PUTTING THE "k" IN CURVATURE: k-PLANE CONSTANT CURVATURE CONDITIONS

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ABSTRACT. This research generalizes the properties known as *constant sectional curvature* and *constant vector curvature* in Riemannian model spaces of arbitrary finite dimension. While these properties have been previously studied using 2-plane sectional curvatures, we generalize them by considering k-plane scalar curvature. We prove that there is a unique algebraic curvature tensor with k-plane constant sectional curvature ε for $2 \leq k \leq n-2$, which coincides with a 2-plane constant sectional curvature tensor. As with $k = 2$, there are model spaces without k-plane constant scalar curvature but with k-plane constant vector curvature for $k > 2$. Through two examples, we explore properties of k-plane constant vector curvature in a given model space. In particular, we demonstrate a method for determining values for ε, bounding values of ε, and generating a connected set of values for ε. Many results are generalizations of known aspects of 2-plane constant curvature conditions. By studying general k-plane curvature, we can further characterize model spaces by generating representative numbers for the various subspaces.

SPLASHING IN THE SHALLOW END

1. Introduction and Background

Classical differential geometry uses the tools of calculus to study local properties of some topological surfaces, called *manifolds*. As manifolds locally resemble Euclidean space, we can use calculus and linear algebra to examine the tangent space of the manifold at a particular point. We can create model spaces to represent this local picture of our manifold, and investigate properties of the model space to describe the curvature of the manifold at a point. In order to study curvature, we need to develop a way to measure distance in the tangent space of a point, which is a vector space. One such metric that describes geometric notions of distance in Euclidean space is the *inner product*:

Definition 1.1. *An* **inner product** *on a vector space $V \subseteq \mathbb{R}^n$ is a function from pairs of vectors to scalars,*

$$\langle\ ,\ \rangle\ :\ V \times V \to \mathbb{R},$$

that is

1. *Symmetric: $\langle x, y\rangle = \langle y, x\rangle$ for all $x, y \in V$,*
2. *Bilinear: $\langle x + x', y\rangle = \langle x, y\rangle + \langle x', y\rangle$ and $\langle \lambda x, y\rangle = \lambda\langle x, y\rangle$ for all $x, y \in V$ and $\lambda \in \mathbb{R}$. Linearity in the second slot follows from symmetry,*
3. *Non-degenerate: for all $x \in V$ there is some y such that $\langle x, y\rangle \neq 0$.*

We say the inner product is *positive-definite* if $\langle x, x\rangle \geq 0$ for all $x \in V$, with the equality if and only if $x = 0$. Note that every positive-definite inner product is non-degenerate.

For the purposes of this research, we assume all inner products to be positive-definite unless explicitly stated otherwise. A *Riemannian manifold* has a positive-definite inner product for the tangent space of each point, whereas a *pseudo-Riemannian manifold* has a non-degenerate inner product for the tangent spaces of each point.

To further characterize the properties of a manifold, we can use the *algebraic curvature tensor* as a powerful algebraic tool to calculate curvature in higher dimensions.

Definition 1.2. *An* **Algebraic Curvature Tensor (ACT)** *is a function from four tangent vectors in $V \subseteq \mathbb{R}^n$ to a scalar,*

$$R : V \times V \times V \times V \to \mathbb{R}$$

with the following properties, for all $x, y, z, w \in V$:

(1) *Multilinearity: $R(x + x', y, z, w) = R(x, y, z, w) + R(x', y, z, w)$ and $R(\lambda x, y, z, w) = \lambda R(x, y, z, w)$ for all $x' \in V$ and $\lambda \in \mathbb{R}$. This shows linearity in the first slot; linearity is similar for the other slots,*
(2) *Skew-symmetry in the first two slots: $R(x, y, z, w) = -R(y, x, z, w)$,*
(3) *Interchange symmetry: $R(x, y, z, w) = R(z, w, x, y)$,*
(4) *The Bianchi Identity: $R(x, y, z, w) + R(z, x, y, w) + R(y, z, x, w) = 0$.*

For basis vectors $\{e_1, \ldots, e_n\}$, denote $R(e_i, e_j, e_k, e_l)$ as R_{ijkl}.

Definition 1.3. *Let ϕ be a symmetric, bilinear function that takes pairs of vectors to scalars. A* **canonical ACT** *with respect to ϕ is defined as*

$$R_\phi(x, y, z, w) = \phi(x, w)\phi(y, z) - \phi(x, z)\phi(y, w).$$

Note that the ACT with respect to the inner product is canonical, with the associated matrix I_n. Denote this particular ACT as $R_{\langle\ ,\ \rangle} = R_*$, so $R_*(x, y, z, w) = \langle x, w\rangle\langle y, z\rangle - \langle x, z\rangle\langle y, w\rangle$.

From the span of canonical ACTs, we can build a vector space that contains all ACTs, denoted $\mathcal{A}(V)$, and all vector space properties follow.

Proposition 1.1. *The following properties hold for all $S, T \in \mathcal{A}(V)$, $x, y, z, w \in V$, and $\lambda \in \mathbb{R}$,*

(1) *Addition: $(S + T)(x, y, z, w) = S(x, y, z, w) + T(x, y, z, w)$,*
(2) *Scalar multiplication: $(\lambda S)(x, y, z, w) = \lambda S(x, y, z, w)$.*

Definition 1.4. *Define the* **kernel of R** *as follows:*

$$\ker(R) = \{v \in V | R(v, y, z, w) = 0,\ \forall y, z, w \in V\}.$$

Theorem 1.5. *Suppose $rank(\phi) \geq 2$. Then*

$$\ker(\phi) = \{v \in V | \phi(v, w) = 0,\ \forall w \in V\} = \ker(R).$$

Proof. See [3]. □

Now understanding algebraic curvature tensors, we have described the main tools necessary to build model spaces:

Definition 1.6. *A* **model space** *$\mathcal{M} = (V, \langle\ ,\ \rangle, R)$ is defined as a vector space $V \subseteq \mathbb{R}^n$, a non-degenerate inner product on V, and an algebraic curvature tensor.*

Given a manifold, a metric, and a specific point on the manifold, we can build a model space from the tangent space, metric, and Riemannian curvature tensor at that point. If we are interested in the curvature of a manifold at a point, we can study the curvature given by a model space.

One curvature measurement is called the *sectional curvature*. This calculation inputs a non-degenerate 2-plane π and returns a number $\kappa(\pi)$. Subce we are working with a positive definite inner product, all subspaces here are non-degenerate.

Definition 1.7. *Let $\mathcal{M}$ be a model space and let $x, y \in V$ be tangent vectors. Let $\pi = span\{x, y\}$ be a non-degenerate 2-plane. The* **sectional curvature** *is a function that takes pairs of tangent vectors to scalars, $\kappa : V \times V \to \mathbb{R}$, where*

$$\kappa(\pi) = \frac{R(x, y, y, x)}{\langle x, x\rangle\langle y, y\rangle - \langle x, y\rangle^2}.$$

The sectional curvature is a geometric invariant. That is to say, this measurement is independent of the basis chosen for π, meaning that for any $u = ax + by$ and $v = cx + dy$ that form another basis for π,

$$\frac{R(x, y, y, x)}{\langle x, x\rangle\langle y, y\rangle - \langle x, y\rangle^2} = \frac{R(u, v, v, u)}{\langle u, u\rangle\langle v, v\rangle - \langle u, v\rangle^2}.$$

The sectional curvature is an important and celebrated measurement. Note that if $\{x, y\}$ is an orthonormal set of vectors, $\kappa(\pi) = R(x, y, y, x)$. Let κ_{ij} denote the sectional curvature of the 2-plane spanned by e_i, e_j.

Another curvature invariant is the *scalar curvature*, which assigns a real number to each point on the manifold based on the sectional curvatures at that point.

Definition 1.8. *Let $\mathcal{M}$ be a model space with orthonormal basis $\{e_1, \ldots, e_n\}$. Define the* **scalar curvature** *(or* **Ricci scalar***) τ by*

$$\tau = \sum_{i,j} \varepsilon_i \varepsilon_j R_{ijji}$$

where $\varepsilon_k = \langle e_k, e_k\rangle = \pm 1$.

Since we are working with a positive definite inner product, calculating τ amounts to summing over the R_{ijji} entries on an orthonormal basis. On such a basis, for all basis vectors e_i, e_j, $\langle e_i, e_j\rangle = 1$ if and only if $i = j$ and equals 0 otherwise.

