Analysis of Length Spectra of Geodesics in the Knot Complement of the Borromean Rings

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Abstract

The purpose of this paper is to give an understanding of the geodesics in the knot complement of the Borromean rings (which we shall denote as 'B'). This is done by relating isometries of the fundamental region of B to isometries of $\mathbb{H}^3$ and the length of the corresponding geodesic. We then give a topological realization of the geodesic. Finally, we investigate how length spectra behave under belt sum.

1 Introduction

A *knot* is a closed curve of zero thickness, embedded in three-space with no self intersections. A *link* is multiple knots interconnected, but still with no self intersections. In this paper we focus on the Borromean rings, a link of 3 components:

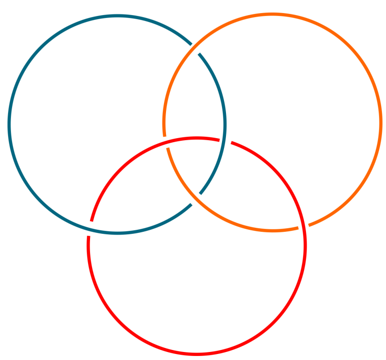


Figure 1: Representation of Borromean Rings

The Borromean rings are a *hyperbolic knot*. To discuss what defines a hyperbolic knot, let us first introduce the concept of a *knot complement*. The knot complement is the three-dimensional space around the knot, but without the knot. For the purposes of this paper, we shall denote the knot complement of the Borromean rings as B. A hyperbolic knot is a knot whose knot complement is a hyperbolic polyhedron. In the case of the Borromean rings, B is two regular, ideal octahedra. This is the geometric realization and structure we will be working with in this paper.

We will also be utilizing belt sum composition in this paper. *Belt sum composition* of two links consists of slicing along one component of each link. Along the slice, there are two planes left with two points each. Open these planes so that they are parallel and identify the points from one link to another.

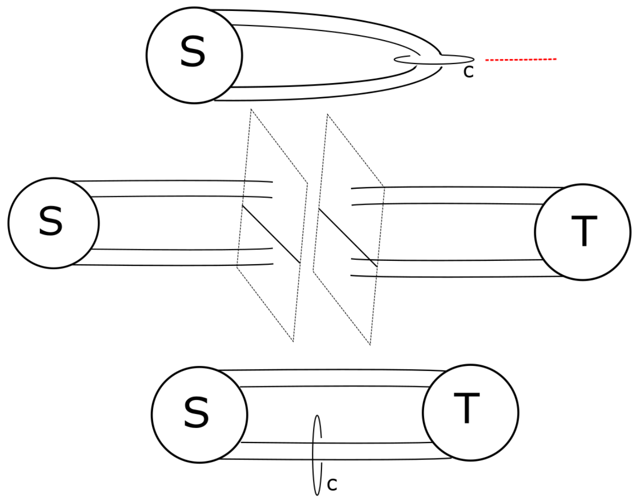


Figure 2: Belt Sum Instructions

Finally we introduce the notions of geodesics and length spectra. For the purposes of our paper, we will be working with simple, closed geodesics in B and the belt sum of two Borromean rings, denoted $B\#B$. This can be imagined by a curve that goes through B, that connects back to its starting point and does not intersect itself. We consider length spectra of the Borromean rings to be the set of geodesic lengths in B. The primary focus of this research is to be able to relate isometries of the fundamental region of B to isometries of $\mathbb{H}^3$, the length of the corresponding geodesic γ, and finally to give a topological realization of γ. We then begin to relate geodesics in B to geodesics in $B\#B$.

2 Preliminaries

To begin, we note that the nature of this work requires a consistent labeling system. Here we will introduce such labeling systems:

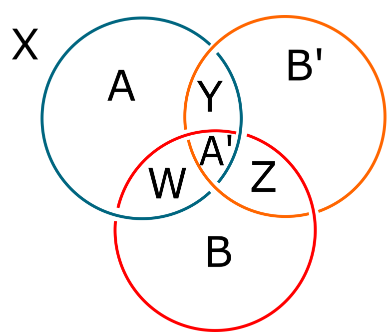


Figure 3: Labeling System Representation

This labeling system allows us to compare this representation of the Borromean rings to the cell decomposition of the Borromean rings, as pictured here:

