ALGEBRAIC CURVATURE TENSORS OF EINSTEIN AND WEAKLY EINSTEIN MODEL SPACES

ROBERTA SHAPIRO

Abstract. This research investigates the restrictions on symmetric bilinear form φ such that $R = R_\varphi$ in Einstein and Weakly Einstein model spaces. It has been determined that if a model space is Einstein and has a positive definite inner product, then: if the scalar curvature $\tau \geq 0$, then the model space has constant sectional curvature, and if $\tau < 0$, Φ can have at most two eigenvalues. Alternatively, given $R = R_\varphi$, a model space is weakly Einstein if and only if R_{φ^2} has constant sectional curvature. Also, it has been found that given an Einstein model space with a non-degenerate metric, if Φ is diagnoalizable, then the above applies, but if $dim(V) = 4$ and there exists a basis such that Φ takes one of several specific Jordan forms, then it must be the case that all eigenvalues of Φ are zero.

1. Introduction

An ***algebraic curvature tensor*** R over a vector space V is defined by $R : V \times V \times V \times V \to \mathbb{R}$ satisfying:

$$R(x, y, z, w) = -R(y, x, z, w) = R(z, w, x, y), \text{ and}$$
$$R(x, y, z, w) + R(z, x, y, w) + R(y, z, x, w) = 0,$$

the latter termed the *Bianchi Identity.* Such a function allows us to study the characteristics of $\mathfrak{M}$. Given an orthonormal basis $\{e_1, e_2, \ldots, e_n\}$, a notational convention is to say $R(e_i, e_j, e_k, e_l) = R_{ijkl}$.

The following are several preliminary definitions to aid in the understanding of the study of algebraic curvature tensors.

Definition 1.1. *Given vector space V, a **symmetric bilinear form** φ: $V \times V \to \mathbb{R}$ is:*

1. *Symmetric: $\varphi(x, y) = \varphi(y, x)$ $\forall x, y \in V$, and*
2. *Linear in the first slot: $\varphi(ax_1 + x_2, y) = a\varphi(x_1, y) + \varphi(x_2, y)$ $\forall x, y \in V$.*

Definition 1.2. *An **inner product** or **metric** $\langle \cdot, \cdot \rangle$ on vector space V is a symmetric bilinear form.*
*A metric is **non-degenerate** if, for all $x \in V$, there exist $w \in V$ such that $\langle x, w\rangle \neq 0$.*
*A metric is **positive definite** if $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0$ if and only if $x = 0$.*

Notice that positive definite implies non-degenerate.

A metric with respect to basis $\{e_1, e_2, \ldots, e_n\}$ will be expressed in the following form: $\langle e_i, e_j\rangle = g_{ij}$. The metric could also be represeted by the matrix G, where $G_{ij} = g_{ij}$. For the purposes of future definitions, $g^{ij} = [G^{-1}]_{ij} = G_{ij} = g_{ij}$ since the metric is non-degenerate and symmetric.

It may be assumed that any metric mentioned in this paper is positive definite unless otherwise stated.

Definition 1.3. *Given vector space V of dimension n, a metric $\langle \cdot, \cdot \rangle$, and an algebraic curvature tensor R, a **model space** $\mathcal{M}$ is defined by:*

$$\mathcal{M} = (V, \langle \cdot, \cdot \rangle, R).$$

Definition 1.4. *A **canonical algebraic curvature tensor** R_φ is an algebraic curvature tensor that can be expressed as*

$$R_\varphi(x, y, z, w) = \varphi(x, w)\varphi(y, z) - \varphi(x, z)\varphi(y, w)$$

where φ is a symmetric bilinear form.

Henceforth, all algebraic curvature tensors will be canonical. That is, $R = R_\varphi$.

Definition 1.5. *The **Ricci tensor** [5] ρ over a vector space V of dimension n and basis $\{e_1, ...e_n\}$ is defined by:*

$$\rho(x, y) = \sum_{i,j=1}^{n} g^{ij} R(x, e_i, e_j, y).$$

Definition 1.6. *The **scalar curvature** [5] of model space $\mathcal{M}$ is defined by:*

$$\tau = \sum_{i=1}^{n} \rangle \rho(e_i, e_i)$$

on orthonormal basis $\{e_1, \dots, e_n\}$ for V.

Definition 1.7. *A model space $\mathcal{M}$ is **Einstein** [5] if the Ricci tensor is a scalar multiple of the metric. That is,*

$$\rho(\cdot, \cdot) = \lambda \langle \cdot, \cdot \rangle.$$

*We will call λ the **Einstein constant.***

Furthermore, $\lambda = \frac{\tau}{n}$.

Definition 1.8. *A model space $\mathcal{M}$ is **weakly Einstein** [1] if, given orthonormal basis $\{e_1, \dots, e_n\}$ of V:*

$$\sum_{a,b,c=1}^{n} R_{abci} R_{abcj} = \mu g_{ij} \qquad i, j = 1, \dots, n$$

*where $\mu = \frac{1}{n} \sum_{w,x,y,z=1}^{n} R^2_{wxyz}$. We will call μ the **weakly Einstein constant**.*

Given these preliminary definitions, it is now possible to establish the conventions for the results.

First, we will discuss symmetric bilinear form φ and its relationship with associated linear operator Φ, whose eigenvalues will provide a base for the remainder of the calculations. In Section 3, we will discover that Einstein model spaces with positive scalar curvature have Φ as a multiple of the identity matrix. On the other hand, when the Einstein and weakly Einstein constants are 0, at most one eigenvalue of Φ will be nonzero. In Section 4, we will find that if a model space is weakly Einstein, all the eigenvalues of Φ will be the same, up to a sign. Section 5 reveals the precise relationship between the dimension of V in an Einstein model space and the eigenvalues of Φ if the scalar curvature is negative. Finally, Section 6 will explore Einstein model spaces with non-degenerate metrics and conclude that in multiple cases, either the results are identical to those of the model spaces with positive definite metrics or all eigenvalues are zero.

2. Diagonalization and Eigenvalues of Φ

Let φ be a symmetric bilinear form. Given a model space $\mathcal{M}$, and an orthonormal basis $\{e_1, e_2, \dots, e_n\}$, it is possible to express φ as the matrix:

$$\varphi = \begin{pmatrix} \varphi(e_1, e_1) & \varphi(e_1, e_2) & \dots & \varphi(e_1, e_n) \\ \varphi(e_2, e_1) & \varphi(e_2, e_2) & \dots & \varphi(e_2, e_n) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi(e_n, e_1) & \varphi(e_n, e_2) & \dots & \varphi(e_n, e_n) \end{pmatrix}.$$

There also exists a unique associated operator $\Phi : V \to V$ defined by:

$$\varphi(x, y) = \langle \Phi x, y \rangle.$$

Furthermore, Φ is self-adjoint due to the symmetry of φ:

$$\varphi(x, y) = \langle \Phi x, y \rangle = \langle x, \Phi^* y \rangle = \langle \Phi^* y, x \rangle = \varphi(y, x) = \langle \Phi y, x \rangle.$$

Thus, $\Phi = \Phi^*$, so Φ is self-adjoint.

In the case that the metric is positive definite, associated matrix Φ can be diagonalized on an orthonormal basis [3]:

$$\Phi = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}.$$

Furthermore, we notice that since $\varphi(e_i, e_j) = \langle \Phi e_i, e_j \rangle = \langle \lambda_i e_i, e_j \rangle = \lambda_i g_{ij} = \lambda_i \delta_{ij}$, the representation of ϕ as a matrix (above) is equivalent to Φ. So, the matrix representation of φ when Φ is diagonal is:

$$\varphi = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}.$$

Every symmetric bilinear form φ has a similar matrix Φ. For instance, given a positive definite metric as the symmetric bilinear form, the associated matrix is the identity, I.

Therefore, if it is given that $R = R_\varphi$, and the basis for V is such that Φ is diagonal, the only possible nonzero entries of R are those given by $R_{ijji} = \lambda_i\lambda_j = -R_{ijij}$, $i \neq j$. This is true since $\varphi(e_a, e_b) = 0$ if $a \neq b$. Furthermore, $R_{iiii} = \varphi(e_i, e_i)\varphi(e_i, e_i) - \varphi(e_i, e_i)\varphi(e_i, e_i) = 0$. Thus, all other entries of R are zero.