Definition 1.9. *Let V be a vector space spanned by an orthonormal basis $\{e_1, \ldots, e_n\}$. The* **Ricci tensor** *is a symmetric, bilinear function from pairs of tangent vectors to a scalar, $\rho : V \times V \to \mathbb{R}$, where*

$$\rho(x, y) = \sum_{i=1}^{n} \varepsilon_i R(x, e_i, e_i, y).$$

Again, since we are working with a positive definite inner product, $\varepsilon_i = 1$ for all $i \in \{1, \ldots, n\}$. Note that summing $\rho(e_i, e_i)$ across all basis vectors is equal to the scalar curvature.

Now that we have sufficiently defined the tools necessary to study curvature, we can work towards our main goal of understanding generalized constant curvature conditions. We begin by defining the special condition whereupon a model space gives the same sectional curvature value for all two-planes:

Definition 1.10. *A model space $\mathcal{M}$ has* **constant sectional curvature** *ε, denoted csc(ε), if $\kappa(\pi) = \varepsilon$ for all non-degenerate 2-planes π.*

Constant sectional curvature is particularly important in that a manifold with this condition is *locally homogeneous*: there is some locally distance-preserving map between any of two points. A manifold with constant sectional curvature is called a *space form*. Examples of such space forms include Euclidean n-space, the n-sphere, and hyperbolic n-space.

As constant sectional curvature is a somewhat uncommon property, we can also study the weaker condition of *constant vector curvature:*

Definition 1.11. *A model space $\mathcal{M}$ has* **constant vector curvature** *ε, denoted cvc(ε), if for every $v \in V$, there is some 2-plane π where $v \in \pi$ and $\kappa(\pi) = \varepsilon$ for all non-degenerate 2-planes π.*

Studying constant vector curvature has generated interesting results about the manifolds and model spaces with this condition. In 2011, Schmidt and Wolfson published the first study of constant vector curvature in 3-manifolds [4]. Since then, constant vector curvature has been completely resolved for 3-dimensional model spaces. Through a combined effort, it has been shown that all 3-dimensional Riemannian model spaces have cvc(ε) for some $\varepsilon \in \mathbb{R}$, however this is not necessarily the case for Lorentzian model spaces with a non-degenerate inner product [9, 10]. In studying both constant sectional curvature and constant vector curvature we can learn more about the structures of model spaces that have each condition.

While these conditions are well understood for model spaces of dimension 3, not much is known about model spaces of higher dimensions. The work that has been done in characterizing higher dimensional Riemannian model spaces considers curvature strictly using 2-planes [8]. It seems natural that we should be able to generalize curvature conditions using k-planes for model spaces of arbitrary finite dimension. In [1], Chen defines the scalar curvature of a k-plane as a sum of some sectional curvatures:

Definition 1.12. *Let $\mathcal{M} = (V, \langle\ ,\ \rangle, R)$ with $\{e_1, \ldots, e_n\}$ orthonormal basis for $V \subseteq \mathbb{R}^n$ and non-degenerate inner product. Define $M_L = (L, \langle\ ,\ \rangle, R_L)$ with $\{f_1, \ldots, f_k\}$ orthonormal basis for $L \subseteq V$, $\langle\ ,\ \rangle_L = \langle\ ,\ \rangle|_L$, and $R_L = R|_L \in A(L)$. Define the* **k-plane scalar curvature** *of L by $\mathcal{K}_{R_L} : L \to \mathbb{R}$, given by*

$$\mathcal{K}_{R_L}(L) = \sum_{j>i=1}^{k} \kappa(f_i, f_j).$$

If it is clear that we are considering $\mathcal{K}_R(L)$ with respect to a certain R, we can omit the subscript and simply write $\mathcal{K}(L)$. We will also denote $\mathcal{K}(L) = \mathcal{K}(span\{e_1, \ldots, e_k\})$ by simply $\mathcal{K}(e_1, \ldots, e_k)$.[1]

Although it is understood that we are evaluating $\mathcal{K}(L)$ with respect to the restricted model space M_L, for the ease of notation we will discuss $\mathcal{K}(L)$ mostly in terms of the given model space $\mathcal{M}$. Additionally, since we are considering a positive definite inner product and computing these quantities on an orthonormal basis, calculating $\mathcal{K}(L)$ amounts to summing over certain R_{ijji} terms. Note that if L is a 2-plane, $\mathcal{K}(L)$ is equal to the sectional curvature. When $k = n$, $\mathcal{K}(L) = \frac{\tau}{2}$. Since the only

[1]In his work, Chen uses τ instead of $\mathcal{K}$.

n-plane curvature is characterized by the curvature of the entire vector space, we are mostly interested in k-planes such that $2 \leq k \leq n-1$.

Just as sectional curvature is a geometric invarient, k-plane scalar curvature generates representative numbers that are independent of the chosen basis for the k-plane. Given this tool for examining k-plane curvatures, we can study constant curvature conditions on model spaces of any dimension. An intuitive adaptation of definitions 1.6 and 1.7 follow:

Definition 1.13. *A model space $\mathcal{M}$ has* **k-plane constant sectional curvature** *ε, denoted k-csc(ε), if $\mathcal{K}(L) = \varepsilon$ for all non-degenerate k-planes L.*

Definition 1.14. *A model space $\mathcal{M}$ has* **k-plane constant vector curvature** *ε, denoted k-csc(ε), if for all $v \in V$ there is some non-degenerate k-plane L containing v such that $\mathcal{K}(L) = \varepsilon$.*

In the following sections we will further explore these k-plane constant curvature conditions. Section 2 introduces some immediate propositions that easily generalize from known results from 2-csc and 2-cvc. In Section 3, we further investigate k-plane constant sectional curvature. We prove that in a model space where all k-planes have curvature 0, for some given value of $3 \leq k \leq n-2$, that model space must have $R = 0$. This result gives several corollaries, including that there is a unique R with k-csc(ε). We move on to the weaker condition of k-cvc in Sections 4 and 5, each section exploring an example model space with a canonical tensor. The first example gives a 6-dimensional model space with k-cvc(ε) for certain values of k and ε. We demonstrate a method for calculating values for ε for various values of k, and show that a model space can have multiple ε values for a given k. The second example gives an n-dimensional model space with a large kernel, and demonstrates bounding possible values for ε. We also show this model space to have k-cvc(ε) for any $\varepsilon \in [0, 1]$. Section 6 gives more general results for model spaces with canonical tensors. Sections 7 and 8 summarize the results and list some open questions.

2. k-Plane Curvature General Results

With this understanding of k-plane curvature conditions, some intuitive properties follow immediately, including generalizations of some results given in [8].

Proposition 2.1. *If a model space $\mathcal{M}$ has k-csc(ε) then it has k-cvc(ε).*

Proof. Suppose $\mathcal{M}$ has k-csc(ε) for some $\varepsilon \in \mathbb{R}$. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis and let $v \in V$. Choose a k-plane L with orthonormal basis $\{f_1, \ldots, f_k\}$ such that $f_1 = \frac{v}{||v||}$. So $\mathcal{K}(L) = \varepsilon$, and such an L exists for all $v \in V$. Hence $\mathcal{M}$ has k-cvc(ε). $\square$

Proposition 2.2. *Let $M_1 = (V, \langle\ ,\ \rangle, R_1)$ have k-csc(ε) and $M_2 = (V, \langle\ ,\ \rangle, R_2)$ have k-cvc(δ). Then $M = (V, \langle\ ,\ \rangle, R = R_1 + R_2)$ has k-cvc($\varepsilon + \delta$).*

Proof. Let $v \in V$ and let L be a k-plane containing v such that $\mathcal{K}_{R_2}(L) = \delta$. Note also that $\mathcal{K}_{R_1}(L) = \varepsilon$. So then

$$\begin{aligned}
\mathcal{K}_R(L) &= \sum_{j>i=1}^{k} R_{ijji} \\
&= \sum_{j>i=1}^{k} (R_1 + R_2)_{ijji} \\
&= \sum_{j>i=1}^{k} ((R_1)_{ijji} + (R_2)_{ijji}) && \text{(by Proposition 1.1.1)} \\
&= \sum_{j>i=1}^{k} (R_1)_{ijji} + \sum_{j>i=1}^{k} (R_2)_{ijji} \\
&= \varepsilon + \delta.
\end{aligned}$$

□

Proposition 2.3. *Suppose $M = (V, \langle\ ,\ \rangle, R)$ has k-cvc(ε). Let $c \in \mathbb{R}$. Then $M = (V, \langle\ ,\ \rangle, cR)$ has k-cvc($c\varepsilon$).*

Proof. Let $v \in V$, and let L be a k-plane containing v such that $\mathcal{K}_R(L) = \varepsilon$. Then

$$\begin{aligned}
\mathcal{K}_{cR}(L) &= \sum_{j>i=1}^{k} (cR)_{ijji} \\
&= \sum_{j>i=1}^{k} c(R_{ijji}) && \text{(by Proposition 1.1.2)} \\
&= c \sum_{j>i=1}^{k} R_{ijji} \\
&= c\epsilon.
\end{aligned}$$

□

These three propositions generalize quite easily from results previously known about cvc. However, in generalizing the following lemma, we see that it is the dimension of the kernel relative to the dimension of the 2-plane is what gives k-cvc(0), rather than simply a non-zero kernel.