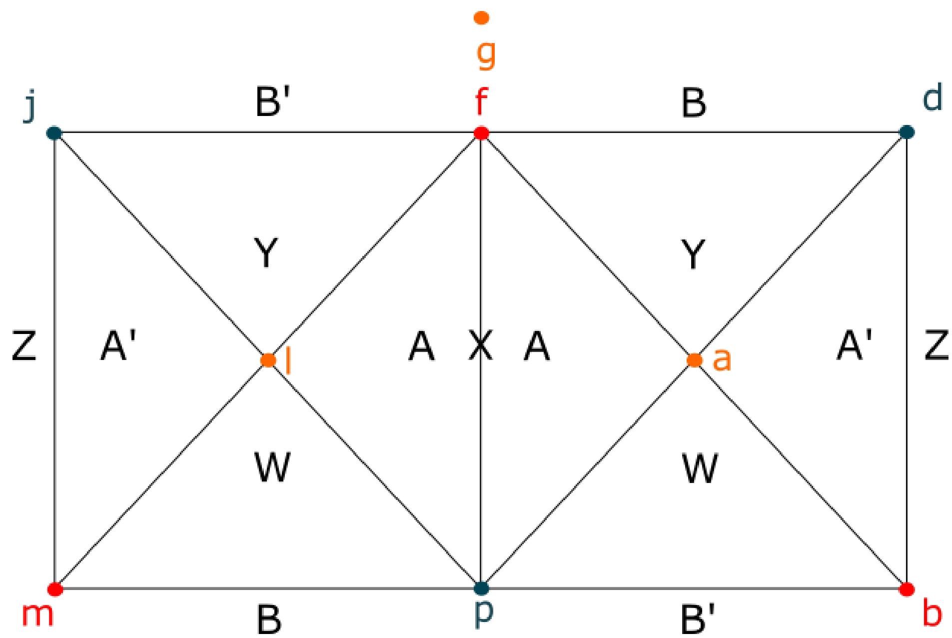


Figure 4: Labeling of the Fundamental Region of B

In this cell decomposition, the vertices (one-cells) correspond colorwise to the edge they come from and the faces on the decomposition are labeled in correspondence to the faces from which they originate. First, note that the geometric structure this decomposes into is two ideal octahedra. These octahedra are viewed from above in the above figure and g is a point at infinity that all outer faces have as a vertex. Second, note that the half decomposition that "comes out of the page", so to speak, is denoted by P^+ (the left-hand side octahedra) and the half decomposition that comes out of the page is denoted by P^- (the right-hand side octahedra).

Further, in order to to calculate lengths of the geodesics in B, we must choose coordinates for the vertices of B. This is what we chosen: $j = -2 + i, m = -2 - i, l = -1, p = -i, f = i, a = 1, d = 2 + i, b = 2 - i$.

3 Methods

To begin, we consider the *fundamental group* of B, denoted $\pi_1(B)$. A fundamental group is a group of closed curves. We consider $\pi_1(B)$ to be the geodesics in B and if a geodesic γ is in B, we say $\gamma \in \pi_1(B)$. In order to find various geodesics in B, we begin by computing the set of generators of $\pi_1(B)$, which we will denote G.

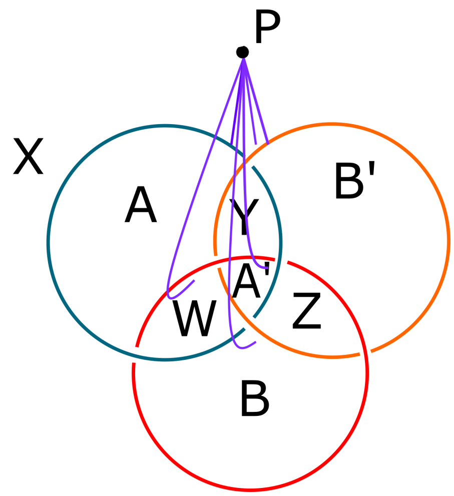


Figure 5: Topological Realizations of the Generators *A'*,*W*,*B*

This group consists of all geodesics, with associated mobius transformation as an isometry of the fundamental region of B, that travel from the face of the ideal octahderon P^+ to the face of the same label in the P^- ideal octahedron. We denote all such mobius transformations as $J...H$, where J is the face originated at in P^+, H is the face terminated at in P^-, and all labels in the middle are the faces intersected. Since the elements in the fundamental group are of the form JXJ (i.e. the only face they intersect in between is face A), we will denote these elements simply as J. Thus the set of generators is $G = \{A, Y, B', W, A', Z, B, A^{-1}, Y^{-1}, B'^{-1}, W^{-1}, A'^{-1}, Z^{-1}, B^{-1}\}$. Since the X faces of P^+ and P^- are already "glued" together, the mobius transformation is trivial.