Proposition 2.1. *Let $\mathcal{M}$ be a model space with $R = R_\varphi$ and a positive definite metric. Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of Φ and $\{e_1, \dots, e_n\}$ an orthonormal basis for V. Then, the following system of equations holds when $\mathcal{M}$ is Einstein:*

$$\begin{array}{rcl} \lambda & = & \lambda_1(\lambda_2 + \lambda_3 + \ldots + \lambda_n) \\ \lambda & = & \lambda_2(\lambda_1 + \lambda_3 + \ldots + \lambda_n) \\ \lambda & = & \lambda_3(\lambda_1 + \lambda_2 + \ldots + \lambda_n) \\ & \vdots & \\ \lambda & = & \lambda_i(\lambda_1 + \lambda_2 + \ldots + \lambda_i - 1 + \lambda_{i+1} + \lambda_n) \\ & \vdots & \\ \lambda & = & \lambda_n(\lambda_1 + \lambda_2 + \ldots + \lambda_{n-1}) \end{array} \tag{1}$$

Furthermore, the converse is true.That is, if System (1) holds in $\mathcal{M}$ with $R = R_\varphi$ and diagonalized Φ on an orthonormal basis, then $\mathcal{M}$ is Einstein.

Proof. Recall that $\mathcal{M}$ is Einstein when the Ricci tensor is a constant multiple (λ) of the metric. Since $R = R_\varphi$,

$$\rho(e_i, e_j) = \lambda\langle e_i, e_j\rangle = \lambda\delta_{ij} = \sum_{k=1}^{n} R(e_i, e_k, e_k, e_j).$$

Only nonzero terms contribute to the sum, so $i = j$. Let $\mathcal{M}$ be Einstein. Then,

$$\lambda = \sum_{k=1}^{n} R(e_i, e_k, e_k, e_i) = \sum_{k=1, k\neq i}^{n} \lambda_i\lambda_k = \lambda_i \sum_{k=1, k\neq i}^{n} \lambda_k \quad \forall i \in \{1, 2, \dots, n\}.$$

Since each step of this proof is reversible, it is true that if System 1 is satisfied, then $\mathcal{M}$ is Einstein. □

Proposition 2.2. *Let $\mathcal{M}$ be a model space with $R = R_\varphi$ and a positive definite metric. Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of Φ and $\{e_1, \dots, e_n\}$ an orthonormal basis for V. Then, the following system of equations holds when $\mathcal{M}$ is weakly Einstein:*

$$\begin{array}{rcl} \tilde{\mu} & = & \lambda_1^2(\lambda_2^2 + \lambda_3^2 + \ldots + \lambda_n^2) \\ \tilde{\mu} & = & \lambda_2^2(\lambda_1^2 + \lambda_3^2 + \ldots + \lambda_n^2) \\ \tilde{\mu} & = & \lambda_3^2(\lambda_1^2 + \lambda_2^2 + \ldots + \lambda_n^2) \\ & \vdots & \\ \tilde{\mu} & = & \lambda_i^2(\lambda_1^2 + \lambda_2^2 + \ldots + \lambda_{i-1}^2 + \lambda_{i+1}^2 + \lambda_n^2) \\ & \vdots & \\ \tilde{\mu} & = & \lambda_n^2(\lambda_1^2 + \lambda_2^2 + \ldots + \lambda_{n-1}^2) \end{array} \tag{2}$$

where $\tilde{\mu} = \frac{\mu}{2}$. Furthermore, the converse is true.That is, if System 2 holds in $\mathcal{M}$ with $R = R_\varphi$ and diagonalized Φ on an orthonormal basis, then $\mathcal{M}$ is weakly Einstein.

Proof. Recall that $\mathcal{M}$ is weakly Einstein if $\sum_{a,b,c=1}^{n} R_{abci}R_{abcj} = \mu g_{ij}$. Since $R_{xyyx} = -R_{yxyx}$ are the only nonzero entries of any given R_{abci}, all other entries may be discarded. Thus, it is the case that in the definition of weakly Einstein, $i = j$ if the term is nonzero. Then,

$$\sum_{a,b,c=1}^{n} R_{abci}R_{abcj} = \sum_{a=1}^{n} R_{iaai}^2 + \sum_{a=1}^{n} R_{aiai}^2 = 2\sum_{a=1}^{n} R_{iaai}^2 = \mu$$

Thus,

$$\sum_{a=1}^{n} R_{iaai}^2 = \frac{\mu}{2} = \tilde{\mu}$$

Since $R_{iiii} = 0$,

$$\tilde{\mu} = \sum_{a=1, a\neq i}^{n} \lambda_i^2\lambda_a^2 = \lambda_i^2 \sum_{a=1, a\neq i}^{n} \lambda_a^2 \qquad \forall i \in \{1, 2, \ldots, n\}$$

Once more, the logic is reversible, so the converse holds true. □

In the following sections, we will apply these equations to Einstein and weakly Einstein model spaces with the purpose of solving for Φ given various parameters for Einstein constant λ, scalar curvature τ, or weakly Einstein constant μ.

3. Constant Sectional Curvature in Einstein and Weakly Einstein Model Spaces with $R = R_\varphi$

Now, we will see why, in Einstein model spaces, when $\tau \geq 0$, a model space has a special property called *constant sectional curvature*. Furthermore, we will find that this same property applies to weakly Einstein model spaces when $\mu = 0$.

Definition 3.1. *A model space $\mathcal{M}$ has* ***constant sectional curvature*** *ε if:*

$$\kappa(u, v) = \frac{R(u, v, v, u)}{\langle v, v\rangle\langle u, u\rangle - \langle u, v\rangle^2} = \varepsilon \qquad \forall u, v \in V$$

where u, v span a non-degenerate 2-plane.

One can notice that this definition simplifies to:

$$\kappa(u, v)\frac{R(u, v, v, u)}{R_{\langle\cdot,\cdot\rangle}(u, v, v, u)} = \varepsilon \qquad \forall u, v \in V.$$

Also, if a model space has $csc(\varepsilon)$, it does so for any basis.

Lemma 3.2. *Given a model space $\mathcal{M}$ and an algebraic curvature tensor R_φ,*

$$R_{c\varphi} = c^2 R_\varphi$$

Proof. This result can be obtained through a straightforward application of the definition of R_φ:

$$\begin{aligned} R_{c\varphi}(x, y, z, w) &= c\varphi(x, w) \cdot c\varphi(y, z) - c\varphi(x, z) \cdot c\varphi(y, w) \\ &= c^2(\varphi(x, w)\varphi(cy, z) - \varphi(x, z) \cdot \varphi(y, w)) \\ &= c^2 R_\varphi(x, y, z, w) \end{aligned}$$

□

Lemma 3.3. *If $\varphi = \omega\langle\cdot,\cdot\rangle$ (so $\Phi = \omega I$) and $R = R_\varphi$, then $\mathcal{M}$ has $csc(\omega^2)$.*

Proof. Let $\varphi = \omega\langle\cdot,\cdot\rangle$. Then,

$$R_\varphi = R_{\omega\langle\cdot,\cdot\rangle} = \omega^2 R_{\langle\cdot,\cdot\rangle}$$

Calculating the sectional curvature yields:

$$\kappa(u, v) = \frac{R_\varphi(u, v, v, u)}{R_{\langle\cdot,\cdot\rangle}(u, v, v, u)} = \frac{\omega^2 R_{\langle\cdot,\cdot\rangle}(u, v, v, u)}{R_{\langle\cdot,\cdot\rangle}(u, v, v, u)} = \omega^2$$

Thus, $\mathcal{M}$ has $csc(\omega^2)$. □

Lemma 3.4. *Let $\mathcal{M}$ be an Einstein model space with orthonormal basis $\{e_1, \ldots, e_n\}$ of V and $\rho(\cdot,\cdot) = \lambda\langle\cdot,\cdot,\rangle$. Let $\{\lambda_i | 1 \leq i \leq n\}$ be eigenvalues of diagonalized Φ, as in Section 2. If $R = R_\varphi$, then the following are equivalent:*

(1) $\lambda = 0$ *and*
(2) $\lambda_i \neq 0$ *for at most one* $i \in \{1, \dots, n\}$.

In this case, $\mathcal{M}$ *has* $csc(0)$.