Lemma 2.1. *Let $\mathcal{M}$ be a model space with $\ker(R) \neq 0$. Then $\mathcal{M}$ has 2-cvc(0) and only 2-cvc(0).*

Proof. See [8]. □

It is clear that what generalizes to higher dimensions is the relationship between the dimensions of the kernel and the k-plane, although it is not generally the case that we will get *only* k-cvc(0), as is illustrated by Example 1 in Section 4.

Proposition 2.4. *Let $M = (V, \langle\ ,\ \rangle, R)$ where $\dim(\ker(R)) \geq k-1$. Then $\mathcal{M}$ has k-cvc(0).*

Proof. Let $\mathcal{M}$ be a model space and suppose $\alpha = \dim(\ker(R)) \geq k-1$. Let $e_1, \ldots, e_{k-1}, \ldots, e_\alpha$ is an orthonormal basis for $\ker(R)$ and $e_{\alpha+1}, \ldots, e_n$ complete an orthonormal basis for V. Let $v \in V$ where $v = x_1e_1 + \cdots + x_ne_n$. Construct a k-plane L spanned by

$$f_1 = e_1, \ldots, f_{k-1} = x_{k-1}e_{k-1} + \cdots + x_\alpha e_\alpha / \sqrt{x_{k-1}^2 + \cdots + x_\alpha^2},$$

$$\text{and } f_k = x_{\alpha+1}e_{\alpha+1} + \cdots + x_ne_n / \sqrt{x_{\alpha+1}^2 + \cdots + x_n^2}.$$

Note that since $\alpha \geq k-1$, $f_{k-1} \neq 0$. In the event that $x_{k-1} = \cdots = x_\alpha = 0$, set $f_{k-1} = e_{k-1}$. Similarly, if $x_{\alpha+1} = \cdots = x_n = 0$, set $f_k = e_n$. So when $\mathcal{K}(L) = \sum_{j>i=1}^{k} R(f_i, f_j, f_j, f_i)$ is written in terms of the e_i's, an $e_i \in \ker(R)$ appears in each R_{ijkl} term. So $\mathcal{K}(L) = 0$ and $\mathcal{M}$ has k-cvc(0). $\square$

Having established these general propositions about k-plane constant curvature conditions, the following sections will explore each property more in-depth.

JUMPING IN THE DEEP END

3. k-plane Constant Sectional Curvature

Now that we have defined a general notion of k-plane constant sectional curvature, we can prove our main result. It seems intuitive that, given $\mathcal{K}(L) = 0$ for any k-plane L, our algebraic curvature tensor must be 0. It is known that a model space with 2-csc(0) must have $R \equiv 0$, and this result is easily proven as R can be written in terms of sectional curvatures [6]. Hence in our proof, we consider $k > 2$.

For $3 \leq k \leq n-2$, the extra two dimensions give a degree of freedom that allows us to first show the R_{ijki} entries to be 0 (for $j \neq k$), and the rest follows. However, when considering $(n-1)$-planes, we cannot isolate R_{ijkl} terms in the same way, and in fact we conjecture that there exists $R \neq 0$ that gives $(n-1)$-csc(0).

Theorem 3.1. *Set $2 \leq k \leq n-2$. Let $M = (V, \langle\ ,\ \rangle, R)$ be a model space. Suppose $\mathcal{K}(L) = 0$ for all k-planes L. Then $R = 0$.*

Proof. Let $\mathcal{M}$ be a model space with $\{e_1, \ldots, e_n\}$ orthonormal basis for V. The result is already known for $k = 2$. Let $3 \leq k \leq n-2$, and suppose $\mathcal{M}$ has k-csc(0). First we will prove that the ACT entries of the form R_{ijjk} are 0 (for $i \neq k$): Consider the k-plane L spanned by

$$\begin{aligned} f_1 &= \cos\theta e_1 + \sin\theta e_2, \\ f_2 &= e_3, \\ &\vdots \\ f_k &= e_{k+1} = e_\alpha. \end{aligned}$$

By hypothesis, the k-plane scalar curvature of L is 0, regardless of the value of θ, so

$$
\begin{aligned}
0 &= \mathcal{K}(\cos\theta e_1 + \sin\theta e_2, e_3, \ldots, e_\alpha) \\
&= \cos^2\theta(R_{1331} + \cdots + R_{1\alpha\alpha 1}) + \sin^2\theta(R_{2332} + \cdots + R_{2\alpha\alpha 2}) \\
&\quad + 2\cos\theta\sin\theta(R_{1332} + \cdots + R_{1\alpha\alpha 2}) + \sum_{j>i=3}^{\alpha} R_{ijji} \\
&= \cos^2\theta(\sum_{j=3}^{\alpha} R_{1jj1} + \sum_{j>i=3}^{\alpha} R_{ijji}) - \cos^2\theta(\sum_{j>i=3}^{\alpha} R_{ijji}) \\
&\quad + \sin^2\theta(\sum_{j=3}^{\alpha} R_{2jj2} + \sum_{j>i=3}^{\alpha} R_{ijji}) - \sin^2\theta(\sum_{j>i=3}^{\alpha} R_{ijji}) \\
&\quad + \sum_{j>i=3}^{\alpha} R_{ijji} + 2\cos\theta\sin\theta(\sum_{j=3}^{\alpha} R_{1jj2}) \qquad \text{(by adding clever 0s).}
\end{aligned}
$$

But now notice that $\sum_{j=3}^{\alpha} R_{1jj1} + \sum_{j>i=3}^{\alpha} R_{ijji} = \mathcal{K}(e_1, e_3, \ldots, e_\alpha)$ and $\sum_{j=3}^{\alpha} R_{2jj2} + \sum_{j>i=3}^{\alpha} R_{ijji} = \mathcal{K}(e_2, e_3, \ldots, e_\alpha)$. Both of these equations equal 0 by supposition, so

$$
\begin{aligned}
0 &= -\cos^2\theta(\sum_{j>i=3}^{\alpha} R_{ijji}) - \sin^2\theta(\sum_{j>i=3}^{\alpha} R_{ijji}) + \sum_{j>i=3}^{\alpha} R_{ijji} + 2\cos\theta\sin\theta(\sum_{j=3}^{\alpha} R_{1jj2}) \\
&= (\cos^2\theta + \sin^2\theta)(-\sum_{j>i=3}^{\alpha} R_{ijji}) + \sum_{j>i=3}^{\alpha} R_{ijji} + 2\cos\theta\sin\theta(\sum_{j=3}^{\alpha} R_{1jj2}) \\
&= 2\cos\theta\sin\theta(\sum_{j=3}^{\alpha} R_{1jj2}).
\end{aligned}
$$

Since the equation must hold true for all values of θ,

$$
0 = \sum_{j=3}^{\alpha} R_{1jj2}. \tag{3.1}
$$

Since we were considering $k \le n-2$, we can construct a new k-plane by setting $f_k = e_n$ and keeping all other f_i basis vectors the same. We repeat the process above to get

$$
0 = \sum_{n-1\neq j=3}^{n} R_{1jj2}. \tag{3.2}
$$

Now, subtracting (3.2) from (3.1):

$$
0 = R_{1\alpha\alpha 2} - R_{1nn2},
$$

which means $R_{1\alpha\alpha 2} = R_{1nn2}$. Note that our choice of e_α and e_n was arbitrary, so we could permute any e_i, e_j basis vectors to get $R_{1ii2} = R_{1jj2}$. Since $0 = \sum_{j=3}^{\alpha} R_{1jj2} = (\alpha-2)R_{1332}$, and since $\alpha > 2$, we can conclude that $R_{1332} = R_{1jj2} = 0$ for all $j \in \{1, \ldots, n\}$. By replacing e_1 and e_2 with e_i and e_k, we can repeat the process to get that $R_{ijjk} = R_{jikj} = 0$ for any distinct $i, j, k \in \{1, \ldots, n\}$.

