# Generalized Cell Decompositions of Nested Lorenz Links

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#### Abstract

This paper focuses on generalizing the cell decompositions of various nested Lorenz links to determine types of hyperbolic and octahedral links. By utilizing a class of generalized fully augmented links called nested links, we are able to determine various types of nested Lorenz links that are hyperbolic based on their generalized cell decompositions. We are also able to determine subsets of these hyperbolic nested Lorenz links that are octahedral. With the information gathered from several generalized cell decompositions, we are able to determine whether or not a subset of nested Lorenz links are hyperbolic based on the position of two specific strands on the Lorenz template.

## **1** Preliminaries

In this paper, we use topology and hyperbolic geometry explore the geometries of Lorenz links. In topology, a knot is a closed curve in three dimensions and a link is a collection of multiple knots that can be interconnected. We specifically investigate Lorenz links, which are defined as all links on the Lorenz template [1]. We often refer to a link on the Lorenz template as a (P,Q) Lorenz link, where P is the number of vertices on the left upper half of the template and Q is the number of vertices on the right upper half. There are several properties the Lorenz template has that are essential to follow when constructing Lorenz links. The first property of the Lorenz template is that each overcrossing strand must have negative slope and each undercrossing strand must have positive slope. Furthermore, two overcrossing (resp. undercrossing) strands never intersect. The other main property is that for each overcrossing strand, the position of the endpoint will be bigger than the position of the start point [1]. We define the strands on the Lorenz template to be one of three types. A L.L. strand ("left-left") occurs when an upper P vertex maps to a lower P vertex. A L.R. strand ("left-right") occurs when an upper P vertex. Men forming the Lorenz link from the Lorenz template, each P vertex joins itself and each Q vertex joins itself, forming the left and right lobes respectively. An example of a link on the Lorenz template with lobes omitted is shown in figure 1.



Figure 1: A (5,3) Link on the Lorenz Template

A hyperbolic link is a link such that its link compliment forms a complete hyperbolic manifold. One effective way to study hyperbolic links is to augment the links. To create a fully augmented link, we encircle each twist region with an unknotted region called a crossing disk [6]. In a fully augmented link, each twist region contains only two strands and thus each crossing disk is always twice punctured. This is beneficial when finding the link compliment, since twice punctured crossing disks form triangles in the cell decomposition. Triangles whose vertices are ideal are totally geodesic and are all isometric to each other in hyperbolic space [5]. This is helpful when creating a hyperbolic manifold.

In a generalized fully augmented link, we allow crossing disks to encircle more than two strands in a given twist region[6]. This causes our crossing disks to potentially be punctured more than twice, which does not guarantee totally geodesic shapes in the cell decomposition. As a result, this causes difficulty when attempting to create a hyperbolic manifold.

Since Lorenz links frequently have more than two strands, we utilize a subclass of generalized fully augmented links called nested links. In a nested Lorenz link, we do not limit ourselves to encircling only twisted regions, but rather allow for any given region of the link to be encircled by crossing disks. We also allow for our crossing disks to encircle more than two strands. To ensure our crossing disks are still only twice punctured, we make the crossing disks coplanar such that the innermost crossing disk is twice punctured by two strands, while all other crossing disks are twice punctured once by a strand and once by another crossing disk. To simplify our figures, we only show the top half of the crossing disk.

This paper focuses on two types of nested Lorenz links: left nested Lorenz links and complete nested Lorenz links. Left nested Lorenz links have crossing disks that encircle the strands that form the left lobe of the Lorenz link. These left nested Lorenz links are denoted  $N(L_{(P,Q)})$ , where P is the number of left vertices and Q is the number of right vertices on the upper half of the Lorenz template. Complete nested Lorenz links have crossing disks encircled around the overcrossing and undercrossing strands found in the middle of the Lorenz link. These complete nested Lorenz Links are denoted  $C(L_{(P,Q)})$  where P is the number of left vertices and Q is the number of right vertices on the upper half of the Lorenz template. An example of a left nested Lorenz link and a complete nested Lorenz link are shown in figures 2 and 3 respectively.



Figure 2: A left nested Lorenz Link

When determining whether or not a nested link is hyperbolic, we utilize two important theorems. The Circle Packing Theorem states that any simple, connected, planar graph G is isomorphic to the nerve of a circle packing [7]. This circle packing gives us the polyhedra necessary for the creation of our hyperbolic manifold. We then invoke Theorem 3.2 to state that the manifold obtained by gluing these polyhedra according to an admissible gluing pattern is a complete hyperbolic manifold [3]. Therefore, if we show a cell decomposition's nerve is simple, connected, and planar, then the link corresponding to that cell decomposition is hyperbolic.