Proof. Suppose $\mathcal{M}$ is Einstein. If $\lambda_i = 0$, then

$$\lambda = \lambda_i \left(\sum_{j=1, j\neq i}^{n} \lambda_j \right) = 0.$$

Conversely, suppose $\lambda = 0$. Let there be i nonzero eigenvalues of Φ, and suppose $i \geq 2$. Without loss of generality, suppose $\lambda_1, \dots, \lambda_i \neq 0$. By System (1), for some $a, b \in \{1, \dots, i\}$, $a \neq b$,

$$\lambda_a \left(\sum_{c=1, c\neq a}^{n} \lambda_c \right) = 0 \text{ and}$$

$$\lambda_b \left(\sum_{c=1, c\neq b}^{n} \lambda_c \right) = 0.$$

Dividing the first equation through by λ_a and the second by λ_b leads to,

$$\sum_{c=1, c\neq a}^{n} \lambda_c = 0 \text{ and}$$

$$\sum_{c=1, c\neq b}^{n} \lambda_c = 0.$$

Then, subtracting the first equation from the second results in

$$\lambda_a - \lambda_b = 0,$$

and simplifying yields

$$\lambda_a = \lambda_b.$$

Since $a, b \in \{1, 2, \dots, i\}$ were arbitrary,

$$\lambda_1 = \lambda_2 = \cdots = \lambda_i = \eta \neq 0.$$

Furthermore, when this is substituted back into any of $1, 2, \dots, i$ of System (1), it becomes clear that

$$\eta \left(\eta(i-1) + \sum_{j=i+1}^{n} \lambda_j \right) = 0.$$

So since $\lambda_j = 0$ for $j > i$,

$$\eta(i-1) + \sum_{j=i+1}^{n} \lambda_j = \eta(i-1) = 0.$$

Thus, either $i = 1$, meaning Φ has one nonzero eigenvalue, or $\eta = 0$, contradicting there being more than one nonzero eigenvalue. Therefore, at most one eigenvalue of Φ is nonzero.

Recall that the only nonzero entries of R_φ are those in the format R_{ijji} or R_{ijij} for $i \neq j$. Also, $R_{ijji} = \varphi(e_i, e_i)\varphi(e_j, e_j) = \Phi_{ii}\Phi_{jj} = \lambda_i\lambda_j$. Since at most one $\lambda_i = 0$ when $\lambda = 0$, $R_{ijji} = 0$ for all i, j, and thus $R_\varphi = 0$ identically. It follows that $\mathcal{M}$ has $csc(0)$. □

Lemma 3.5. *Let* $\mathcal{M}$ *be a weakly Einstein model space with orthonormal basis* $\{e_1, \dots, e_n\}$ *of* V *and* μ *defined by:* $\sum_{a,b,c=1}^{n} R_{abci}R_{abcj} = \mu g_{ij}$. *Let* $\tilde{\mu} = \frac{\mu}{2}$. *Let* $\{\lambda_i | 1 \leq i \leq n\}$ *be eigenvalues of diagonalized* Φ, *as in Section 2. If* $R = R_\varphi$, *then the following are equivalent:*

(1) $\mu = 0$ *and*
(2) $\lambda_i \neq 0$ *for at most one* $i \in \{1, \dots, n\}$.

In this case, $\mathcal{M}$ *has* $csc(0)$.

Proof. The proof for this lemma is identical to that Lemma 3.4, but with λ_i^2 instead of λ_i. □

Now that we found Einstein model spaces with $\lambda = 0$ to have $csc(0)$, we will show that Einsein model spaces with $\lambda > 0$ also have constant sectional curvature. When $n = dim(V) = 2$, the model space is trivially flat, so it has constant sectional curvatue. Lemma 3.6 will consider the non-trival case of $n \geq 3$.

Lemma 3.6. *Given $R = R_\varphi$, if $\mathcal{M}$ is Einstein with $\lambda > 0$ and $n \geq 3$, then $\Phi = cI$. Furthermore, $c = \pm\sqrt{\frac{\lambda}{n-1}}$, so $\mathcal{M}$ is $csc(\frac{\lambda}{n-1})$.*

Proof. Let $\lambda > 0$. Suppose that $\varphi \neq c\langle\cdot,\cdot\rangle$. Then, there exists $i \neq j$ such that $\lambda_i \neq \lambda_j$. Due to the symmetries of System (1), we may assume without loss of generality that $\lambda_1 \neq \lambda_2$. Then, subtracting the first two equations of System (1), we find that:

$$(\lambda_1 - \lambda_2)(\lambda_3 + \lambda_4 + \cdots + \lambda_n) = 0.$$

Since $\lambda_1 \neq \lambda_2$, $\lambda_3 + \lambda_4 + \cdots + \lambda_n = 0$. It is now evident that

$$-\lambda_l = \sum_{k=3,k\neq i}^{n} \lambda_k \qquad \forall l \in \{3,\ldots,n\}.$$

Therefore, substituting that into all remaining equations from System (1) yields

$$\lambda = \lambda_l(\lambda_1 + \lambda_2 + \sum_{k=3,k\neq l}^{n} \lambda_k) = \lambda_l(\lambda_1 + \lambda_2 - \lambda_l) \qquad \forall l \in \{3,\ldots,n\}.$$

Summing both sides of equations over $l = 3,\ldots,n$, we find that:

$$\sum_{l=3}^{n} \lambda = \sum_{l=3}^{n} \lambda_i(\lambda_1 + \lambda_2 - \lambda_l) = \sum_{l=3}^{n} \lambda_l(\lambda_1 + \lambda_2) - \sum_{l=3}^{n} \lambda_l^2, \text{ so}$$

$$(n-2)\lambda = (\lambda_1 + \lambda_2)\left(\sum_{l=3}^{n} \lambda_l\right) - \left(\sum_{l=3}^{n} \lambda_l^2\right) = -\left(\sum_{l=3}^{n} \lambda_l^2\right).$$

Since $\lambda > 0$ and $-\left(\sum_{l=3}^{n} \lambda_l^2\right) \leq 0$, this equation is inconsistent. Therefore, it must be the case that $\varphi = c\langle\cdot,\cdot\rangle$ and $\Phi = cI$.
Since all the eigenvalues of Φ are equal, we can write:

$$\lambda = \lambda_i\left((n-1)\,\lambda_i\right) = (n-1)\lambda_i^2$$

Therefore, solving for λ_i,

$$\lambda_i = \pm\sqrt{\frac{\lambda}{n-1}}.$$

It follows that $c = \pm\sqrt{\frac{\lambda}{n-1}}$, so, by Lemma 3.3, $\mathcal{M}$ has $csc(\frac{\lambda}{n-1})$. □

Theorem 3.7. *Given $R = R_\varphi$, if $\mathcal{M}$ is Einstein with scalar curvature $\tau \geq 0$, $\mathcal{M}$ has constant sectional curvature.*

Proof. Since $\tau \geq 0$, $\lambda \geq 0$.
Let $\tau = 0$. Then, by the results of Lemma 3.4, $\mathcal{M}$ is $csc(0)$.
Let $\tau > 0$. Then, by Lemma 3.6, $\mathcal{M}$ has constant sectional curvature. □

The results from this section are significant since they enumerate every possible φ when $\tau \geq 0$ for Einstein spaces, and $\mu = 0$ for weakly Einstein spaces. We will now take a closer look at the remaining cases for weakly Einstein model spaces.

4. Weakly Einstein Model Spaces in $dim(V) = n$ with $R = R_\varphi$

In this section, we will determine that if a model space is weakly Einstein, accociated linear operator Φ has identical eigenvalues, up to sign. The following theorem encapsulates this result.

Theorem 4.1. *Let φ^2 be the symmetric bilinear form with associated linear operator Φ^2. Let $\mathcal{M}$ be a model space with $R = R_\varphi$. Let $\tilde{\mathcal{M}}$ be a model space with $R = R_{\varphi^2}$ and the same metric as $\mathcal{M}$. Then, $\mathcal{M}$ is weakly Einstein if and only if R_{φ^2} has constant sectional curvature.*

Proof. Suppose $\mathcal{M}$ is weakly Einstein. Then, there exists $\tilde{\mu}$ such that System (2) holds. Let $\eta_i = \lambda_i^2$. Then, we know the following to be true:

$$\Phi^2 = \begin{pmatrix} \lambda_1^2 & 0 & \dots & 0 \\ 0 & \lambda_2^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n^2 \end{pmatrix} = \begin{pmatrix} \eta_1 & 0 & \dots & 0 \\ 0 & \eta_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \eta_n \end{pmatrix}$$

Furthermore, the following system holds:

$$\begin{array}{lllll} \tilde{\mu} & = & \lambda_1^2(\lambda_2^2 + \lambda_3^2 + \ldots + \lambda_n^2) & = & \eta_1(\eta_2 + \eta_3 + \ldots + \eta_n) \\ \tilde{\mu} & = & \lambda_2^2(\lambda_1^2 + \lambda_3^2 + \ldots + \lambda_n^2) & = & \eta_2(\eta_1 + \eta_3 + \ldots + \eta_n) \\ \tilde{\mu} & = & \lambda_3^2(\lambda_1^2 + \lambda_2^2 + \ldots + \lambda_n^2) & = & \eta_3(\eta_1 + \eta_2 + \ldots + \eta_n) \\ \vdots & & & & \\ \tilde{\mu} & = & \lambda_n^2(\lambda_1^2 + \lambda_2^2 + \ldots + \lambda_{n-1}^2) & = & \eta_n(\eta_1 + \eta_2 + \ldots + \eta_{n-1}). \end{array}$$

This system clearly satisfies the requirements set by System 1, signifying that model space $\tilde{\mathcal{M}}$ is Einstein with $R = R_{\varphi^2}$. Since $\tilde{\mu} > 0$, Theorem 3.8 states that $\mathcal{M}$ must have constant sectional curvature.