Next, we will show that the R_{ijji} entries must be 0. Set $f_1 = e_i,\ f_2 = \cos\theta e_j + \sin\theta e_k,\ f_3 = \cos\theta e_k - \sin\theta e_j$ for some $e_i, e_j, e_k \in \{e_1, \ldots, e_n\}$. Extend f_1, f_2, f_3 to an orthonormal basis β_f for V. Since $R_{ijjk} = 0$ on any orthonormal basis, including β_f, we get

$$\begin{aligned}
0 &= R(f_2, f_1, f_1, f_3) \\
&= \cos^2\theta R_{ijki} - \sin\theta\cos\theta R_{ijji} - \sin^2\theta R_{ikji} + \sin\theta\cos\theta R_{ikki} \\
&= -\sin\theta\cos\theta R_{ijji} + \sin\theta\cos\theta R_{ikki}.
\end{aligned}$$

Since this equation must hold for all values of theta, we get $0 = R_{ikki} - R_{ijji}$, meaning that $R_{ijji} = R_{ikki}$. Since $0 = \mathcal{K}(L)$ for all k-planes L, including the coordinate plane spanned by $e_1, \ldots, e_k$, we get that

$$\sum_{j>i=1}^{k} R_{ijji} = \frac{k(k-1)}{2} R_{1221} = 0,$$

which implies $R_{1221} = R_{ijji} = 0$ for all $i, j \in \{1, \ldots, n\}$.

Finally, we will show that the R_{ijkl} entries are also 0. Set $f_1 = \cos\theta e_i + \sin\theta e_l$ and $f_2 = \cos\theta e_j + \sin\theta e_k$. As before, extend f_1, f_2 to an orthonormal basis for V. Then, knowing that on any orthonormal basis $R_{ijji} = 0 = R_{ijki}$ for all indices,

$$\begin{aligned}
0 &= R(f_1, f_2, f_2, f_1) \\
&= 2\cos^2\theta\sin^2\theta R_{ijkl} + 2\cos^2\theta\sin^2\theta R_{ikjl} \\
&= 2\cos^2\theta\sin^2\theta(R_{ijkl} + R_{ikjl}).
\end{aligned}$$

Since this equation holds for all theta, $0 = R_{ijkl} + R_{ikjl}$. We can then permute e_i and e_j without loss of generality to get $0 = R_{jikl} + R_{jkil}$. Subtracting this result from our first one, we get

$$\begin{aligned}
0 &= R_{ijkl} + R_{ikjl} - R_{jikl} - R_{jkil} \\
&= R_{ijkl} - R_{kijl} + R_{ijkl} - R_{jkil} && \text{(by property 2 of Definition 1.2)} \\
&= 2R_{ijkl} - (R_{kijl} + R_{jkil}) \\
&= 2R_{ijkl} - (-R_{ijkl}) && \text{(by property 3 of Definition 1.2)} \\
&= 3R_{ijkl}.
\end{aligned}$$

Hence $R_{ijkl} = 0$ for all $i, j, k, l \in \{1, \ldots, n\}$. Since we have shown all components of R to be 0, we can conclude $R = 0$. □

Corollary 3.1.1. *Set $2 \leq k \leq n-2$. Suppose $\mathcal{K}_{R_1}(L) = \mathcal{K}_{R_2}(L)$ for all k-planes L. Then $R_1 = R_2$.*

Proof. Set $R = R_1 - R_2$. Then for any $L = span\{e_1, \ldots, e_k\}$

$$
\begin{aligned}
\mathcal{K}_R(L) &= \sum_{j>i=1}^{k} R(e_i, e_j, e_j, e_i) \\
&= \sum_{j>i=1}^{k} (R_1 - R_2)(e_i, e_j, e_j, e_i) \\
&= \sum_{j>i=1}^{k} (R_1(e_i, e_j, e_j, e_i) - R_2(e_i, e_j, e_j, e_i)) \qquad \text{by properties of } \mathcal{A}(V) \\
&= \sum_{j>i=1}^{k} R_1(e_i, e_j, e_j, e_i) - \sum_{j>i=1}^{k} R_2(e_i, e_j, e_j, e_i) \\
&= \mathcal{K}_{R_1}(L) - \mathcal{K}_{R_2}(L) \\
&= 0.
\end{aligned}
$$

Then by Theorem 3.1, $0 = R = R_1 - R_2$, meaning that $R_1 = R_2$. □

Corollary 3.1.2. *There is a unique R giving k-csc(ε) where $R = \frac{2\varepsilon}{k(k-1)} R_*$ for $2 \leq k \leq n-1$.*

Proof. Let $\mathcal{M}$ be a model space with $R = \frac{2\varepsilon}{k(k-1)} R_*$. Let $L = span\{e_1, \ldots, e_k\}$ for some orthonormal basis $\{e_1, \ldots, e_n\}$. Then

$$
\begin{aligned}
\mathcal{K}(L) &= \sum_{j>i=1}^{k} R_{ijji} \\
&= \sum_{j>i=1}^{k} \frac{2\varepsilon}{k(k-1)} (R_*)_{ijji} && \text{(by Proposition 1.1.2)} \\
&= \frac{2\varepsilon}{k(k-1)} \sum_{j>i=1}^{k} 1 && \text{(since the inner product is positive definite)} \\
&= \frac{2\varepsilon}{k(k-1)} (\frac{1}{2}) \sum_{i=1}^{k} \sum_{j=1}^{k-1} 1 && \text{(since } j > i \text{ and for each of the k } i\text{'s, there is one } j \text{ where } i = j) \\
&= \frac{\varepsilon}{k(k-1)} (k-1) \sum_{i-1}^{k} 1 \\
&= \frac{\varepsilon}{k} k \\
&= \varepsilon.
\end{aligned}
$$

Now suppose there is some $R_1 \neq R$ with k-csc(ε). Then, for all k-planes L, $\mathcal{K}_R(L) = \mathcal{K}_{R_1}(L)$. But then by Corollary 3.1.1, $R = R_1$. Hence R is unique. □

An equivalent statement is to say that $\mathcal{M}$ has k-csc(ε) if and only if $R = \frac{2\varepsilon}{k(k-1)} R_*$. Note that this means $\mathcal{M}$ has 2-csc(γ) if and only if $R = \gamma R_*$.

Corollary 3.1.3. *$\mathcal{M}$ has k-csc(ε) if and only if it has j-csc(δ), where $\delta = \varepsilon \frac{j(j-1)}{k(k-1)}$.*

Proof. Suppose $\mathcal{M}$ has k-csc(ε) for some $k \leq n-2$ and $\varepsilon \in \mathbb{R}$. By Corollary 3.1.1, $R = \frac{2\varepsilon}{k(k-1)}R_*$. So $\mathcal{M}$ has 2-csc(γ) where $\gamma = \frac{2\varepsilon}{k(k-1)}$. Set $\delta = \epsilon\frac{j(j-1)}{k(k-1)}$. Then $\gamma = \frac{2\delta}{j(j-1)}$, so $\gamma R_* = \frac{2\delta}{j(j-1)}R_*$. Hence by Corollary 3.1.1, $\mathcal{M}$ has j-csc(δ). The argument from j-csc(δ) to k-csc(ε) is similar by swapping k and ϵ with j and δ. □

These corollaries align neatly with what is known about the 2-csc condition. In any given model space, the complete collection of curvature equations uniquely determines the tensor. Further, the unique tensor giving k-csc(ε) coincides with the 2-plane constant sectional curvature tensor. Considering any model space, if all k-dimensional subspaces have constant curvature for any $k \leq n-2$, then subspaces of any dimension must also have constant curvature for some constant that is determined by some proportion of dimensions.

With this work completed for $2 \leq k \leq n-2$, we turn to $k = n-1$. When we suppose a model space has $(n-1)$-csc(0), we discover that the model space is Ricci-flat, and there is a duality between the curvature of certain planes. However we could not prove that $R = 0$, and in fact we conjecture that it could be non-zero.

Theorem 3.2. *Suppose a model space $\mathcal{M}$ has $(n-1)$-csc(0). Then the Ricci scalar $\tau = 0$ and the Ricci tensor $\rho = 0$.*

Proof. Suppose $\mathcal{M}$ has k-csc(0) for $k = n-1$. Recall

$$\frac{\tau}{2} = \sum_{j>i=1}^{n} R_{ijji} = \sum_{j>i=1}^{n-1} R_{ijji} + \sum_{j>i=1}^{n-1} R_{inni},$$

and note $\sum_{j>i=1}^{n-1} R_{ijji} = \mathcal{K}(L) = 0$ for L spanned by $\{e_1, \ldots, e_{n-1}\}$, so $\frac{\tau}{2} = \sum_{j>i=1}^{n-1} R_{inni} = \rho(e_n, e_n)$. By permuting the basis vectors, we get $n > 2$ equations, one for each e_i, where $\frac{\tau}{2} = \rho(e_i, e_i)$. Summing over the equations, we get $\frac{n\tau}{2} = \sum_{i=1}^{n} \rho(e_i, e_i)$. But recall $\sum_{i=1}^{n} \rho(e_i, e_i) = \tau$, and since $n > 2$, $\frac{n}{2}\tau = \tau$ implies $0 = \tau = \rho(e_i, e_i)$ for all $i \in \{e_1, \ldots, e_n\}$ on any orthonormal basis. There exists a particular basis upon which ρ is diagonalized since ρ is a symmetric, bilinear form, so the ρ_{ii} entries are the only possible non-zero entries. But $\rho_{ii} = 0$ on this orthonormal basis, and so $\rho = 0$. □