To illustrate the method for calculating mobius transformations in this context, we will demonstrate with the generator Z. To begin the mobius transformation, we want to map the three vertices from one Z face to the other, matching color to color and corresponding each one of these mappings to either $0, 1, \infty$.

$$j \to d(0)$$
$$m \to b(1)$$
$$g \to g(\infty)$$

We then arrive to these mobius transformations:

$$M(x) = \frac{(x - j)}{(m - j)}$$
$$N(w) = \frac{(w - d}{(b - d)}$$

Solving for w, we achieve the mobius transformation from one B face to the other:

$$w = \frac{x}{ix + 1}$$

This mobius transformation represents the curve $Z \in \pi_1(B)$. It is known that there is a homomorphism taking curves in $\pi_1(B)$ to isometries of $\mathbb{H}^3$ which are expressed as matrices in $SL(2,\mathbb{C})$, whose entries are coefficients of the mobius transformation. Thus we can also represent these curves as matrices:

$$Z = \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix}$$

Using this method we find all matrix representations of the generators of $\pi_1(B)$. These matrices are available in Appendix 1. Once we have the generators as matrices, we can represent any geodesic as a combination of multiplying the generating matrices. Once you have the matrix representation of any geodesic, you can find the length by:

$$\ell = 2 \operatorname{arccosh} \left(\pm \frac{Tr(M)}{2} \right)$$

Following the processes described above, we used *Maple Software* to generate the length of every geodesic possible from multiplying 2 and 3 generators. In writing this code, several identities about the trace of a matrix were used to generate less duplicate data. It is known that $Tr(AB) = Tr(BA)$ and $Tr(ABC) = Tr(BCA) = Tr(CAB)$, where A, B, C are matrices. This code is avaiable in Appendix 2 and a portion of the generated data is available in Appendix 3. The goal of generating this data was to have a length spectra that associated the length of a geodesic with its generator representation. We achieve another length spectra using the *SnapPy* program. While this program does not give us a generator representation for the geodesics in the length spectra, it does give us the multiplicity of a geodesic of a specific length. The *SnapPy* outputs we used to inform our research are avaiable in Appendix 4.

Another goal of this research was to investigate how length spectra behave under belt sum composition. This lead to us to construct a similar formalization for the belt sum of the Borromean Rings, denoted $B\#B$. This is explored in our next section.

4 Belt Sum of the Borromean Rings

We construct a similar formalization of $B\#B$ as we did to the Borromean rings in order to investigate how geodesics are altered under this operation. We begin with a topological realization of the belt sum of two Borromean rings.

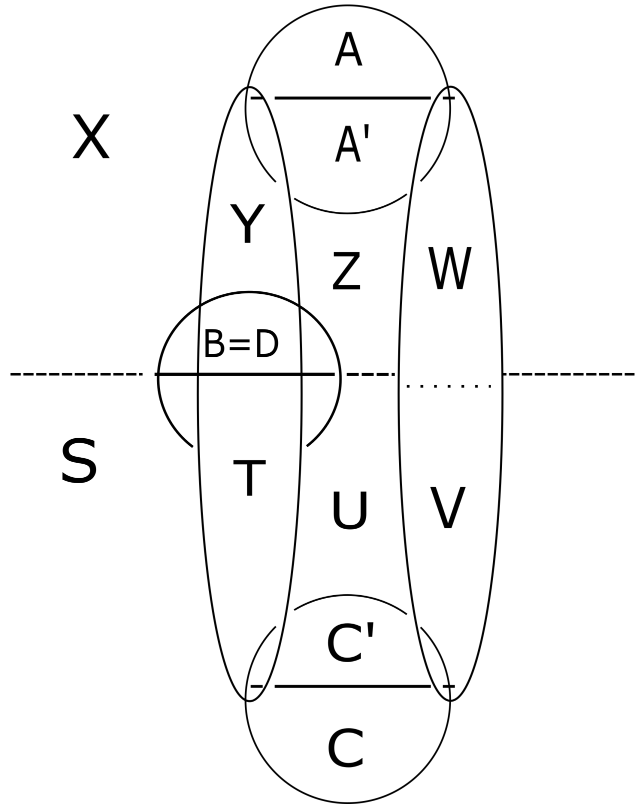


Figure 6: Labeling System for $B\#B$

The cell decomposition of $B\#B$ is then performed to receive the fundamental region.