Conversely, suppose $\tilde{\mathcal{M}}$ has constant sectional curvature. In [4], Gilkey proves that constant sectional curvature implies $\Phi = cI$, so it must be the case that $\Phi^2 = \eta I$ for some η. Here, it is known that $\eta \geq 0$. Clearly, this fulfills the requirements for $\mathcal{M}$ to be Einstein, as enumerated in System 1. Then, Φ^2 can be expressed as a diagonal matrix with eigenvalues η, while Φ can be written as:

$$\Phi = \begin{pmatrix} \pm\sqrt{\eta} & 0 & \dots & 0 \\ 0 & \pm\sqrt{\eta} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \pm\sqrt{\eta} \end{pmatrix}.$$

Then, the equations in System 2 are fulfilled with $\tilde{\mu} = (n-1)\eta$. Therefore, $\mathcal{M}$ is weakly Einstein. □

Theorem 4.1 has several important implications. First, given a weakly Einstein model space, we know that the eigenvalues of Φ must be equal with the exception of sign. A simple calculation demonstrates that we know how many possible matrices Φ yield weakly Einstein model spaces.

Corollary 4.2. *Let $\mathcal{M}$ a model space with $R = R_\varphi$, and Φ be diagonal with respect to an orthonormal basis. Given that all eigenvalues are nonzero, if $dim(V) = n$ is even, then there are $\frac{n}{2} + 1$ unique sets of eigenvalues of φ, and when n is odd, there are $\frac{n-1}{2} + 1$ unique sets eigenvalues, both excluding negation of all entries. If at most one eigenvalue is nonzero, there is one set of possible eigenvalues excluding scaling, and the same is true if all eigenvalues are zero.*

Proof. In all cases, we must be careful to not double count any permutations of the basis vectors. Therefore, suppose that $\lambda_1, \ldots, \lambda_i < 0$ and $\lambda_{i+1}, \ldots, \lambda_n > 0$.To account for negation, $i \leq \frac{n}{2}$.

Let n be even. Then, there are $\frac{n}{2}$ possible sets of eigenvalues of Φ from above and one more from the possibility that all are the same sign, for a total of $\frac{n}{2} + 1$.

Let n be odd. Then, to account for the above constraints, there are $\frac{n-1}{2}$ possibilities with some negative eigenvalues and another from the possibility that all eigenvalues are the same sign. This totals to $\frac{n-1}{2} + 1$ possible sets of eigenvalues.

The case that only one eigenvalue is nonzero is trivial, though that eigenvalue can take on any value. The case that all eigenvalues are zero is trivial as well.

By Lemma 3.2, $R_{-\varphi} = (-1)^2 R_\varphi = R_\varphi$, so negation of all entries of Φ leads to an identical algebraic curvature tensor. $\square$

Despite the fact that if a model space is weakly Einstein then, then $\tilde{\mathcal{M}} = (V, \langle\cdot,\cdot\rangle, R)$ with $R = R_{\varphi^2}$ has constant sectional curvature, this may not be the case for $\mathcal{M} = (V, \langle\cdot,\cdot\rangle, R)$ with $R = R_\varphi$ might not have constant sectional curvature.

Example 4.3. *Let $\mathcal{M} = (V, \langle\cdot,\cdot\rangle, R)$ be of dimension four with $R = R_\varphi$. Let Φ have eigenvalues $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (1, 1, -1, -1)$. It is easy to check that $\mathcal{M}$ is indeed weakly Einstein by referencing System (2).*

Now, to show that $\mathcal{M}$ does not have constant sectional curvature, we will compute curvature κ with two distinct sets of vectors.

$$\kappa(e_1, e_2) = \frac{R_{1221}}{g_{11}g_{22} - g_{12}^2} = \frac{\lambda_1\lambda_2}{1-0} = 1$$
$$\kappa(e_1, e_3) = \frac{R_{1331}}{g_{11}g_{33} - g_{13}^2} = \frac{\lambda_1\lambda_3}{1-0} = -1$$

Since $1 \neq -1$, $\mathcal{M}$ does not have constant sectional curvature.

Now that we have completely solved for the possible canonical algebraic curvature tensors in weakly Einstein model spaces of dimension n, we will do the same for Einstein model spaces.

5. Einstein Model Spaces in $dim(V) = n$ with $R = R_\varphi$

The case in which $\tau, \lambda \geq 0$ in Einstein spaces has already been solved in Section 3, leaving the case in which $\tau, \lambda < 0$. To solve for the eigenvalues of Φ, we will first establish that there exist at most 2 distinct eigenvalues of Φ.

Theorem 5.1. *Given an Einstein model space $\mathcal{M}$ with and diagonalized φ with respect to an orthonormal basis, if $R = R_\varphi$, then Φ can have at most 2 distinct eigenvalues.*

Proof. If $dim(V) \leq 2$, Φ is smaller than or equal to a 2×2 matrix, and therefore can have at most 2 eigenvalues.

Suppose then that $dim(V) \geq 3$, and φ has at least 3 distinct eigenvalues. Due to the symmetries of System 1, we can let $\lambda_1 = X$, $\lambda_2 = Y$, and $\lambda_3 = Z$, where $X, Y,$ and Z are unique nonzero constants. (The case in which any one of $\{X, Y, Z\}$ is zero is covered in Lemma 3.4.) Manipulating the first three equations yields:

$$\begin{aligned}
\tfrac{\lambda}{X} &= Y + Z + \lambda_4 + \ldots + \lambda_n \\
\tfrac{\lambda}{Y} &= X + Z + \lambda_4 + \ldots + \lambda_n \\
\tfrac{\lambda}{Z} &= X + Y + \lambda_4 + \ldots + \lambda_n
\end{aligned}$$

Subtracting the second equation from the first and simplifying shows that:

$$\tfrac{\lambda}{X} - \tfrac{\lambda}{Y} = Y - X$$

So, $\lambda = XY$. Similar operations for the second and third equations, as well as the first and third equations, lead to the conclusion that $\lambda = XY = YZ = XZ$. Since $X, Y,$ and Z are nonzero, $X = Y = Z$, which contradicts their being distinct. Thus, Φ can have at most 2 distinct eigenvalues. $\square$

Let x, y be distinct eigenvalues of Φ. Let j be the number of times x is an eigenvalue of Φ and k be the number of times y is an eigenvalue of Φ, so $j + k = n$. Since $\lambda = xy$, as in Theorem 5.1, one of $\{x, y\}$ must be negative and the other positive, and we may assume that $x > 0$ and $y < 0$ without loss of generality. The following system of equations can be compiled from the equations presented in previous sections:

$$\lambda = xy \tag{3}$$

$$n = j + k \tag{4}$$

$$(j-1)x + (k-1)y = 0 \tag{5}$$

Equation 5 is derived from System 1, which states that, in this case,

$$\lambda = x(y + (j-1)x + (k-1)y) = xy + x((j-1)x + (k-1)y)$$

Thus, $x((j-1)x + (k-1)y) = 0$, and dividing by x yields: $(j-1)x + (k-1)y = 0$.

Note that $j, k \neq 1$ since that would imply that either x or y equals zero, which contradicts their being nonzero. If j or k is zero, Φ has only one eigenvalue, which simplifies to the $\Phi = cI$ case covered in Section 3.

Theorem 5.2. *Let $\mathcal{M}$ be an Einstein model space with $R = R_\varphi$, and let Φ be diagonalized with eigenvalues $\lambda_1, \ldots, \lambda_n$. Let scalar curvature $\tau < 0$. Then,*

$$(\lambda_1, \ldots, \lambda_j, \lambda_{j+1}, \ldots, \lambda_n) = \sqrt{n|\tau|}\sqrt{\frac{k-1}{j-1}}(1, \ldots, 1, -\frac{j-1}{k-1}, \ldots, -\frac{j-1}{k-1}). \tag{6}$$

Proof. Since $\tau < 0$, we know that $\lambda < 0$.