Theorem 3.3. *Suppose a model space $\mathcal{M}$ has $(n-1)$-csc(0). Set $2 \leq k \leq n-2$ and let $\{e_1, \ldots, e_n\}$ be an orthonormal basis for V. Choose k of the e_i to span a plane L, and let $L^\perp$ be the plane spanned by the remaining $n-k$ coordinate e_i. Then $\mathcal{K}(L) = \mathcal{K}(L^\perp)$.*

Proof. Let a model space $\mathcal{M}$ have $(n-1)$-csc(0), then consider the $(n-1)$-plane spanned by $\{e_1, e_3, \ldots, e_n\}$. So

$$\begin{aligned}
0 &= \mathcal{K}(L) \\
&= \sum_{j=3}^{n} R_{1jj1} + \sum_{j>i=3}^{n} R_{ijji} \\
&= (\rho(e_1, e_1) - R_{1221}) + \sum_{j>i=3}^{n} R_{ijji} \\
&= -\kappa(e_1, e_2) + \mathcal{K}(e_3, \ldots, e_n).
\end{aligned}$$

This shows that the sectional curvature of the 2-plane spanned by e_1, e_2 is equal to the k-plane scalar curvature of the $(n-2)$-plane spanned by the remaining basis vectors of V. By permuting the basis vectors, this equality holds for any way we choose to divide the e_i basis vectors into a collection of 2 and $n-2$. We claim that the equality in fact holds for any division of the e_i into a collection of k and $n-k$ for all $k \geq 2$. Suppose the property holds for some $m \geq 2$. Then, by the induction hypothesis,

$$
\begin{aligned}
0 &= -\mathcal{K}(e_1, \ldots, e_m) + \mathcal{K}(e_{m+1}, \ldots, e_n) \\
&= -(\sum_{j>i=1}^{m} R_{ijji}) + \sum_{j>i=m+1}^{n} R_{ijji} \\
&= -(\sum_{j>i=1}^{m} R_{ijji}) + \sum_{j=m+2}^{n} R(e_{m+1}, e_j, e_j, e_{m+1}) + \sum_{j>i=m+2}^{n} R_{ijji} \\
&= -(\sum_{j>i=1}^{m} R_{ijji}) + (\rho(e_{m+1}, e_{m+1}) - \sum_{j=1}^{m} R(e_{m+1}, e_j, e_j, e_{m+1})) + \sum_{j>i=m+2}^{n} R_{ijji} \\
&= -(\sum_{j>i=1}^{m} R_{ijji} + \sum_{j=1}^{m} R(e_{m+1}, e_j, e_j, e_{m+1})) + \sum_{j>i=m+2}^{n} R_{ijji} \\
&= -(\sum_{j>i=1}^{m+1} R_{ijji}) + \sum_{j>i=m+2}^{n} R_{ijji} \\
&= -\mathcal{K}(e_1, \ldots, e_{m+1}) + \mathcal{K}(e_{m+2}, \ldots, e_n)
\end{aligned}
$$

which shows the desired equality for $(m+1)$-planes and $(n-(m+1))$-planes. Hence the property holds for all $m \geq 2$ by the Principle of Mathematical Induction. □

A model space with $(n-1)$-csc(0) must have these properties, but it is not yet known whether the tensor in such a model space must be the zero tensor. Working with 4-planes in $\mathbb{R}^5$ led us to construct the following tensor as a possible non-zero candidate for a model space with $(n-1)$-csc(0).

Conjecture 3.3.1. *There exists $R \neq 0$ such that $\mathcal{M}$ has $(n-1)$-csc(0). The proposed R is has the following components on an orthonormal basis: $R_{1221} = R_{3443} = 1$, $R_{1331} = R_{2442} = -1$, and otherwise is 0.*

Due to the high degree of symmetry in this tensor, it seems to satisfy all the requirements. In order to prove or disprove the conjecture, we look for particular $(n-1)$-planes that disproportionately skew some R_{ijkl} components. If we cannot find a plane that gives non-zero k-plane scalar curvature using the proposed tensor, then we have proven the conjecture. If we do find such a plane, then we will have proved that R must be 0, since this plane will allow us to isolate some tensor components as equal to 0, and the rest would follow easily.

4. k-plane Constant Vector Curvature: First Example

Having fully characterized model spaces with k-plane constant sectional curvature (for $2 \leq k \leq n-2$), we now investigate model spaces with k-plane constant vector curvature for $3 \leq k \leq n-1$. As previously mentioned, model spaces with k-csc also have k-cvc. But there are many model spaces that only have the weaker condition of k-cvc. We use two examples of such model spaces to demonstrate interesting aspects of this curvature condition. In particular, we investigate methods for determining k-cvc values, including establishing bounds and showing the values to lie within a connected set.

In order to show a model space has k-cvc(ε), we choose an arbitrary vector and construct a k-plane that contains the vector. The plane is specifically constructed to have a curvature value that does not depend on the specific components of our chosen vector. That is to say, the construction of our plane should work indiscriminately for a vector with non-zero components any number of dimensions. The method used in the first example is to decompose v based on components in the eigenspaces of a symmetric, bilinear form ϕ, where $R = R_\phi$.

Definition 4.1. *Let $A : V \to V$ be a linear transformation. A non-zero $v \in V$ is an* **eigenvector** *of A with* **eigenvalue** $\lambda \in \mathbb{R}$ *if $Av = \lambda v$.*

The eigenvalues of a linear transformation are basis independent, as λ is an eigenvalue if and only if $\det(A - \lambda I_n) = 0$ and the determinant calculation does not depend on the choice of a basis of V. We can also discuss eigenvalues in the context of symmetric, bilinear forms. It is known that for any ϕ that is a symmetric, bilinear function which takes pairs of vectors to scalars, if the vector space has a non-degenerate inner product, then there is a self-adjoint linear transformation A such that $\phi(x, y) = \langle Ax, y\rangle$. Hence the eigenvalues of ϕ are precisely the eigenvalues of A. So for orthonormal eigenvectors $f_i,\ f_j$, we know $\phi(f_i, f_j) = \langle Af_i, f_j\rangle = \lambda_i\langle f_i, f_j\rangle$, which is equal to 0 unless $i = j$. Given an eigenvalue λ_i, there is an *eigenspace* spanned by the associated eigenvector v_i. The eigenspace associated with any λ_i is denoted as E_i.

Definition 4.2. *Let ϕ be a symmetric, bilinear form. The* **spectrum** *of ϕ, denoted spec(ϕ), is the collection of eigenvalues of ϕ, repeated according to multiplicity.*

In this research, the geometric multiplicity of λ_i (the number of times λ_i appears in the spectrum) is equal to the algebraic multiplicity of λ_i ($\dim(E_i)$) for all $\lambda_i \in$ *spec*(ϕ). In our first example, each eigenvalue has multiplicity of 2.

Proposition 4.1. *Let $\mathcal{M} = (V, \langle\ ,\ \rangle, R_\phi)$ be a model space. If $f_i,\ f_j$ are unit vectors in the eigenspaces for $\lambda_i,\ \lambda_j$, respectively, then $\kappa(f_i, f_j) = \lambda_i\lambda_j$.*

Proof. By the Spectral Theorem, there is some change of basis that diagonalizes ϕ, so $\phi(e_i, e_j) = 0$ for $i \neq j$. Suppose $f_i \in E_i$ and $f_j \in E_j$ are unit vectors. Then

$$\begin{aligned}\kappa(f_i, f_j) &= R_\phi(f_i, f_j, f_j, f_i)\\ &= \phi_{ii}\phi_{jj} - \phi_{ij}^2\\ &= \lambda_i\lambda_j.\end{aligned}$$

□

This proposition allows $\mathcal{K}(L)$ to be determined in terms of products of eigenvalues.

Example 4.3. Let $\mathcal{M} = (V, \langle\ ,\ \rangle, R)$ be a model space such that $\{e_1, \ldots, e_6\}$ is an orthonormal basis for V, the inner product is positive definite, and $R = R_\phi$ where ϕ is represented by

$$\begin{bmatrix} I_2 & 0_2 & 0_2\\ 0_2 & -I_2 & 0_2\\ 0_2 & 0_2 & 0_2\end{bmatrix}$$

where I_2 is the 2×2 identity matrix and 0_2 is the 2×2 matrix whose entries are all 0. So $\phi(e_i, e_j)$ is the ij^{th} entry of the array above. Let $E_1 = span\{e_1, e_2\}$ be the eigenspace for $\lambda_1 = 1$, $E_2 = span\{e_3, e_4\}$ be the eigenspace for $\lambda_2 = -1$, and $E_3 = span\{e_5, e_6\} = \ker(R)$ be the eigenspace for $\lambda_3 = 0$. Since $\{e_1, \ldots, e_6\}$ is an orthonormal basis, it follows that $E_i \perp E_j$ for $i \neq j$. We will show that $\mathcal{M}$ has multiple k-cvc values for different choices of k. This amounts to constructing different k-planes containing an arbitrary $v \in V$ whose curvature is a constant that does not depend on specific components of v. One way to approach this problem is to construct basis vectors that are contained entirely in some eigenspace. Such a construction is often possible in multiple ways for a given value of k.