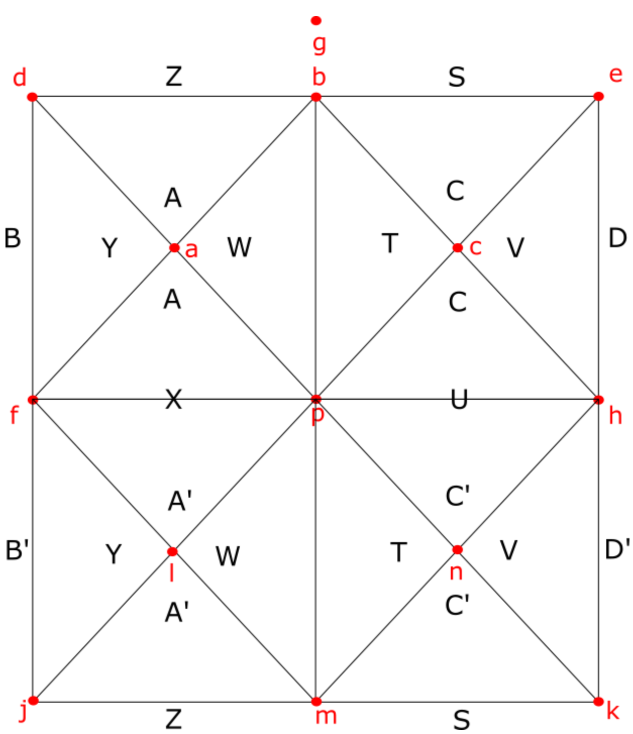


Figure 7: Fundamental Region of $B\#B$

Now that we have the fundamental region of $B\#B$, we can use the same methods as before to find the generators of the fundamental group, $\pi_1(B\#B)$ and the length spectra with generator representation using the same *Maple Software* code as before. The matrices representing these generators are available in Appendix 5. The *SnapPy* output for $B\#B$ is available in Appendix 4.

Since we now have methods for locating geodesics in both B and $B\#B$, we can begin to investigate the relation between the two. This is explored in the next section, along with the topological realization of geodesics in the knot complements.

5 Examples

We begin by investigating the systole. The systole of B has length $\ell = 2.1225$ and, as evidenced by the *SnapPy* output, there are 12 geodesics that realize this length. Here is one example of a systole:

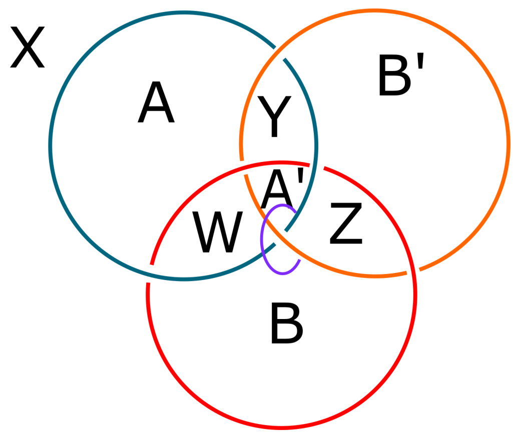
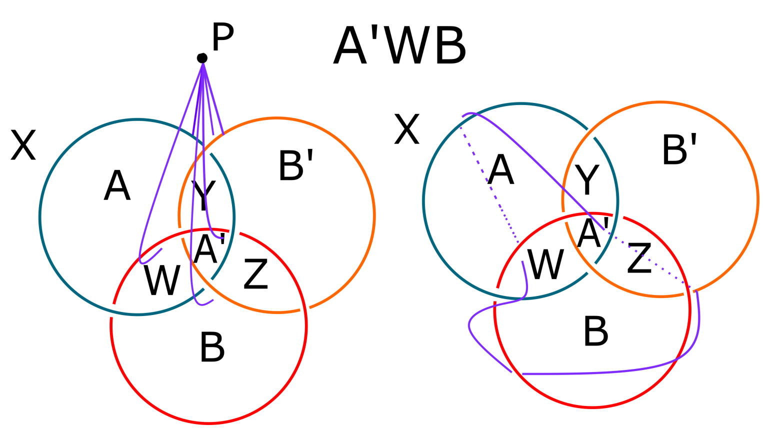


Figure 8: A'B Systole

Further investigation shows that the twelve systoles are the geodesics that run through two faces that share exactly one vertex.

For more complicated geodesics, we can form the topological realization by returning to the fundamental group. To start, draw a point P in the X face. With the generator representation, for each generator, draw a geodesic from point P, under the face, and then back to point P. From these geodesics, the topological geodesic can be realized. Here is the an example with the geodesic $A'WB$:



$A'WB$ has length $\ell = 2.6339$, a length that does not appear in the length spectra of $B\#B$.