When studying hyperbolic links, Purcell suggests that we study octahedral links. According to Purcell, the geometry of fully augmented links (and nested links) is completely determined by the circle packing, which can be computed from the nerve [6]. It can be hard to obtain a circle packing from an extremely complicated nerve, and can also make geometric estimates, such as volume, harder to obtain exactly. To overcome this when investigating nested Lorenz links, we can restrict ourselves to studying the octahedral subclass of links. Octahedral links are links whose polyhedra are a union of regular ideal octahedra [6]. These octahedral links have simpler geometries than that of other links. We identify several types of octahedral



Figure 3: A Complete Nested Lorenz Link

nested Lorenz links to aid in the further geometric exploration of nested Lorenz links. To determine whether or not a link is octahedral, we use Proposition 3.8 [6]. This states that our link is octahedral if and only if the nerve is obtained by centrally subdividing  $K_4$ .

# 2 Left Nested Lorenz Links

As described in the preliminary section, we can left nest various Lorenz links and determine if the nested links are hyperbolic, octahedral, or both. To begin, we topologically manipulate the cell decompositions of these links to create more straightforward and beneficial cell decompositions that are isotopic to the originals. Figures 4 and 5 show an example of this process.

After shrinking the crossing disk arcs in our link, the resulting image is a curved spine with standard lettered triangles (representing the back of our crossing disks) on the top half of the spine and prime lettered triangles (representing the front of our crossing disks) on the bottom half. Each triangle's vertex either has a L.L. strand (colored green) or a L.R. strand (colored red). These strands are connected to another vertex determined by the Lorenz template. An example is shown in figure 4.

We proceed by topologically flattening out the spine of our cell decomposition as shown in figure 4. After flattening the spine, we pull the self-intersecting L.R. strands below the spine as shown in 5. We then rotate each triangle that is connected to a L.R. strand 180 degrees to make our cell decomposition planar. The resulting cell decomposition is shown in figure 5. We label the regions created by the L.L. strands  $X_j$ and the regions created by the L.R. strands  $Y_i$ . The region of the plane  $\hat{C}$  is denoted by W.

This cell decomposition is isotopic to the original, and has some noticeable properties that can be useful. Because of our Lorenz template requirement that there are no R.R. strands, there will always be P - Qregions of type  $X_j$  and Q regions of type  $Y_i$  for any  $N(L_{(P,Q)})$  link. Furthermore, due to left nesting our Lorenz link there will always be 2(P-1) triangles in our cell decomposition. This method of topologically manipulating the cell decomposition holds for any  $N(L_{(P,Q)})$  link that has no R.R. strands. We will utilize this method for the remainder of the left nested section.

For left nested Lorenz links, if Q = 1 then there is only one mapping for P + 1 that does not violate the requirements of the Lorenz template. This mapping is actually the  $\pi_*(i)$  permutation that is used to define torus knots [2]. The only reason this definition did not extend to torus links is due to a restriction that (P,Q) must be coprime. We do not use this restriction, so the definition extends to torus links.



Figure 4: Topologically Manipulating the Cell Decomposition (1)



(Moving the L.R. Strands)

 $\begin{array}{c|c} W & X_2 \\ \hline W & B & A & X_1 & B' \\ \hline D & C & Y_1 & A' & C' \\ \hline & Y_2 & \\ & Y_3 & \end{array}$ 

(Rotating the L.R. Trianlges)

Figure 5: Topologically Manipulating the Cell Decomposition (2)

**Lemma 2.1.**  $L_{(P,1)}$  is a (P,1) torus link.

### **Proof:**

If we map  $P + 1 \mapsto 1$ , then our (P,1) Lorenz link will follow the  $\pi_*(i)$  mapping and will be a Torus link by definition. If we map P + 1 to any vertex i (where  $2 \leq i \leq P$ ) then vertices 1 through i - 1 are forced to map to themselves as shown in figure 6. This violates the Lorenz template requirement that the endpoint of each overcrossing strand must always be bigger than that of the initial point [1]. Therefore  $L_{(P,1)}$  is a Torus link.