In Section 4 of [8], a similar approach is used to determine possible 2-cvc values for a 3-dimensional model space. By orthogonally decomposing v into unit vectors v_i in the eigenspace corresponding to eigenvalue λ_i, Beveridge determines 2-cvc values as products of the eigenvalues. We can further split each v_i between different f_i basis vectors and calculate the curvature value ε as a sum of products of the eigenvalues. It seems likely that further developing this method of decomposing into eigenspaces is most promising for generally determining k-cvc values for an arbitrary model space.

Proposition 4.2. *The model space $\mathcal{M}$ of Example 4.3 has the following properties:*

1. *2-cvc*(0) *and only 2-cvc*(0)*,*
2. *3-cvc*(0) *and 3-cvc*(-1)*,*
3. *4-cvc*(-1)*,*
4. *5-cvc*(-1) *and 5-cvc*(-2)*.*

(1) $\mathcal{M}$ has 2-cvc(0) and only 2-cvc(0).

Proof. Set $k = 2$. Let $v \in V$. Note that $\dim(\ker(R)) \geq 2$. So by Lemma 2.7, $\mathcal{M}$ has 2-cvc(0) and only 2-cvc(0). To show this explicitly, we can decompose $v = av_1 + bv_2 + cv_3$ where $a, b, c \in \mathbb{R}$ and $v_i \in E_i$ are unit vectors. Construct an orthonormal basis for $L = span\{f_1, f_2\}$ where $f_1 = av_1 + bv_2/\sqrt{a^2+b^2}$ and $f_2 = v_3$. So $v = (\sqrt{a^2+b^2})f_1 + cf_2$ and is contained in L. In the event that $a = b = 0$, set $f_1 = e_1$. Similarly, if $c = 0$, set $f_2 = e_5$. Then

$$\mathcal{K}(L) = R(f_1, f_2, f_2, f_1) = 0 \qquad \text{since } f_2 \in \ker(R).$$

Since this construction of L works for arbitrary $v \in V$, $\mathcal{M}$ has 2-cvc(0). □

(2) $\mathcal{M}$ has 3-cvc(0) and 3-cvc(-1).

Proof. Set $k = 3$. Let $v \in V$ be expressed as before. Again, by Proposition 2.4, we know $\mathcal{M}$ has 2-cvc(0). We can show this explicitly by constructing L similarly as we did in (a), but set $f_1 = av_1 + bv_2/\sqrt{a^2+b^2}$, $f_2 = e_5$, and $f_3 = e_6$. In the event that $a = b = 0$, set $f_1 = e_1$. Since $f_2, f_3 \in \ker(\phi) = \ker(R)$ by Theorem 1.5, $\mathcal{K}(L) = 0$.

However, $\mathcal{M}$ also has 3-cvc(-1). Construct $L = span\{f_1, f_2, f_3\}$ where $f_1 = v_1$, $f_2 = v_2$, and $f_3 = v_3$. If $a = 0$, set $f_1 = e_1$. If $b = 0$, set $f_2 = e_3$. If $c = 0$, set

$f_3 = e_5$. Since $f_3 \in \ker(R)$,

$$\begin{aligned}\mathcal{K}(L) &= R(f_1, f_2, f_2, f_1) + R(f_1, f_3, f_3, f_1) + R(f_2, f_3, f_3, f_2)\\ &= \lambda_1\lambda_2 + 0\\ &= -1\end{aligned}$$

which shows $\mathcal{M}$ has 3-cvc(-1). ☐

(3) $\mathcal{M}$ has 4-cvc(-1).

Proof. Set $k = 4$. Let $v \in V$ as before. We can show $\mathcal{M}$ has 4-cvc(-1), and in fact there are multiple constructions of 4-planes that give curvature -1, for any v. One way is to construct L such that $f_1 = e_1,\ f_2 = e_2,\ f_3 = v_2$, and $f_4 = v_3$. As before, in the event that $b = 0$ or $c = 0$, we can set $f_2 = v_3$ or $f_3 = e_5$ as needed. Then $\mathcal{K}(L) = \lambda_1^2 + 2\lambda_1\lambda_2 = 1 - 2 = -1$. ☐

(4) $\mathcal{M}$ has 5-cvc(-1) and 5-cvc(-2).

Proof. Set $k = 5$. Let $v \in V$ as before. First, we can show that $\mathcal{M}$ has 5-cvc(-1) by constructing L such that $f_1 = v_1$ and $f_i = e_{i+1}$ for $i \in \{2, \ldots, 5\}$. If $a = 0$, set $f_1 = e_1$. Then

$$\begin{aligned}\mathcal{K}(L) &= 2\lambda_1\lambda_2 + 2\lambda_1\lambda_3 + \lambda_2^2 + 4\lambda_2\lambda_3 + \lambda_3^2\\ &= -2 + 0 + 1 + 0 + 0\\ &= -1.\end{aligned}$$

Next, show $\mathcal{M}$ has 5-cvc(-2). Construct L such that $f_i = e_i$ for $i \in \{1, \ldots, 4\}$ and $f_5 = v_3$. So $f_3 \in \ker(R)$ and

$$\begin{aligned}\mathcal{K}(L) &= \lambda_1^2 + 4\lambda_1\lambda_2 + \lambda_2^2 + 0\\ &= 1 - 4 + 1\\ &= -2.\end{aligned}$$

☐

5. k-Plane Constant Vector Curvature: Second Example

Example 5.1. Let $M = (V, \langle\ ,\ \rangle, R)$ be a model space such that $\dim(V) = n$, the inner product is positive definite, and $R = R_\phi$ where ϕ is represented by

$$\begin{bmatrix} I_2 & 0_2 & \ldots & 0_2\\ 0_2 & 0 & \ldots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0_2 & 0 & \ldots & 0\end{bmatrix}$$

where I_2 and 0_2 are expressed as in Example 4.3. Note that $\lambda_1 = 1$ where $\dim(E_1) = 2$ and $\lambda_2 = 0$ where $\dim(E_2) = n - 2$.

Proposition 5.1. *The model space $\mathcal{M}$ in Example 5.1 has the following properties:*

(1) *k-cvc(0) for $k \geq 2$,*
(2) *k-cvc(1) for $k \geq 3$,*
(3) *If $\mathcal{M}$ has 3-cvc(ε), then $\varepsilon \in [0, 1]$,*
(4) *For $k \geq 4$, if $\mathcal{M}$ has k-cvc(ε) then $\varepsilon \in [0, k-1)$,*

(5) *$\mathcal{M}$ has k-cvc$([0,1])$.*[2]

(1) $\mathcal{M}$ has k-cvc(0) for $k \geq 2$.

Proof. Since $\ker(R) \neq 0$, by Lemma 2.7, $\mathcal{M}$ has 2-cvc(0) and only 2-cvc(0). Set $k \geq 3$, and let $v \in V$ be decomposed as $v = av_1 + bv_2$, where $a, b \in \mathbb{R}$, and $v_1 \in E_1 = span\{e_1, e_2\}$, $v_2 \in E_2 = span\{e_3, \ldots, e_n\}$ be unit vectors. Construct a k-plane L_0 containing v, by setting $f_1 = v_1$, and $f_2 = v_2$. Since $\dim(E_2) = n - 2$, we can create an orthonormal basis for E_2 by finding vectors $w_1, \ldots, w_{n-3}$ perpendicular to v_2. Complete the orthonormal basis for L by setting $f_3 = w_1, \ldots, f_k = w_k$. If $b = 0$, then set $f_i = e_{i+1}$ for $i \in \{2, \ldots, k\}$. Then $f_2, \ldots, f_k \in \ker(R)$, so $\mathcal{K}(L) = \sum_{j>i=1}^{k} R_{ijji} = 0$. □

(2) $\mathcal{M}$ has k-cvc(1) for $k \geq 3$.