6 Results

Proposition. Let M be a matrix with associated geodesic $\gamma \in \pi_1(B)$ of length ℓ. A geodesic $\gamma^* \in \pi_1(B\#B)$ of length ℓ is not guaranteed.

Proof. Suppose to the contrary that for all $\gamma \in \pi_1(B)$ of length ℓ, there exists $\gamma^* \in \pi_1(B\#B)$ of length ℓ. A counter example to that, as mentioned above, is the geodesic $A'WB$ of length $\ell = 2.6339$ of which there is no geodesic of length ℓ in the length spectra of $B\#B$.

This proposition contextualizes why the following theorem is not trivial.

Theorem. Let M be a matrix such that it corresponds to a geodesic $\gamma \in \pi_1(B)$ of length ℓ. Then the matrix M^2 corresponds to a geodesic $\gamma^* \in \pi_1(B\#B)$ of length 2ℓ.

Proof. Begin by tessellating $\mathbb{H}^3$ with the fundamental region of $B\#B$. Any valid geodesic $\gamma^* \in \pi_1(B\#B)$ is associated with a matrix M^* that is an isometry of the tessellation. Now let every fundamental region of $B\#B$, call them F, be a refinement of two fundamental regions of B, call them G and H. Any valid geodesic $\gamma \in \pi_(B)$ is associated with a matrix M that is an isometry of the tessellation of fundamental regions of B. However, γ may not a valid geodesic in $\pi_1(B)$. This leads us to two cases,

- *Case 1.* First consider the case where the matrix M maps a G fundamental region of B to another fundamental region labeled G. Then the associated geodesic $\gamma \in \pi_(B\#B)$ because that would M would be an isometry of $\mathbb{H}^3$. This would trivially give you a geodesic $\gamma^* \in \pi_1(B\#B)$ of length 2ℓ.

- *Case 2.* Consider the case where M maps a fundamental region marked G to a fundamental region marked H. This is not an isometry of $\mathbb{H}^{\nvDash}$, so the corresponding geodesic γ is not in $\pi_(B\#B)$. Let M be any mapping that takes you from one fundamental region of B to another one of the opposite labeling. Thus $\gamma^2 \in \pi_(B\#B)$ since M^2 is an isometry of $\mathbb{H}^3$ and γ^2 has length 2ℓ.

7 Further Research

- *Conjecture.* Let M be a matrix such that $M \in \pi_1(B)$. If $M \notin \pi_1(B\#B)$, then the geodesic, γ, corresponding to M intersects all thrice punctured spheres in B.

- Extending the theorem to matrices M^n in $B\#B$ that will correspond to a geodesic γ of length $n\ell$, where $n \in 2\mathbb{Z}$.

- Observe and justify geodesics of length $n\ell$ where $n \in (Z)\ 2(Z)$ in the length spectra of $B\#B$.

- Relate the multiplicity of symmetry of the Borromean rings and the geometry of B.

- Is $\pi_1(B\#B) \subset \pi_1(B)$?

8 References

[1] Colin C. Adams. Thrice punctured spheres in hyperbolic 3-manifolds. 1985.

[2] Heiner Zieschang Gerhard Burde, Michael Heusener. *Knots*. 2013.

[3] Alan W. Reid Colin MacLachlan. *The Arthimetic of Hyperbolic 3-Manifolds*. 2003.

[4] Sally M. Miller. Geodesic knots in the figure-eight knot complement. 2001.

9 Acknowledgements

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10 Appendices

10.1 Generating Matrices for the Fundamental Group of B

$$A = \begin{pmatrix} 2-i & -4i \\ -\frac{1}{2} & i \end{pmatrix}, A' = \begin{pmatrix} -2-i & -4i \\ \frac{1}{2} & i \end{pmatrix}, W = \begin{pmatrix} 1 & -4i \\ 0 & 1 \end{pmatrix},$$

$$Y = \begin{pmatrix} 1-2i & -4i \\ i & 1+2i \end{pmatrix}, B = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{pmatrix}, Z = \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix},$$