Figure 6: The  $L_{(P,1)}$  Template

For  $N(L_{(P,Q)})$  that have no R.R. strands, there are four cases of generalized cell decompositions. These cases are determined by the two rightmost vertices in the lower P-section of the Lorenz template. Furthermore, only one of these cases yields non-hyperbolic left nested Lorenz links. The three hyperbolic cases for  $N(L_{(P,Q)})$ are shown in figures 7 and 8. The non-hyperbolic case is shown in figure 9. To simplify the images, we omitted the lobes and crossing disks of each case.



Figure 7: Left Nested Lorenz Links Case 1 and 2

**Theorem 2.2** (The Lorenz Hyperbolicity Theorem). Let  $N(L_{(P,Q)})$  be a left nested Lorenz link satisfying the following: a) The Lorenz template contains no R.R. strands

b)  $P \neq Q$ 

c)  $Q \ge 2$ 

Then the two rightmost vertices in the lower P-section of the Lorenz template determine if  $N(L_{(P,Q)})$  is hyperbolic. More precisely,  $N(L_{(P,Q)})$  is not hyperbolic if and only if  $P + Q - 1 \mapsto P - 1$ ,  $P + Q \mapsto P$ .



Figure 8: Left Nested Case 3

### **Proof:**

To show that cases 1, 2, and 3 are hyperbolic, we will prove that their nerves are simple, connected, and planar. To show case 4 is not hyperbolic, we will prove that this type of left nested Lorenz link always bounds an annulus.

Case 4:  $P + Q - 1 \mapsto P - 1, P + Q \mapsto P$ 

Case 4  $N(L_{(P,Q)})$  links always result in a Lorenz template as shown in figure 9. These links always bound an annulus (denoted by the red strands) and are therefore not hyperbolic [7].



Figure 9: Case 4 Bounded Annulus

Case 1:  $P - Q \mapsto P - 1$ ,  $P + Q \mapsto P$ 

Case 1  $N(L_{(P,Q)})$  links always result in a cell decomposition shown in figure 10.

Strands 1, 2, and 3 in figure 10 represent a fixed connection that exists in all case 1 links regardless of triangle arrangement (with respect to the case 1 limitations). Strand 1 results from the P+Q  $\mapsto$ P mapping. Similarly, strand 3 results from the P-Q  $\mapsto$ P-1 mapping. Strand 2 results from the requirement of Lorenz links that on each overcrossing strand the position of the endpoint must always be bigger than that of the



Figure 10: Case 1  $N(L_{(P,Q)})$  General Cell Decomposition

initial point [1]. Since we do not allow R.R. strands in our Lorenz template, the P+1  $\mapsto$ 1 mapping will always occur; resulting in strand 2.

In the arbitrary region (denoted by '?' in figure 10) there can be any arrangement of vertex up and vertex down triangles (with respect to the case 1 limitations).



Figure 11: Case 1  $N(L_{(P,Q)})$  Adjacencies

**Simple:** We will use figure 11 to show that the case  $1 N(L_{(P,Q)})$  nerve will be simple. In figure 11, the  $Y_1$  region is adjacent to all  $X_j$  regions as well as the  $Y_2$ , and W regions denoted by the red lines.  $Y_1$  clearly does not have multiple adjacencies with a single region as shown.

Each  $Y_i$  region (where  $2 \le i \le Q - 1$ ) is adjacent to the  $Y_{i+1}$ ,  $Y_{i-1}$ , W, and some number of  $X_1$  through  $X_j$ (where  $j \le P - Q$ ) regions based on the formation of the arbitrary region. These adjacencies are denoted by the orange lines. Recall that  $Y_2$ 's adjacency to the  $Y_1$  region is denoted by the red line, and any  $Y_i$  region adjacent to the  $X_1$  region is denoted by the green line. Each  $Y_i$  does not have multiple adjacencies to the  $Y_{i+1}$ ,  $Y_{i-1}$ , and W regions as shown.

The  $Y_Q$  region is adjacent to the  $Y_{Q-1}$  region denoted by the orange line.  $Y_Q$  is also adjacent to the W region, and some number of the  $X_1$  through  $X_j$   $(j \leq P - Q)$  regions based on the formation of the arbitrary region. These adjacencies are denoted by the yellow lines.  $Y_Q$  does not have multiple adjacencies to  $Y_{Q-1}$  and W as shown.

The  $X_1$  region is adjacent to the  $Y_1$  region denoted by the red line.  $X_1$  is also adjacent to the  $X_2$  region, and some number of  $Y_2$  through  $Y_k$  (where  $k \leq Q$ ) based on the formation of the arbitrary region. These adjacencies are denoted by the green lines.  $X_1$  does not have multiple adjacencies with  $X_2$  or  $Y_1$ .