Proof. Set $k \geq 3$. Let $v \in V$ be expressed as before. Construct L_1 by setting $f_1 = e_1$, $f_2 = e_2$, $f_3 = v_2$, $f_4 = w_1, \ldots, f_k = w_{k-3}$ where $\{v_2, w_1, \ldots, w_{n-3}\}$ is an orthonormal basis for E_2 as before. If $b = 0$, set $f_i = e_i$ for $i \in \{3, \ldots, k\}$. So $f_3, \ldots, f_k \in \ker(R)$. Then R_{1221} is the only non-zero R_{ijji} term in $\mathcal{K}(L)$, so $\mathcal{K}(L) = 1$. □

(3) If $\mathcal{M}$ has 3-cvc(ε), then $\varepsilon \in [0, 1]$.

Proof. Suppose $\mathcal{M}$ has 3-cvc(ε) for some $\varepsilon \in \mathbb{R}$. Set $v = e_3$. Let $L = span\{f_1, f_2, f_3\}$ be a 3-plane containing v such that $\mathcal{K}(L) = \varepsilon$. Without loss of generality, since v is contained in L, we can construct an orthonormal basis for L such that $f_1 = v$. Let $f_2 = a_1 e_1 + \cdots + a_n e_n$ and $f_3 = b_1 e_1 + \cdots + b_n e_n$. Since $f_1 \in \ker(R)$, $\mathcal{K}(L) = R(f_2, f_3, f_3, f_2)$. By [7], the sectional curvature values are bounded by the products of eigenvalues, so for $\pi = span\{f_2, f_3\}$, $0 \leq \kappa(\pi) = R(f_2, f_3, f_3, f_2) \leq 1$. Hence $0 \leq \mathcal{K}(L) = \varepsilon \leq 1$. □

(4) For $k \geq 4$, if $\mathcal{M}$ has k-cvc(ε) then $\varepsilon \in [0, k-1)$.

Proof. Suppose $\mathcal{M}$ has k-cvc(ε) for some $\varepsilon \in \mathbb{R}$. Set $v = e_1$. Let L be a k-plane containing v such that $\mathcal{K}(L) = \varepsilon$. As before, without loss of generality, we can set $f_1 = v$, and extend to an orthonormal basis $\{f_1, \ldots, f_k\}$ for L. So each $f_i = a_{i2} e_2 + \cdots + a_{in} e_n$ for some $a_{ij} \in \mathbb{R}$ where $\sum_{j=2}^{n} a_{ji}^2 = 1$. Since $e_3, \ldots, e_n \in \ker(R)$,

$$\begin{aligned}
\varepsilon = \mathcal{K}(L) &= a_{22}^2 R_{1221} + \cdots + a_{k2}^2 R_{1221} \\
&= a_{22}^2 + \cdots + a_{k2}^2 \\
&< 1 + \cdots + 1 \\
&= k - 1.
\end{aligned}$$

The strict inequality holds since f_i are orthonormal, so if any of the $a_{i2}^2 = 1$ then $a_{j2} = 0$ for all $j \neq i$. We get a lower bound of 0, as $\mathcal{K}(L)$ is a sum of squares of real numbers and so cannot be negative. Hence $0 \leq \mathcal{K}(L) = \varepsilon < k - 1$. □

(5) Let $v \in V$. For all $\varepsilon \in [0, 1]$, there is a k-plane L containing v and some θ such that $\mathcal{K}(A_\theta L) = \varepsilon$. Hence $\mathcal{M}$ has k-cvc([0,1]).

[2] We do not claim that $\mathcal{M}$ does not have k-cvc(δ) for $\delta \notin [0, 1]$.

Proof. For any $v \in V$, we can rotate $(k-1)$-planes in $v^{\perp}$ to get a connected set of curvature values. We do so by creating a linear transformation represented by a matrix in $\mathrm{SO}(n)$, the group of orthogonal matrices with determinant 1. Let the linear transformation $A_\theta : [0, \frac{\pi}{2}] \to SO(n)$ be represented by

$$A_\theta = \begin{bmatrix} I_{k-1} & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & I_{n-k-1} \end{bmatrix}$$

where

$$R = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

This matrix rotates the k^{th} vector into the $(k+1)^{th}$, and the $(k+1)^{th}$ into the $-k^{th}$. We can construct two k-planes L_0, L_1 containing v such that $\mathcal{K}(L_0) = 0$ and $\mathcal{K}(L_1) = 1$, and use the above special orthogonal matrix to rotate between the two. Thus we can obtain a connected set of k-cvc values, that is to say, for every $\varepsilon \in [0,1]$, there is some k-plane containing v such that $\mathcal{K}(L) = \varepsilon$. For this example, we are considering $k \geq 3$, $n \geq 4$.

We again decompose V into E_1 and $E_2 = \ker(\phi) = \ker(R)$, the eigenspace associated with $\lambda_1 = 1$. Then decompose v as $v = av_1 + bv_2$, where $v_i \in E_i$ are unit vectors. Since E_1 is 2-dimensional, $span\{v_1, u\} = E_1$ for some unit vector $u \perp v_1$. Similarly, we can construct an orthonormal basis $\{v_2, w_1, \ldots, w_{n-3}\}$ that spans the $(n-2)$-dimensional E_2. Now, set $f_1 = v_1,\ f_2 = v_2,\ f_3 = w_1,\ \ldots,\ f_k = w_{k-2}, f_{k+1} = u$. Extend this to an orthonormal basis for all of V. Then for the k-plane $L = span\{f_1, \ldots, f_k\}$,

$$\begin{aligned}
\mathcal{K}(A_\theta L) &= \mathcal{K}(A_\theta f_1, \ldots, A_\theta f_k) \\
&= \mathcal{K}(v_1, v_2, w_1, \ldots, w_{k-3}, \cos\theta w_{k-2} + \sin\theta u) \\
&= R(v_1, \cos\theta w_{k-2} + \sin\theta u, \cos\theta w_{k-2} + \sin\theta u, v_1) && \text{(since } v_2, \ldots, w_{k-3} \in E_2\text{)} \\
&= R(v_1, \sin\theta u, \sin\theta u, v_1) && \text{(since } w_{k-2} \in E_2\text{)} \\
&= \sin^2\theta R(v_1, u, u, v_1) \\
&= \sin^2\theta
\end{aligned}$$

since $span\{v_1, u\} = E_1$ which has sectional curvature 1. If $a = 0$, then $v \in E_2$. So $E_1 = span\{e_1, e_2\}$. Set $f_1 = e_1$ and $f_{k+1} = e_2$, and keep all other f_i the same. Extend this to an orthonormal basis for V. Then, as before, $\mathcal{K}(A_\theta L) = R(e_1, \sin\theta e_2, \sin\theta e_2, e_1) = \sin^2\theta R_{1221} = \sin^2\theta$.

If $b = 0$, then $v \in E_1$. So $V_1 = span\{e_3, \ldots, e_n\}$. Now, set $f_2 = e_3, \ldots, f_k = e_{k+1}$ and keep all other f_i the same. Extend this to an orthonormal basis for V. Then, as before, $\mathcal{K}(A_\theta L) = R(v_1, \sin\theta u, \sin\theta u, v_1) = \sin^2\theta$.

Note $\mathcal{K}(A_0 L) = 0$ and $\mathcal{K}(A_{\frac{\pi}{2}} L) = 1$. Further, $L \mapsto \mathcal{K}(L)$ is continuous, and the Grassmannian $\mathrm{Gr}_{k,n}$ is connected, where $\mathrm{Gr}_{k,n}$ is the space that parametrizes all k-dimensional linear subspaces of the n-dimensional vector space. Hence by the Intermediate Value Theorem, for all $\varepsilon \in [0,1]$, there is some θ such that $\mathcal{K}(A_\theta L) = \varepsilon$, where L contains an arbitrary $v \in V$. Further, we explicitly determine this θ to be

$\arcsin(\sqrt{\varepsilon})$. Let $\varepsilon \in [0,1]$, and set $\theta = \arcsin(\sqrt{\varepsilon})$. Then

$$\begin{aligned}\mathcal{K}(A_\theta L) &= \sin^2\theta \\ &= \sin^2(\arcsin(\sqrt{\varepsilon})) \\ &= \sqrt{\varepsilon}^2 \\ &= \varepsilon.\end{aligned}$$

□

6. k-Plane Constant Vector Curvature: General Results

Based on the work done in these examples, we get results that apply more generally to model spaces with canonical curvature tensors. First, we show that we can construct model spaces with a compact interval of k-cvc values out of two other model spaces.