$$B' = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{pmatrix}$$

10.2 *Maple Software* Code

```
#The ComplexLength procedure takes two matrices in as arguments and then
#computes the length of associated geodesic of the matrices product.

ComplexLength:=proc(A,B)
T:=Multiply(A,B):         #Uses the matrix multiplication function in Maple
tr:=Trace(T):             #Takes the trace of the product
cLength:=2arccosh(tr/2): #Finds the length of associated geodesic
cLength:=evalf(%)         #Evaluate to decimal approximation
end proc:

#The LengthSpectra procedure takes in an array of matrices and multiplies
#different combinations of these matrices to generate a length spectra
#with generator representations.

LengthSpectra:=proc(A)
LS:=[]:
i=1:
for i from 1 to (nops(A)-2) do
        for j from i+1 to (nops(A)-1) do
                for k from j+1 to nops(A) do
                        cLength:=ComplexLength(Multiply(A[i],A[j]),A[k]):
                        LS:=[op(LS), cLength]:
                end do:
        end do:
end do:                 #This nested loop moves through the array of matries
for i from 1 to nops(A) do
        for j from 1 to nops(A) do
                cLength:=ComplexLength(Multiply(A[i],A[i]),A[j]):
                LS:=[op(LS),cLength]:
        end do:
end do:
LS
end proc:
```

10.3 Generated Length Spectra with Generator Representation

A B z	2.122550124
A D z	2.122550124
A b z	2.122550124
A z d	2.122550124
P W z	2.122550124
P Y z	2.122550124
P B Z	2.122550124
P Z D	2.122550124
P Z w	2.122550124
P Z y	2.122550124
P Z b	2.122550124
P Z d	2.122550124
W B z	2.122550124
W Z a	2.122550124
W D z	2.122550124
W a z	2.122550124
W b z	2.122550124
W z d	2.122550124
Y B z	2.122550124
Y Z a	2.122550124
Y D z	2.122550124
Y a z	2.122550124
Y b z	2.122550124
Y z d	2.122550124
B Z a	2.122550124
B Z w	2.122550124
B Z y	2.122550124
B p z	2.122550124
Z D a	2.122550124
Z D w	2.122550124
Z D y	2.122550124
Z a w	2.122550124
Z a y	2.122550124
Z a b	2.122550124
Z a d	2.122550124
Z w b	2.122550124
Z w d	2.122550124
Z y b	2.122550124
Z y d	2.122550124
D p z	2.122550124
A B y	2.633915794
A D y	2.633915794
A y b	2.633915794
Y b d	3.057141838
B D a	3.057141838
B D p	3.057141838
B D w	3.057141838
B D y	3.057141838
A W p	3.525494348
A Y p	3.525494348
A p w	3.525494348
A p y	3.525494348
Y B w	3.525494348
Y D w	3.525494348
Y w b	3.525494348
Y w d	3.525494348
A B Z	3.648397404
A Z D	3.648397404
A Z b	3.648397404
A Z d	3.648397404
P B z	3.648397404
P D z	3.648397404
P b z	3.648397404
P z d	3.648397404
W B Z	3.648397404
W Z D	3.648397404
W Z b	3.648397404
W Z d	3.648397404
Y B Z	3.648397404
Y Z D	3.648397404
Y Z b	3.648397404
Y Z d	3.648397404
B Z p	3.648397404
B a z	3.648397404
B w z	3.648397404
B y z	3.648397404
Z D p	3.648397404
Z p b	3.648397404
Z p d	3.648397404
D a z	3.648397404
D w z	3.648397404
D y z	3.648397404
A P B	3.937275852
A P D	3.937275852
A P b	3.937275852
A P d	3.937275852

A y d	2.633915794
P W B	2.633915794
P W D	2.633915794
P Y b	2.633915794
P Y d	2.633915794
P B w	2.633915794
P D w	2.633915794
W B a	2.633915794
W D a	2.633915794
W a b	2.633915794
W a d	2.633915794
Y B p	2.633915794
Y D p	2.633915794
B a y	2.633915794
B p w	2.633915794
D a y	2.633915794
D p w	2.633915794
A W B	2.88727095
A W D	2.88727095
A Y b	2.88727095
A Y d	2.88727095
A B w	2.88727095
A D w	2.88727095
A w b	2.88727095
A w d	2.88727095
P W b	2.88727095
P W d	2.88727095
P Y B	2.88727095
P Y D	2.88727095
P B y	2.88727095
P D y	2.88727095
P y b	2.88727095
P y d	2.88727095
W B p	2.88727095
W D p	2.88727095
W p b	2.88727095
W p d	2.88727095
Y B a	2.88727095
Y D a	2.88727095
Y a b	2.88727095
Y a d	2.88727095
B a w	2.88727095
B p y	2.88727095
A W Y	3.937275852
A w y	3.937275852
P W Y	3.937275852
P w y	3.937275852
W Y a	3.937275852
W Y p	3.937275852
B a p	3.937275852
D a p	3.937275852
A W y	4.245100248
A Y w	4.245100248
A p b	4.245100248
A p d	4.245100248
P a b	4.245100248
P a d	4.245100248
W Y z	4.245100248
W Z y	4.245100248
W p y	4.245100248
W y z	4.245100248
Y Z w	4.245100248
Y p w	4.245100248
Y w z	4.245100248
Z w y	4.245100248
A Z w	4.277244172
A Z y	4.277244172
A w z	4.277244172
A y z	4.277244172
P W Z	4.277244172
P Y Z	4.277244172
W Z p	4.277244172
Y Z p	4.277244172
Z p w	4.277244172
Z p y	4.277244172
A W b	4.397146056
A W d	4.397146056
A Y B	4.397146056
A Y D	4.397146056
P w b	4.397146056
P w d	4.397146056
Y p b	4.397146056
Y p d	4.397146056
A P w	4.741097074
A P y	4.741097074
W Y B	4.741097074