Manipulating Equation 5 and combining it with Equation 3 yields:

$$\frac{\lambda}{x} = y = -\frac{j-1}{k-1} \cdot x$$

Since $\lambda < 0$, $-\lambda = |\lambda|$, so $x^2 = |\lambda|\frac{k-1}{j-1}$. Thus,

$$x = \sqrt{|\lambda|\frac{k-1}{j-1}}, \qquad y = -\sqrt{|\lambda|\frac{j-1}{k-1}}$$

Hence,

$$(\lambda_1, \ldots, \lambda_j, \lambda_{j+1}, \ldots, \lambda_n) = (\sqrt{|\lambda|\frac{k-1}{j-1}}, \ldots, \sqrt{|\lambda|\frac{k-1}{j-1}}, -\sqrt{|\lambda|\frac{j-1}{k-1}}, \ldots, -\sqrt{|\lambda|\frac{j-1}{k-1}})$$

Factoring the right hand side and substituting in $n|\tau|$ for λ leads to the conclusion that:

$$(\lambda_1, \ldots, \lambda_j, \lambda_{j+1}, \ldots, \lambda_n) = \sqrt{|n\tau|}\sqrt{\frac{k-1}{j-1}}(1, \ldots, 1, -\frac{j-1}{k-1}, \ldots, -\frac{j-1}{k-1})$$

□

Corollary 5.3. *Let $\mathcal{M}$ be an Einstein model space with $R = R_\varphi$ and let j and k be as in Theorem 5.2. Then, switching the values of j and k yields the same solution up to permutation and negation.*

Proof. Beginning with Equation 6 and swapping j and k, we get:

$$\sqrt{|\lambda|}\sqrt{\frac{j-1}{k-1}}(1, \ldots, 1, -\frac{k-1}{j-1}, \ldots, -\frac{k-1}{j-1})$$

Multiplying the $\sqrt{\frac{j-1}{k-1}}$ into the parentheses, then factoring out $-\sqrt{\frac{k-1}{j-1}}$ leads to:

$$-\sqrt{|\lambda|}\sqrt{\frac{k-1}{j-1}}(1, \ldots, 1, -\frac{j-1}{k-1}, \ldots, -\frac{j-1}{k-1})$$

Therefore, the two solutions are identical up to a permutation and a negation. □

Corollary 5.4. *Let $\mathcal{M}$ be an Einstein model space with $R = R_\varphi$. If $dim(V) = n$ is even, then there are $\frac{n-2}{2}$ unique sets of eigenvalues, up to permutation and negation. If $dim(V) = n$ is odd, then there are $\frac{n-3}{2}$ unique sets of eigenvalues, up to permutation and negation.*

Proof. Suppose n is even. Then, $j \in \{2, 3, \ldots, n-3, n-2\}$. This provides for $n-3$ total possibilities for j. However, $j = a$ leads to the same solution as $j = n - a$ by Corollary 5.3. Furthermore, there is one case in which $j = k$. Therefore, the total number of solutions is given by $\frac{n-3}{2} + \frac{1}{2} = \frac{n-2}{2}$.

Alternately, suppose n is odd. Once more, there are $n-3$ possible values for j, and each is double counted. Hence, the possible number of solutions for n odd is $\frac{n-3}{2}$. □

In 2010, it was proven that in four dimensions, if a model space is Einstein, it is also weakly Einstein [2]. However, it is not true in higher dimensions.

Corollary 5.5. *In model spaces, the Einstein condition does not imply a weakly Einstein condition for $dim(V) \geq 5$.*

Proof. Take the construction of the eigenvalues of Φ to be dictated by $j = 2$, as described in Theorem 5.2. Then, it is clear that in dimensions other than 4, the eigenvalues are not negatives of each other, as required by Theorem 4.1 for a model space to be weakly Einstein. Thus, a model space with such an $R = R_\varphi$ is Einstein but not weakly Einstein. The following is a concrete example of this concept. $\square$

Example 5.6. ***Einstein does not imply weakly Einstein.*** *Let $\mathcal{M}$ have $R = R_\varphi$ defined by the eigenvalues of Φ being:*

$$(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) = \sqrt{\frac{1}{2}}(1, 1, 1, -2, -2).$$

Clearly, this satisfies being Einstein with $j = 3, k = 2$. However,

$$\left(\sqrt{\frac{1}{2}}\right)^2 \left(\sqrt{\frac{1}{2}}^2 + \sqrt{\frac{1}{2}}^2 + (-2\sqrt{\frac{1}{2}})^2 + (-2\sqrt{\frac{1}{2}})^2\right)$$
$$\neq \left(-2\sqrt{\frac{1}{2}}\right)^2 \left(\sqrt{\frac{1}{2}}^2 + \sqrt{\frac{1}{2}}^2 + \sqrt{\frac{1}{2}}^2 + (-2\sqrt{\frac{1}{2}})^2\right) = \frac{\mu}{2},$$

which is the necessary condition for weakly Einstein. Therefore, Einstein does not imply weakly Einstein.

6. Einstein Model Spaces in Higher Signature Settings

The **signature** of a metric refers to the sigs of the entries of G, the matrix representing the metric. Model spaces in higher signature settings therefore imply having a non-degenerate metric. In this section, we will investigate the effect of the signature of a metric on the algebraic curvature tensor of a model space.

Any matrix A may be expressed in a form called a **Jordan-Normal (Jordan) form** [3]. The basis under which A is Jordan-Normal is called the **Jordan basis**. Given any Jordan basis, matrix may take only one Jordan-Normal form [3]. Which Jordan form the matrix takes is determined by the rank of the matrix and the number of repeated eigenvalues.

Thus far, we have considered the diagonalized form of Φ, which is the first Jordan form. As we previously discussed, a positive definite metric leads to a diagonalizable Φ, which is why no other Jordan forms have been considered. The following are Jordan forms for matrices of dimension 4 with real eigenvalues:

$$\text{Type I: } \begin{pmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{pmatrix} \qquad \text{Type II: } \begin{pmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & \lambda_2 \end{pmatrix}$$

$$\text{Type III: } \begin{pmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{pmatrix} \qquad \text{Type IV: } \begin{pmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 \\ 0 & 0 & \lambda_1 & 1 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix}$$

It is possible for $\lambda_i = \lambda_j$ in any of the above cases.

However, a model space with a non-degenerate metric may still have a diagonalizable Φ.

6.1. Einstein Model Spaces with Non-degenerate Metrics and Diagonalizable Φ.

Given an Einstein model space $\mathcal{M}$ with diagonalizable Φ, it is possible to express the metric as $g_{ij} = \epsilon_i \delta_{ij}$, where $\epsilon_i = \pm 1$. Thus, $\varphi(e_i, e_j) = \epsilon_i \lambda_i \delta_{ij}$. Let λ be the constant associated with the definition of Einstein spaces. Computing the Ricci tensor yields the following equations:

$$\begin{aligned} \rho(e_i, e_i) &= g^{11}R_{i11i} + g^{22}R_{i22i} + g^{33}R_{i33i} + \cdots + g^{nn}R_{inni} \\ &= \epsilon_1(\epsilon_i\lambda_i)(\epsilon_1\lambda_1) + \epsilon_2(\epsilon_i\lambda_i)(\epsilon_2\lambda_2) + \cdots + \epsilon_{i-1}(\epsilon_i\lambda_i)(\epsilon_{i-1}\lambda_{i-1}) \\ &\quad + \epsilon_{i+1}(\epsilon_i\lambda_i)(\epsilon_{i+1}\lambda_{i+1}) + \cdots + \epsilon_n(\epsilon_i\lambda_i)(\epsilon_n\lambda_n) \\ &= \epsilon_i\lambda_i(\textstyle\sum_{j=1, j\neq i}^{n} \lambda_j) \\ &= \lambda\epsilon_i. \end{aligned} \tag{7}$$

Clearly, $\rho(e_i, e_j) = 0$ for $i \neq j$ since $g_{ij} = 0$.