Theorem 6.1. *For any compact interval $[a,b]$ in $\mathbb{R}$, there exists $M = (V, \langle\ ,\ \rangle, R)$ such that $\mathcal{M}$ has k-cvc$([a,b])$ for $k \geq 3$.*

Proof. Let $[a,b] \in \mathbb{R}$ be compact. Let $M_1 = (V, \langle\ ,\ \rangle, R_1)$ have k-csc(a), and let $M_2 = (V, \langle\ ,\ \rangle, R_2)$ where $R_2 = (b-a)R_\phi$ for R_ϕ as in Example 5.1. So by Proposition 2.3 and Example 5.1 (a), M_2 has k-cvc$((b-a)[0,1])$, in other words, it has k-cvc$([0,b-a])$. Construct $M = (V, \langle\ ,\ \rangle, R)$ such that $R = R_1 + R_2$. Then by Proposition 2.2, $\mathcal{M}$ has k-cvc$(a + [0, b-a])$, that is to say, $\mathcal{M}$ has k-cvc$([a,b])$. □

Note that we say $\mathcal{M}$ has "at least" k-cvc$([a,b])$. While we can prove that for any $\varepsilon \in [a,b]$, there is some L containing arbitrary v such that $\mathcal{K}(L) = \varepsilon$, the proof does not establish that $\mathcal{M}$ is *not* k-cvc(δ) for some $\delta \notin [a,b]$. An important next step is developing a method to show that $\mathcal{M}$ has *only* k-cvc$([a,b])$. Additionally, this proof only works for $k \geq 3$ in a model space of arbitrary n dimension. Since $\ker(R_2) \neq 0$, by Lemma 2.6 M_2 has only 2-cvc(0), and not the interval $[0,1]$.

It seems the range of possible values for ε would be determined by the dimensions of the various eigenspaces. We can establish very loose bounds for ε based on extremal products of eigenvalues of ϕ.

Theorem 6.2. *Suppose a model space $\mathcal{M} = (V, \langle\ ,\ \rangle, R_\phi)$ has k-cvc(ε) for some $\varepsilon \in \mathbb{R}$. Then*

$$\binom{k}{2}\min\{\lambda_i\lambda_j | i \neq j\} \leq \varepsilon \leq \binom{k}{2}\max\{\lambda_i\lambda_j | i \neq j\}.$$

Proof. Let $\mathcal{M}$ has k-cvc(ε) for some $\varepsilon \in \mathbb{R}$. By [7], we know the sectional curvatures are bounded by products of eigenvalues of ϕ. Since calculating the k-plane scalar curvature amounts to summing over $\binom{k}{2}$ sectional curvatures, we know we could sum at least $\binom{k}{2}$ minimal sectional curvatures and at most $\binom{k}{2}$ maximal sectional curvatures. □

Clearly for model spaces with a canonical ACT there is some relationship between the eigenvalues of ϕ and the possible k-cvc values. We conjecture that the multiplicity of eigenvalues can determine whether a model space has k-cvc(ε) for some $\varepsilon \in \mathbb{R}$.

Conjecture 6.2.1. *For $\mathcal{M} = (V, \langle\ ,\ \rangle, R_\phi)$, if there are no more than k distinct eigenvalues, then $\mathcal{M}$ has k-cvc(ε) for some $\varepsilon \in \mathbb{R}$.*

In wrapping up our investigation of the k-cvc condition, we see that model spaces with this property are not as easily characterized as model spaces with the k-csc property. The variety of k-cvc values we can get in a given model space exemplifies how this condition can generate representative numbers that help to characterize model spaces.

7. Conclusions

In this paper, we seek to understand k-plane curvature, and specifically the conditions defined as k-plane constant sectional curvature and k-plane constant vector curvature. Working in Riemannian model spaces, we are able to generalize some results previously known from the 2-plane constant curvature conditions. By showing that every model space with k-csc(0) for some $k \leq n-2$ must have $R = 0$, we prove there is a unique ACT with k-csc(ε). This ACT coincides with the 2-csc tensor. Additionally, we show that a complete collection of curvature equations uniquely determines R, and a model space that has k-plane constant sectional curvature for some $k \leq n-2$ also has j-plane constant sectional curvature.

In order to study k-plane constant vector curvature, we give two examples. The first example demonstrates a method for determining possible k-plane constant vector curvature values. These values can be found in terms of products of the eigenvalues of the symmetric, bilinear form used to define our canonical tensor in these examples. In the second example, we bound ε based on the sectional curvatures of the model space. We rotate basis vectors to show that our model space has a connected set of k-cvc values. This result implies that for any compact interval of real numbers, there is a model space that has at least k-cvc for that interval. In a model space with a canonical tensor, we give loose bounds on possible k-cvc values and conjecture some sufficient conditions for the k-cvc condition. Further study of this constant curvature condition should investigate a method for determining when a model space does or does not have k-cvc(ε) for some $\varepsilon \in \mathbb{R}$.

8. Open Questions

(1) What is a method for determining possible k-cvc values, given your model space? See [10] for methods for determining possible 2-cvc values in 3-dimensional model spaces. Difficulties arise when k becomes small relative to n, as $v \in L$ could potentially have many non-zero components relative to an orthonormal basis. Perhaps it would be easiest to first consider canonical ACTs, and $(n-1)$-planes, so that the non-zero entries of v can be accounted for individually in the basis vectors of L. Or, is it possible to neatly construct an L that has v as its first basis vector? It seems a helpful tactic could be decomposing v into its components in the eigenspaces. So then if we could construct a basis for L such that each basis vector is in one eigenspace, it seems each R_{ijji} would be independent from the coefficients x_i's in v. This gives us $\mathcal{K}(L) = \varepsilon$ of the type we are looking for.

(2) Similarly, develop some method for showing that a model space does *not* have k-cvc(ε), for some $2 \leq k \leq n-1$ and $\varepsilon \in \mathbb{R}$. The examples given do not quite answer this question, as we instead try to find bounds on possible k-cvc values. Our approach relies on bounding sectional curvatures, and gives estimates that are clearly too loose. Is there some way to prove that, given some $e_i \notin \ker(R)$, there are some restrictions on how e_i could appear

in the f_i basis for L? That is to say, is there some bound on how much a basis vector can contribute to the curvature?

(3) If $\mathcal{M}$ has k-cvc(ε) and k-cvc(δ), does $\mathcal{M}$ have k-cvc($[\varepsilon, \delta]$)? It seems possible using the linear transformation A_θ as in Example 4.4 (perhaps multiple times for various f_i), so we could change the basis vectors of L to create such a set of connected values. The trick is doing so in such a way that we are guaranteed $v \in L$. It seems that decomposing v into components in the eigenspaces would again be a helpful approach.

(4) Does every model space have k-cvc(ε) for some ε and some k? If not, does it for a particular $k < n$?

(5) As shown in [6], it is possible to completely characterize R by 2-plane curvatures. It is possible to characterize R in terms of some k-plane curvature equations? For a particular k relative to n?

(6) It is known that the Ricci tensor completely determines R for 3-dimensional model spaces, however it is possible to have different higher dimensional curvature tensors with the same Ricci tensor. There is a method to determine an ACT given a Ricci tensor, but this answer is non-unique. It seems that, after answering (7), the $\mathcal{K}(L)$ equations could determine $\frac{\tau}{2}$ restricted to L, which could determine R. Is this true? Or is there some other kind of k-Ricci entity that could characterize R.

(7) Generalizing the property of extremal constant vector curvature as discussed in [8, 10], and : It seems that this condition could easily generalize for k-planes:

Definition 8.1. *A model space $\mathcal{M}$ has k-**plane extremal constant vector curvature** ε if $\mathcal{M}$ has k-cvc(ε) and ε is a bound on possible k-plane curvature values.*

Investigate this condition further. It is obvious that every model with k-csc(ε) would also have k-ecvc(ε). For some model spaces, would we get k-ecvc for only certain values of k?

(8) We say a model space is *Einstein* if $\rho = c\langle\ ,\ \rangle$ for some $c \in \mathbb{R}$. On page 34 of [1], Chen references a property he calls *k-Einsteinian*. We can roughly describe this property as follows:

Definition 8.2. *Let $\mathcal{M} = (V, \langle\ ,\ \rangle, R)$ with $V \subseteq \mathbb{R}^n$ and a positive definite inner product. Let V_l be an l-plane section of V, and let L be a k-plane section of V_l. Let $\rho_L = \rho|_L$. Let $v \in L$ be a unit vector, and extend v to an orthonormal basis for L, $\{v, e_2, \ldots, e_k\}$. We say V_L is k-Einsteinian if, for all k-planes L, $\sum_{j=2}^{k} \rho_L(e_j, e_j) = \varepsilon$ for some $\varepsilon \in \mathbb{R}$.*

A model space $\mathcal{M}$ is k-Einsteinian when $V_l = V$. Investigate this property further. It seems a model space is k-Einsteinian if and only if it has k-csc(ε). Which known results regarding Einsteinian model spaces generalize for k-Einsteinian model space? How could we generalize the condition known as *weakly Einsteinian* to some k-weakly Einstein condition?

(9) This research assumed a positive definite inner product. What happens when the inner product is non-degenerate?

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