10.4 *SnapPy* Outputs

```
In[1]: baroRing=Manifold()
Starting the link editor.
Select Tools->Send to SnapPy to load the link complement.


New triangulation received from PLink!
In[2]: baroRing.length_spectrum(cutoff=5)
Out[2]:
mult length                                 topology      parity
6    2.122550123810088 - 1.809113788604778*I circle        orientation-preserving
6    2.122550123810071 + 1.809113788604755*I circle        orientation-preserving
4    2.633915793849654 - 2.75185841738858 E-14*I circle        orientation-preserving
8    2.887270950357629 + 3.141592653589778*I circle        orientation-preserving
12   3.057141838962033 - 2.287435480804908*I circle        orientation-preserving
12   3.057141838961999 + 2.287435480804803*I circle        orientation-preserving
18   3.525494348078214 - 4.25298794745958 E-14*I circle        orientation-preserving
24   3.648397404387757 - 0.675402218531149*I circle        orientation-preserving
24   3.648397404387703 + 0.675402218531039*I circle        orientation-preserving
24   3.728323088315768 - 2.526385354528534*I circle        orientation-preserving
24   3.728323088315881 + 2.526385354528182*I circle        orientation-preserving
24   3.937275851585900 - 1.212275644774905*I circle        orientation-preserving
24   3.937275851586136 + 1.212275644774472*I circle        orientation-preserving
39   4.245100247619760 - 2.664957729970601*I circle        orientation-preserving
39   4.245100247620414 + 2.664957729969705*I circle        orientation-preserving
30   4.277244172632873 - 1.598556269346402*I circle        orientation-preserving
30   4.277244172632378 + 1.598556269345788*I circle        orientation-preserving
24   4.367170433129136 - 0.947749555929585*I circle        orientation-preserving
24   4.367170433129171 + 0.947749555929235*I circle        orientation-preserving
28   4.397146055841905 - 2.233852233664029*I circle        orientation-preserving
28   4.397146055841755 + 2.233852233662814*I circle        orientation-preserving
72   4.584863339121355 - 1.27190641996834 E-12*I circle        orientation-preserving
72   4.611018062487973 - 1.873624922313131*I circle        orientation-preserving
72   4.611018062486565 + 1.873624922310445*I circle        orientation-preserving
24   4.626441883539021 - 0.402390159402002*I circle        orientation-preserving
24   4.626441883539171 + 0.402390159401402*I circle        orientation-preserving
48   4.661949306097785 - 2.754006380480926*I circle        orientation-preserving
48   4.661949306099448 + 2.754006380479029*I circle        orientation-preserving
48   4.741097074636114 - 0.773129285040031*I circle        orientation-preserving
48   4.741097074635617 + 0.773129285039490*I circle        orientation-preserving
60   4.905827485005551 - 1.093949160566394*I circle        orientation-preserving
60   4.905827485005415 + 1.093949160565935*I circle        orientation-preserving
48   4.919663043246225 - 2.073594514403572*I circle        orientation-preserving
48   4.919663043246396 + 2.073594514399223*I circle        orientation-preserving
48   4.955777460576573 - 3.97805785906392 E-13*I circle        orientation-preserving
24   4.983559705289729 + 3.141592653589580*I circle        orientation-preserving
```

```
In[9]: baroBelt.length_spectrum(cutoff=5.4)
Out[9]:
mult length                               topology      parity
4    2.122550123809819 - 1.809113788605040*I circle        orientation-preserving
4    2.122550123809950 + 1.809113788604717*I circle        orientation-preserving
16   3.057141838962576 - 2.287435480805406*I circle        orientation-preserving
16   3.057141838961751 + 2.287435480804512*I circle        orientation-preserving
28   3.525494348078057 - 4.00452081587321 E-13*I circle        orientation-preserving
16   3.648397404387106 - 0.675402218531437*I circle        orientation-preserving