Theorem 6.1. *Let $\mathcal{M}$ be a model space with diagonalized Φ with respect to an orthonormal basis, $R = R_\varphi$, and non-degenerate metric. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis for V. The following are then true:*

(1) *If the scalar curvature* $\tau \geq 0$*, then* $\mathcal{M}$ *has constant sectional curvature.*
(2) *Let* $dim(V) = n$*, and* $j, k \in \{2, \ldots, n-2\}$ *such that* $j + k = n$*. If* $\tau < 0$*, then*

$$(\lambda_1, \ldots, \lambda_j, \lambda_{j+1}, \ldots, \lambda_n) = \sqrt{|\lambda|}\sqrt{\frac{k-1}{j-1}}(1, \ldots, 1, -\frac{j-1}{k-1}, \ldots, -\frac{j-1}{k-1}).$$

Proof. Equation 7 can be rewritten as:

$$\lambda g_{ii} = \epsilon_i \lambda = \epsilon_i \lambda_i (\sum_{j=1, j\neq i}^{n} \lambda_j).$$

So,

$$\lambda = \lambda_i (\sum_{j=1, j\neq i}^{n} \lambda_j).$$

This equation is identical to the case in which the metric is positive definite, and therefore it has the same solutions. Thus, by Theorem 3.8, the first claim is true, and by Theorem 5.2, the second claim is true. □

6.2. **Einstein model spaces in higher signature settings and Type I Φ.**
Given the Einstein condition (with a corresponding λ) and Φ in the Jordan-Normal form

$$\Phi = \begin{pmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{pmatrix},$$

it is possible to solve for the metric g by using the symmetric bilinear form φ. Let $\{e_1, e_2, e_3, e_4\}$ be an orthonormal basis for V. The equations are as follow:

$$\begin{aligned} \varphi(e_1, e_2) &= \langle \Phi e_1, e_2 \rangle &= \lambda_1 g_{12} \\ &= \langle \Phi e_2, e_1 \rangle &= \lambda_1 g_{12} + g_{11} \end{aligned} \tag{8}$$

Thus, we know that $g_{11} = 0$.

$$\begin{aligned} \varphi(e_1, e_3) &= \langle \Phi e_1, e_3 \rangle &= \lambda_1 g_{13} \\ &= \langle \Phi e_3, e_1 \rangle &= \lambda_2 g_{13} \end{aligned} \tag{9}$$

From Equation 9, we know that either $\lambda_1 = \lambda_2$ or $g_{13} = 0$.

$$\begin{aligned} \varphi(e_1, e_4) &= \langle \Phi e_1, e_4 \rangle &= \lambda_1 g_{14} \\ &= \langle \Phi e_4, e_1 \rangle &= \lambda_3 g_{14} \end{aligned} \tag{10}$$

Equation 10 tells us that either $\lambda_1 = \lambda_3$ or $g_{14} = 0$.

$$\begin{aligned} \varphi(e_2, e_3) &= \langle \Phi e_2, e_3 \rangle &= g_{13} + \lambda_1 g_{23} \\ &= \langle \Phi e_3, e_2 \rangle &= \lambda_2 g_{23} \end{aligned} \tag{11}$$

From Equation 9, we know that $\lambda_1 = \lambda_2$ or g_{13}. Suppose $\lambda_1 = \lambda_2$. Then, $g_{13} = 0$. If not, $g_{13} = 0$ from above. Thus, we know $g_{13} = 0$. Then, we know that it must be the case that either $\lambda_1 = \lambda_2$ or $g_{23} = 0$.

$$\begin{aligned} \varphi(e_2, e_4) &= \langle \Phi e_2, e_4 \rangle &= g_{14} + \lambda_1 g_{24} \\ &= \langle \Phi e_4, e_2 \rangle &= \lambda_3 g_{24} \end{aligned} \tag{12}$$

By the same logic as above, from Equation 10, we know that $g_{14} = 0$. Then, either $\lambda_1 = \lambda_3$ or $\lambda_{24} = 0$.

$$\begin{aligned} \varphi(e_3, e_4) &= \langle \Phi e_3, e_4 \rangle &= \lambda_2 g_{34} \\ &= \langle \Phi e_4, e_3 \rangle &= \lambda_3 g_{34} \end{aligned} \tag{13}$$

Therefore, either $\lambda_2 = \lambda_3$ or $g_{34} = 0$.

For the purpose of generalizing as much as possible, we will suppose we know nothing of the eigenvalues of Φ. Thus, all that we know of the matrix G (as defined by $G_{ij} = g_{ij}$) is:

$$G = \begin{pmatrix} 0 & g_{12} & 0 & 0 \\ g_{21} & g_{22} & g_{23} & g_{24} \\ 0 & g_{32} & g_{33} & g_{34} \\ 0 & g_{42} & g_{43} & g_{44} \end{pmatrix}$$

Note that $g_{12} = g_{21} \neq 0$ since the metric is non-degenerate. Now, we must create a series of change of bases that will place G in a helpful form.

Lemma 6.2. *Let $\{e_1, \dots, e_4\}$ be a Jordan-Normal basis of V. Given a change of basis that involves any $f_i = e_i + xe_j$, such that e_j is an eigenvector of Φ and such that e_i and e_j have identical associated eigenvalues. Then, Φ is preserved.*

Proof. Let μ be the eigenvalue associated with e_i and e_j. Then,

$$\Phi f_i = \Phi(e_i + xe_j) = \Phi e_i + x\Phi e_j = \mu e_i + x\mu e_j = \mu(e_i + xe_j) = \mu f_i.$$

Thus, f_i is preserved as an eigenvector of Φ. Similarly, all other f_j are preserved. Therefore, the matrix Φ is preserved. □

Lemma 6.3. *Let $\{e_1, \dots, e_n\}$ be a Jordan basis for V. Given a change of basis that involves any $f_i = Ce_i$, such that $C \neq 0$ and e_i is an eigenvector of Φ. Then, Φ is preserved.*

Proof. Let μ be the eigenvalue associated with e_i. Then,

$$\Phi f_i = \Phi(Ce_i) = n\Phi e_i = C\mu e_i = \mu(Ce_i) = \mu f_i.$$

By similar reasoning as in Lemma 6.2, the matrix Φ is preserved. □

Lemma 6.4. *Let $\{e_1, e_2, e_3, e_4\}$ be a Jordan basis for V. Then, the basis $\{f_1, f_2, f_3, f_4\}$ defined by:*

$$\begin{aligned} f_1 &= e_1 \\ f_2 &= -\tfrac{g_{22}}{2g_{12}} e_1 + e_2 \\ f_3 &= e_3 \\ f_4 &= e_4 \end{aligned}$$

preserves Φ and sets $\langle f_2, f_2 \rangle$ to 0 without altering the remaining 0 entries of the metric.

Proof. First, the change of basis does not alter Φ by Lemma 6.2. To confirm that $\langle f_2, f_2 \rangle = 0$ and none of the other zero entries of G are affected, we must check all of the inner products.

$$\begin{aligned} \langle f_1, f_2 \rangle &= \langle e_1, -\tfrac{g_{22}}{2g_{12}} e_1 + e_2 \rangle = -\tfrac{g_{22}}{2g_{12}} g_{11} + g_{12} = g_{12} \neq 0 \\ \langle f_2, f_2 \rangle &= (-\tfrac{g_{22}}{2g_{12}})^2 g_{11} + g_{22} + 2 \cdot -\tfrac{g_{22}}{2g_{12}} g_{12} = 0 \end{aligned}$$

Of the remainder of the entries, those that contain an inner product with f_2 do not matter since the values are not yet set, and all the others are preserved since $\langle e_i, e_j \rangle = \langle f_i, f_j \rangle$ for $i, j \neq 2$. □

Now, the matrix Φ is unaltered, while

$$\tilde{G} = \begin{pmatrix} 0 & g_{12} & 0 & 0 \\ g_{21} & 0 & g_{23} & g_{24} \\ 0 & g_{32} & g_{33} & g_{34} \\ 0 & g_{42} & g_{43} & g_{44} \end{pmatrix}$$

with basis $\{f_1, \dots, f_n\}$ of V. All g_{ij} in the following lemma are in reference to this updated metric.

Lemma 6.5. *Let $\{f_1, \dots, f_n\}$ be a basis as described above. Then, the basis $\{h_1, h_2, h_3, h_4\}$ defined by:*

$$\begin{aligned} h_1 &= \tfrac{1}{|g_{12}|} f_1 \\ h_2 &= f_2 \\ h_3 &= f_3 \\ h_4 &= f_4 \end{aligned}$$

preserves Φ and sets $g_{12} = g_{21} = \epsilon_1 = \pm 1$ without altering the remaining 0 entries of the metric.

Proof. By Lemma 6.3, this change of basis does not alter Φ. As stated earlier, $g_{12} = g_{21} \neq 0$. Then,

$$\langle h_1, h_2 \rangle = \langle \frac{1}{|g_{12}|} f_1, f_2 \rangle = \frac{1}{|g_{12}|} \langle f_1, f_2 \rangle = \frac{1}{|g_{12}|} \cdot g_{12} = \pm 1 = \epsilon_1.$$

Note that $\langle h_1, h_i \rangle$ for $i \neq 2$ remains 0 and g_{22} is completely unaffected. □

Once more, following this change of basis, Φ is still in the above Jordan form. The updated matrix representation of the metric is:

$$G' = \begin{pmatrix} 0 & \epsilon_1 & 0 & 0 \\ \epsilon_1 & 0 & g_{23} & g_{24} \\ 0 & g_{32} & g_{33} & g_{34} \\ 0 & g_{42} & g_{43} & g_{44} \end{pmatrix}$$

with basis $\{h_1, \ldots, h_n\}$, as described in Lemma 6.5. Henceforth, all $g_{ij} = G'_{ij}$.

