$$\implies 1 - 4a^2b^2 - a^4 - b^4 - 1 - 2a^2 + 2b^2 + 2a^2b^2 = 0$$
  
$$\implies (a^2 + b^2)^2 + 2(a^2 - b^2) = 0.$$

This equation is only zero when a = b = 0 or in the event that  $a^2 < b^2$ , both of which violate the conditions set on v. So by Lemma 2.2, we have a non-degenerate pairing.

For the case that  $a^2 \leq b^2$ , we select  $w = be_1 - ae_2 + ce_3$ . The proof follows similarly, but now we end up with

$$(a^2 + b^2)^2 + 2(b^2 - a^2) = 0$$

which has solutions only for a = b = 0 or  $a^2 > b^2$ . Both of these situations violate the conditions on this v.

Having exhausted all possible combinations of v, we have shown that if M has  $cvc(\varepsilon)$ , then  $\varepsilon = -\alpha$ .

This is the only family of model spaces, then, that always have  $cvc(\varepsilon)$ . There is no restriction on the ordering of the eigenvalues in this case, unlike the real cases.

## 7. General Results

We turn now from a form-specific analysis to some general results that can be gleaned from prior results. To do this, we first convert all of our metrics to orthonormal bases. Type I is already on an appropriate orthonormal basis, so we instead provide change of bases for Types II, III, and IV. A suitable change of basis for Type II is:

$$g_1 = \frac{1}{\sqrt{2}}(e_1 - e_3)$$
  

$$g_2 = e_2$$
  

$$g_3 = \frac{1}{\sqrt{2}}(e_1 + e_3)$$

This will change the form of A to:

$$\begin{bmatrix} \lambda & \frac{1}{\sqrt{2}} & 0\\ -\frac{1}{\sqrt{2}} & \lambda & \frac{1}{\sqrt{2}}\\ 0 & \frac{1}{\sqrt{2}} & \lambda \end{bmatrix}$$

A suitable change of basis for Types III and IV are:

$$g_1 = \frac{1}{\sqrt{2}}(e_1 - e_2)$$
  

$$g_2 = \frac{1}{\sqrt{2}}(e_1 + e_2)$$
  

$$g_3 = e_3$$

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This will change the form of A in Type III, but preserves the form of A in Type IV. Type III becomes:

$$\begin{bmatrix} \lambda_1 - \frac{1}{2} & \frac{1}{2} & 0\\ -\frac{1}{2} & \lambda_1 + \frac{1}{2} & 0\\ 0 & 0 & \lambda_2 \end{bmatrix}$$

In all cases, the metric is now given by:

(18) 
$$[\varphi_{ij}] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The Ricci and curvature entries are obtained in the same way as before. We provide the curvature tensor entries by type below:

## Type I:

$$\begin{array}{ll} R_{1221} = \frac{\lambda_3 - \lambda_1 - \lambda_2}{2} & R_{2332} = \frac{\lambda_2 + \lambda_3 - \lambda_1}{2} & R_{1332} = 0 \\ R_{1331} = \frac{\lambda_2 - \lambda_3 - \lambda_1}{2} & R_{2113} = 0 & R_{1223} = 0 \end{array}$$

## Type II:

$$R_{1221} = -\frac{\lambda}{2} \qquad R_{2332} = \frac{\lambda}{2} \qquad R_{1332} = -\frac{1}{\sqrt{2}} \\ R_{1331} = -\frac{\lambda}{2} \qquad R_{2113} = -\frac{1}{\sqrt{2}} \qquad R_{1223} = 0$$

Type III:

$$\begin{aligned} R_{1221} &= \frac{\lambda_2}{2} - \lambda_1 & R_{2332} = \frac{1 + \lambda_2}{2} & R_{1332} = -\frac{1}{2} \\ R_{1331} &= \frac{1 - \lambda_2}{2} & R_{2113} = 0 & R_{1223} = 0 \end{aligned}$$

# Type IV:

$$R_{1221} = \frac{\lambda_2}{2} - \tilde{a} \qquad R_{2332} = \frac{\lambda_2}{2} \qquad R_{1332} = -\tilde{b} \\ R_{1331} = -\frac{\lambda_2}{2} \qquad R_{2113} = 0 \qquad R_{1223} = 0$$

With these in hand, we can state three general results. Note that the proofs of these results are not provided, since they have been demonstrated by the results found throughout this paper.