16   3.648397404388232 + 0.675402218530833*I circle        orientation-preserving
16   3.728323088317558 - 2.526385354529531*I circle        orientation-preserving
16   3.728323088314929 + 2.526385354527860*I circle        orientation-preserving
32   3.937275851588702 - 1.212275644775789*I circle        orientation-preserving
32   3.937275851587777 + 1.212275644772532*I circle        orientation-preserving
66   4.245100247624102 - 2.664957729971985*I circle        orientation-preserving
66   4.245100247620073 + 2.664957729969045*I circle        orientation-preserving
20   4.277244172639455 - 1.598556269349105*I circle        orientation-preserving
20   4.277244172630872 + 1.598556269344722*I circle        orientation-preserving
112  4.584863339122678 - 1.48080213621589 E-12*I circle        orientation-preserving
112  4.611018062500465 - 1.873624922316285*I circle        orientation-preserving
112  4.611018062482312 + 1.873624922309678*I circle        orientation-preserving
16   4.626441883537721 - 0.402390159402303*I circle        orientation-preserving
16   4.626441883538991 + 0.402390159398970*I circle        orientation-preserving
32   4.661949306095568 - 2.754006380483456*I circle        orientation-preserving
32   4.661949306095668 + 2.754006380477729*I circle        orientation-preserving
64   4.741097074637723 - 0.773129285041169*I circle        orientation-preserving
64   4.741097074637054 + 0.773129285037098*I circle        orientation-preserving
40   4.905827485013677 - 1.093949160569197*I circle        orientation-preserving
40   4.905827485006052 + 1.093949160564404*I circle        orientation-preserving
32   4.919663043270825 - 2.073594514409326*I circle        orientation-preserving
32   4.919663043249039 + 2.073594514398455*I circle        orientation-preserving
64   5.009888296461917 - 2.815595707925855*I circle        orientation-preserving
64   5.009888296462884 + 2.815595707915411*I circle        orientation-preserving
112  5.097291156929847 - 1.361426130294234*I circle        orientation-preserving
112  5.097291156933908 + 1.361426130284950*I circle        orientation-preserving
128  5.199648387591342 - 2.223097846677934*I circle        orientation-preserving
128  5.199648387543863 + 2.223097846659490*I circle        orientation-preserving
152  5.267831587703877 - 2.39742167986502 E-12*I circle        orientation-preserving
64   5.288653572794358 - 0.286635066112156*I circle        orientation-preserving
64   5.288653572787484 + 0.286635066100816*I circle        orientation-preserving
64   5.298392355646262 - 1.580795493626778*I circle        orientation-preserving
64   5.298392355606592 + 1.580795493614189*I circle        orientation-preserving
32   5.307854671069033 - 2.860558841457185*I circle        orientation-preserving
32   5.307854671070369 + 2.860558841441120*I circle        orientation-preserving
64   5.348546891616057 - 0.561643883529712*I circle        orientation-preserving
64   5.348546891620640 + 0.561643883515022*I circle        orientation-preserving
```

10.5 Generating matrices for the Fundamental Group of $B\#B$

$$A = \begin{pmatrix} 2-i & -4i \\ -\frac{1}{2} & i \end{pmatrix}, A' = \begin{pmatrix} -2-i & -4i \\ \frac{1}{2} & i \end{pmatrix}, W = \begin{pmatrix} 1 & -4i \\ 0 & 1 \end{pmatrix},$$

$$Y = \begin{pmatrix} 1-2i & -4i \\ i & 1+2i \end{pmatrix}, B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, Z = \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix},$$

$$B' = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, T = \begin{pmatrix} 1 & -4i \\ 0 & 1 \end{pmatrix}, C = \begin{pmatrix} -2-i & -4i \\ -\frac{1}{2} & i \end{pmatrix},$$

$$S = \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix}, V = \begin{pmatrix} 1+2i & -4i \\ i & 1-2i \end{pmatrix}, C' = \begin{pmatrix} 2-i & 4i \\ \frac{1}{2} & i \end{pmatrix},$$