Now, suppose $\lambda_2 \neq \lambda 3$. Then, by Equation 13, $g_{34} = 0$. However, this may not be the case. Setting g_{34} to 0 is covered in the following lemma.

Lemma 6.6. *Let $\{h_1, h_2, h_3, h_4\}$ be a basis for V as described above. Then, there exists a basis $\{l_1, l_2, l_3, l_4\}$ such that $\langle l_3, l_4 \rangle = 0$ while preserving Φ and all other known entries of the metric.*

Proof. Suppose $g_{44} = 0$. Then, we first want to create a change of basis such that $g_{44} \neq 0$.

Suppose that $g_{33} \neq -2g_{34}$. Then, use the following change of basis for V.

$$\begin{aligned} a_1 &= h_1 \\ a_2 &= h_2 \\ a_3 &= h_3 \\ a_4 &= h_3 + h_4 \end{aligned}$$

Then,

$$\langle a_4, a_4 \rangle = g_{44} + g_{33} + 2g_{34} = g_{33} + 2g_{34} \neq 0, \text{ and}$$
$$\langle a_1, a_4 \rangle = \langle h_1, h_3 + h_4 \rangle = g_{13} + g_{14} = 0 = g_{14},$$

which is exactly as desired. All other inner products are either not affected or we do not care about the results.

Now, suppose that $g_{33} = -2g_{34}$. Then, use the following change of basis for V.

$$\begin{aligned} b_1 &= h_1 \\ b_2 &= h_2 \\ b_3 &= h_3 \\ b_4 &= h_3 + 2h_4 \end{aligned}$$

In this case,

$$\langle b_4, b_4 \rangle = 4g_{44} + 4g_{34} + g_{33} = 2g_{34} \neq 0,$$

which, once more, is the desired result.

By combining Lemmas 6.2 and 6.3, both change of bases preserve Φ. Now, adopt the change of basis that corresponds with G', and call the new metric:

$$G" = \begin{pmatrix} 0 & \epsilon_1 & 0 & 0 \\ \epsilon_1 & 0 & g_{23} & g_{24} \\ 0 & g_{32} & g_{33} & g_{34} \\ 0 & g_{42} & g_{43} & d \end{pmatrix}, \qquad d \neq 0.$$

For simplicity, call whichever basis applies $\{c_1, c_2, c_3, c_4\}$.

Now, apply the following change of basis:

$$\begin{aligned} l_1 &= c_1 \\ l_2 &= c_2 \\ l_3 &= c_3 - \tfrac{g_{34}}{d} c_4 \\ l_4 &= c_4 \end{aligned}$$

By Lemma 6.2, this change of basis preserves Φ. Furthermore,

$$\langle l_3, l_4 \rangle = \langle c_3 - \frac{g_{34}}{d} c_4, c_4 \rangle = g_{34} - \frac{g_{34}}{d} g_{44} = g_{34} - \frac{g_{34}}{d} d = 0,$$

as desired. The only other relevant inner product to check is:

$$\langle l_1, l_3\rangle = \langle c_1, c_3 - \frac{g_{34}}{d}c_4\rangle = g_{13}\frac{g_{34}}{d}g_{14} = 0,$$

as desired. $\square$

The updated matrix representation of the metric, with inserted variable names for ease of notation, is now:

$$\widehat{G} = \begin{pmatrix} 0 & \epsilon_1 & 0 & 0 \\ \epsilon_1 & 0 & a & b \\ 0 & a & c & 0 \\ 0 & b & 0 & d \end{pmatrix}, \qquad d \neq 0.$$

Nothing is known about a, b, and c. From now on, $g_{ij} = \hat{G}_{ij}$. Furthermore, this is also the case if $\lambda_2 \neq \lambda_3$, so we are now ready to tackle further cases.

From Equation 12, if $\lambda_1 \neq \lambda_3$, $g_{24} = 0$. Suppose then that $\lambda_1 = \lambda_3$. We will now see that there exists a change of basis such that $\lambda_{24} = 0$.

Lemma 6.7. *Let $\{l_1, l_2, l_3, l_4\}$ be a basis for V as described in the previous lemma. Then, the basis defined by:*

$$\begin{aligned} m_1 &= l_1 \\ m_2 &= l_2 \\ m_3 &= l_3 \\ m_4 &= l_4 - \tfrac{b}{\epsilon_1}l_1 \end{aligned}$$

preserves Φ and sets $\langle m_2, m_4\rangle$ to 0 while keeping all other known entries of the metric constant.

Proof. By Lemma 6.2, Φ is preserved. Now,

$$\langle m_2, m_4\rangle = \langle l_2, l_4 - \frac{b}{\epsilon_1}l_1\rangle = b - \frac{b}{\epsilon_1}\epsilon_1 = 0,$$

as desired. Furthermore,

$$\langle m_1, m_4\rangle = \langle l_1, l_4 - \frac{b}{\epsilon_1}l_1\rangle = 0 \text{ and}$$

$$\langle m_3, m_4\rangle = \langle l_3, l_4 - \frac{b}{\epsilon_1}l_1\rangle = 0.$$

All other inner products are unaffected or their values are not fixed. $\square$

The updated matrix representation of the metric, with variable names for ease of notation, is now:

$$G^* = \begin{pmatrix} 0 & \epsilon_1 & 0 & 0 \\ \epsilon_1 & 0 & a & 0 \\ 0 & a & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix}$$

Note that a, c, and d are not as in $\widehat{G}$.

When $\lambda_1 \neq \lambda_2$, $a = 0$, as per Equation 11. Suppose then that $\lambda_1 = \lambda_2$. We must now find a change of basis that makes $a = 0$.

Lemma 6.8. *Let $\{m_1, m_2, m_3, m_4\}$ be a basis for V as in the previous lemma. Then, the basis defined by:*

$$\begin{aligned} p_1 &= m_1 \\ p_2 &= m_2 \\ p_3 &= m_3 - \tfrac{a}{\epsilon_1}m_1 \\ p_4 &= m_4 \end{aligned}$$

preserves Φ and sets $\langle p_2, p_4\rangle$ to 0 while keeping all other known values of the metric constant.

Proof. This proof proceeds with identical logic to that of Lemmas 6.4-7. $\square$

The updated matrix representation of the metric is now:

$$\bar{G} = \begin{pmatrix} 0 & \epsilon_1 & 0 & 0 \\ \epsilon_1 & 0 & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix}$$

which preserves Φ. Once again, note that c, d are not as in G^*.

There is now one final step to simplifying this matrix as completely as possible: setting the value of every nonzero entry to ± 1.

Lemma 6.9. *Let $\{p_1, p_2, p_3, p_4\}$ be a basis for V as defined above. Then, the basis defined by:*

$$\begin{aligned} q_1 &= p_1 \\ q_2 &= p_2 \\ q_3 &= \sqrt{\tfrac{1}{|c|}}p_3 \\ q_4 &= \sqrt{\tfrac{1}{|d|}}p_4 \end{aligned}$$

preserves Φ and sets $\langle q_3, q3\rangle$ and $\langle q_4, q_4\rangle$ to ± 1.

Proof. First, $c, d \neq 0$ since g is non-degenerate.

Clearly, the following is true:

$$\langle q_3, q3\rangle = \langle\sqrt{\frac{1}{|c|}}p_3, \sqrt{\frac{1}{|c|}}p_3\rangle = \frac{1}{|c|}\langle p_3, p_3, \rangle = \frac{c}{|c|} = \pm 1 = \epsilon_2.$$

Similarly,

$$\langle q_4, q_4\rangle = \frac{d}{|d|} = \pm 1 = \epsilon_3.$$

The remainder of the proof follows the logic of Lemmas 6.4-8. □

Theorem 6.10. *Let $\mathcal{M}$ be a model space with non-degenerate inner product (metric) g and $R = R_\varphi$. Then, if Φ has one generalized eigenvalue, then there exists a change of basis such that it is possible to express G, where $G_{ij} = g_{ij}$, as:*

$$G = \begin{pmatrix} 0 & \epsilon_1 & 0 & 0 \\ \epsilon_1 & 0 & 0 & 0 \\ 0 & 0 & \epsilon_2 & 0 \\ 0 & 0 & 0 & \epsilon_3 \end{pmatrix}$$

without altering Φ.