**Corollary 7.1.**  $cvc(\varepsilon)$  is well defined in the 3-dimensional setting.

**Theorem 7.1.** Let  $M = (V, \langle \cdot, \cdot \rangle, R)$  be a 3-dimensional Lorentzian model space on the derived orthonormal basis. If M is  $cvc(\varepsilon)$ , then  $\varepsilon = -R_{1221}$ .

**Theorem 7.2.** Let  $M = (V, \langle cdot, cdot \rangle, R)$  be a 3-dimensional Lorentzian model space on any basis. If M is  $cvc(\varepsilon)$ , then

$$\varepsilon = \sum_{i=1}^{n} (\pm 1) \frac{Re(\lambda_i)}{2} d_i.$$

Where n is the number of distinct eigenvalues and  $d_i$  is the multiplicity of each eigenvalue.

One note to make is that we take some liberties with the meaning of "multiplicity" in this case. We consider the multiplicity of the complex eigenvalue to be 2 by counting both itself and its conjugate.

#### 8. Conclusions

In 3 dimensions there is only one family of model spaces which do not have  $cvc(\varepsilon)$  for any  $\varepsilon$ . These model spaces are of the form  $J(\lambda, 3)$  and so have one eigenspace. In this form,  $e_1$  and  $e_3$  form a hyperbolic pair while  $e_2$ is the spacelike unit vector; whenever  $e_1$  is paired with  $e_2$ , the model space fails to have  $cvc(\varepsilon)$ . However, when  $e_2$  is paired with  $e_3$ , everything works out fine. Future research should examine what, if any, significance arises from this choice of pairing.

The other model spaces constitute two or three eigenspaces. In the case of two eigenspaces, the value for  $\varepsilon$  turns out to be nearly identical. In fact, to unify the two cases one may be able to say that  $\varepsilon$  is the real part of the first eigenvalue minus half the second. Further research is again needed in this area. In the case of three eigenspaces, there are a fair number of conditions and cases to consider. One conjecture is that the orientation, size, and positioning of the eigenspaces may play a role in deciding  $cvc(\varepsilon)$ ; this may be a good case to start with in testing that conjecture.

## 9. Open Questions

- (1) The major unanswered question that came from this research is whether these results hold in higher dimensions. A great place to start to see if these results are further generalizable would be to look at certain types of 4-dimensional model spaces whose curvature tensor entries are generated by Ricci tensor entries.
- (2) One conjecture is that  $R_{1221}$  is invariant under certain change of bases, and so one could generalize Theorem 1 to include more situations than just orthonormal bases. A good place to start would be to wonder what the relationship is between the hyperbolic basis and the orthonormal basis derived in this paper.
- (3) The formula given for  $\varepsilon$  in Theorem 2 is very closely related to the trace of the Jordan form. However, there is a difference of a negative sign on the third term (and in one instance, the second term for the

diagonalized case). Our conjecture at this point is that perhaps the eigenvalues must be ranked according to modulus, and that the largest of those ranked will be negative in the trace. In addition, the Jordan block containing the timelike vector is excluded from this ranking (and so is always positive).

#### 10. Acknowledgements

Thank you to Dr. Corey Dunn for his patience and wisdom throughout the research experience, as well as his invaluable mentorship. Thanks also go out to Dr. Rollie Trapp, in conjunction with Dr. Dunn, for organizing the REU at CSUSB. This research was made possible both by funding from the NSF grant DMS-1461286 and the generous support of California State University, San Bernardino.

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