Proof. This follows from Lemmas 6.4-9. □

Lemma 6.11. *Let $\mathcal{M}$ be an Einstein model space, λ being the constant in the definition of Einstein, with the remainder of the conditions dictated in Theorem 6.10. Let $\{e_1, e_2, e_3, e_4\}$ be the basis that achieves those conditions. Then, the only nonzero entries of the Ricci tensor are $\rho(e_1, e_2) = \rho(e_2, e_1)$, $\rho(e_3, e_3)$, and $\rho(e_4, e_4)$, and they yield the following equations:*

$$\begin{aligned} \lambda &= \lambda_1(-\lambda_1 + \lambda_2 + \lambda_3) \\ \lambda &= \lambda_2(2\lambda_1 + \lambda_3) \\ \lambda &= \lambda_3(2\lambda_1 + \lambda_2). \end{aligned} \tag{14}$$

Proof. Since $\mathcal{M}$ is an Einstein model space, we know that $\rho(e_i, e_j) = \lambda\langle e_i, e_j\rangle = \sum_{a,b=1}^{4} g^{ij} R_{iabj}$. Clearly, $g_{ij} = 0$ when $(i, j) \neq (1, 2), (2, 1), (3, 3)$, or $(4, 4)$, so the same is true of ρ. Furthermore, calculating ρ yields:

$$\begin{array}{rcl}
\rho(e_1, e_2) & = & \epsilon_1 \lambda \\
& = & \sum_{a,b=1}^{4} g^{ab} R_{1ab2} \\
& = & \epsilon_1 \cdot 0 + \epsilon_1(\epsilon_1\lambda_1)^2 + \epsilon_2(\epsilon_1\lambda_1)(\epsilon_2\lambda_2) + \epsilon_3(\epsilon_3\lambda_3)(\epsilon_2\lambda_2) \\
& = & \epsilon_1\lambda_1^2 + \epsilon_1\lambda_2^2 + \epsilon_1\lambda_3^2 \\
& = & \rho(e_2, e_1) \\
\rho(e_3, e_3) & = & \epsilon_2\lambda \\
& = & \sum_{a,b=1}^{4} g^{ab} R_{3ab3} \\
& = & \epsilon_1(\epsilon_1\lambda_1)(\epsilon_2\lambda_2) + \epsilon_1(\epsilon_1\lambda_1)(\epsilon_2\lambda_2) + \epsilon_2 \cdot 0 + \epsilon_3(\epsilon_3\lambda_3)(\epsilon_1\lambda_1) \\
& = & \epsilon_2\lambda_1^2 + \epsilon_2\lambda_2^2 + \epsilon_2\lambda_3^2 \\
\rho(e_4, e_4) & = & \epsilon_2\lambda \\
& = & \sum_{a,b=1}^{4} g^{ab} R_{4ab4} \\
& = & \epsilon_1(\epsilon_1\lambda_1)(\epsilon_3\lambda_3) + \epsilon_1(\epsilon_1\lambda_1)(\epsilon_3\lambda_3) + \epsilon_2(\epsilon_2\lambda_2)(\epsilon_3\lambda_3) + \epsilon_3 \cdot 0 \\
& = & \epsilon_3\lambda_1^2 + \epsilon_3\lambda_2^2 + \epsilon_3\lambda_3^2.
\end{array}$$

Cancelling out the epsilons leads to the final system of equations:

$$\begin{array}{rcl}
\lambda & = & \lambda_1(-\lambda_1 + \lambda_2 + \lambda_3) \\
\lambda & = & \lambda_2(2\lambda_1 + \lambda_3) \\
\lambda & = & \lambda_3(2\lambda_1 + \lambda_2).
\end{array}$$

□

Theorem 6.12. *Let $\mathcal{M}$ be an Einstein model space with $R = R_\varphi$ and a non-degenerate inner product (metric) g and basis $\{e_1, e_2, e_3, e_4\}$ for V such that:*

$$g_{ij} = \begin{cases} \epsilon_1 & (i,j) = (1,2) \text{ or } (2,1) \\ \epsilon_2 & (i,j) = (3,3) \\ \epsilon_3 & (i,j) = (4,4) \\ 0 & \text{otherwise} \end{cases}.$$

Let $\lambda_1, \lambda_2, \lambda_3$ be eigenvalues for Φ. Then, scalar curvature $\tau = 0$ and $\lambda_1, \lambda_2, \lambda_3 = 0$.

Proof. First, it is helpful to calculate $\rho(e_2, e_2)$, although we know its value:

$$\rho(e_2, e_2) = \epsilon_2 R_{2332} + \epsilon_3 R_{2442} = \epsilon_2(\epsilon_1)(\epsilon_2\lambda_2) + \epsilon_3(\epsilon_1)(\epsilon_3\lambda_3) = \epsilon_1(\lambda_2 + \lambda_3) = 0.$$

Since $\epsilon_1 \neq 0$, $\lambda_2 = -\lambda_3$.

Substituting this result into the first equation of System (14) yields:

$$\lambda_1 = \sqrt{|\lambda|}.$$

The second two equations in System (14) may be rewritten as:

$$\lambda = \lambda_2(2\lambda_1 - \lambda_2) = 2\lambda_1\lambda_2 - \lambda_2^2$$

$$\lambda = -\lambda_2(2\lambda_1 + \lambda_2) = -2\lambda_1\lambda_2 - \lambda_2^2.$$

Subtracting the second equation from the first and dividing through by 4 results in:

$$\lambda_1\lambda_2 = 0.$$

Then, either $\lambda_1 = 0$ or $\lambda_2 = 0$. If $\lambda_1 = 0$, then $\lambda = 0$, and substituting this into the updated equations above shows that $\lambda_2\lambda_3 = 0$, so $\lambda_2, \lambda_3 = 0$. If $\lambda_2 = 0$, then $\lambda_3 = 0$ and $\lambda = 0$, so $\lambda_1 = 0$. Therefore, $\lambda_1, \lambda_2, \lambda_3, \lambda, \tau = 0$ since $\lambda = \frac{\tau}{n}$. □

7. Examples

Here are several examples of model spaces that fulfill the conditions discussed in previous sections.

Example 7.1. ***Einstein model space.*** *Let V be a four-dimensional vector space and let $\langle\cdot,\cdot\rangle$ be a positive definite inner product on V. Let $R = R_\varphi$ be defined by the following Φ:*

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Let $\mathcal{M} = (V, \langle\cdot,\cdot\rangle, R_\varphi)$.

Then, the first two equations of System (1) are:

$$\lambda = 1(1-1-1) = -1,$$

while the second two equations are:

$$\lambda = -1(1+1-1) = -1.$$

Therefore, the equations are satisfied, so $\mathcal{M}$ is Einstein.

Example 7.2. ***Weakly Einstein model space****. Let $\mathcal{M}$ be a model space as described in Example 7.1.*

Then, each equation from System (2) is:

$$\tilde{\mu} = 1^2(1^2 + (-1)^2 + (-1)^2) = (-1)^2(1^2 + 1^2 + (-1)^2) = 3,$$

meaning $\mathcal{M}$ is weakly Einstein.

As seen in Examples 7.1-2, it is possible for a model space to be both Einstein and weakly Einstein.

Example 7.3. ***Weakly Einstein but not Einstein.*** *Let $\mathcal{M} = (V, \langle\cdot,\cdot,\rangle, R_\varphi$, with $dim(V) = 5$. Let the eigenvalues of Φ be $(1, 1, 1, -1, -1)$. Then, the equations from System (2) are:*

$$\tilde{\mu} = 1^2(1^2 + 1^2 + (-1)^2 + (-1)^2) = (-1)^2(1^2 + 1^2 + 1^2 + (-1)^2) = 4.$$

Therefore, $\mathcal{M}$ is weakly Einstein.

However, the two distinct equations from System (1) are:

$$\lambda = 1(1+1-1-1) = 0$$

$$\lambda = -1(1+1+1-1) = -2.$$

Since $0 \neq -2$, $\mathcal{M}$ is not Einstein.

Example 7.4. ***Einstein but not wealy Einstein.*** *This is covered in Example 5.6.*

8. Open Questions

(1) Section 6 focuses nearly exclusively on $dim(V) = 4$ Type I Φ. Are there analogous results for the other Jordan forms, and are there results in $dim(V) = n$?
(2) This paper demonstrates the necessary characteristics of φ for a model space with $R = R_\varphi$ to be Einstein or weakly Einstein. Are the converse results true - that is, under what conditions is $R = R_\varphi$ if $\mathcal{M}$ is Einstein?
(3) If the above is not true, then are there characteristics of a model space that easily lend information regarding whether the algebraic curvature tensor is canonical?
(4) Does Einstein imply weakly Einstein in manifolds of dimension greater than four?

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