Each  $X_r$  region  $(2 \le r \le P - Q - 1)$  is adjacent to the  $Y_1$  region denoted by the red line, and some number of the  $Y_2$  through  $Y_k$   $(k \le Q)$  regions denoted by the orange and yellow lines. The adjacency between  $X_2$ and  $X_1$  is denoted by the green line. Each  $X_r$  region is adjacent to the  $X_{r+1}$  and  $X_{r-1}$  regions denoted by the blue line. As shown, each  $X_r$  does not have multiple adjacencies with the  $Y_1, X_{r+1}$ , and  $X_{r-1}$  regions.

The  $X_{P-Q}$  region is adjacent to the  $Y_1$  region denoted by the red line, and the  $X_{P-Q-1}$  region denoted by the blue line.  $X_{P-Q}$  is also adjacent to the W region denoted by the pink line, and some number of  $Y_2$ through  $Y_k$  ( $k \leq Q$ ) regions based on the formation of the arbitrary region denoted by the orange and yellow lines.  $X_{P-Q}$  does not have multiple adjacencies with the  $Y_1$ ,  $X_{P-Q-1}$ , and W regions as shown.

W does not have multiple adjacencies to a single region as shown.

It remains to be shown that there are no multiple adjacencies between regions bordering the arbitrary region. Given a vertex up triangle, the strand connecting to its vertex creates two  $X_j$  regions. Thus the corresponding  $Y_i$  region is adjacent to both  $X_j$  regions exactly once through the ideal points located at the other two vertices of the vertex up triangle. Similarly, a vertex down triangle creates two  $Y_i$  regions such that the corresponding  $X_j$  region is adjacent to each  $Y_i$  region exactly once through the ideal points located at the other two vertices of the vertex down triangle. Therefore, if there are multiple adjacencies between regions bordering the arbitrary region, there must be a triangle that does not have a strand connected to its vertex. By construction, all of the triangle vertices are either ideal points or connected to a strand. Therefore, there are no multiple adjacencies between regions bordering the arbitrary region.

We have shown that there are no multiple adjacencies between two regions in our generalized cell decomposition for case 1. Therefore, the nerve for a case 1  $N(L_{P,Q})$  link will always be simple.

**Connected:** To prove the nerve is connected, we must find a Hamiltonian cycle in our case 1  $N(L_{(P,Q)})$  general cell decomposition. A Hamiltonian cycle for case 1 is shown in figure 12. Therefore, the nerve for a case 1  $N(L_{(P,Q)})$  link will always be connected.

**Planar:** Since our general cell decomposition lies on a plane, the nerve for a case 1  $N(L_{(P,Q)})$  link will always be planar.

We have shown that our case 1 nerve is always simple, connected, and planar. Therefore, by the Circle Packing Theorem and Theorem 3.2, case 1  $N(L_{(P,Q)})$  links are hyperbolic [7][3].

Case 2:  $P - Q - 1 \mapsto P - 1, P - Q \mapsto P$ 

There are some important facts for case 2  $N(L_{(P,Q)})$  links. Since we need at least two L.L. strands, case 2



Figure 12: Case 1  $N(L_{(P,Q)})$  Hamiltonian Cycle

requires  $P - Q \ge 2$ . Furthermore, if P - Q = 2, due to having no R.R. strands the  $N(L_{(P,Q)})$  link is forced to follow the  $\pi_*(i)$  mapping and is octahedral [4]. Thus, we restrict case 2 to  $N(L_{(P,Q)})$  such that P - Q > 2.

Case 2  $N(L_{(P,Q)})$  links always result in a cell decomposition shown in figure 13. Similar to those in figure 10, the numbered strands in figure 13 represent a fixed connection that exists in all case 2 links regardless of triangle arrangement (with respect to the case 2 limitations). Strands 1,3, and 4 result from the case 2 P-Q-1  $\mapsto$ P-1, P-Q  $\mapsto$ P mapping. Strand 2 results from the same Lorenz template requirement as in figure 10. The arbitrary region for case 2 is denoted by '?' in figure 13.

**Simple:** Figure 13 is very similar to figure 10 from case 1. Comparing the general cell decomposition of case 2 to that of case 1, we note that changing the connections of fixed strands 1, 3, and 4 does not introduce any multiple adjacencies between the affected regions. Furthermore, the reasoning for no multiple adjacencies between regions bordering the arbitrary region used in case 1 holds for case 2. Therefore, using the same method that was used in case 1, we conclude that the case 2  $N(L_{(P,Q)})$  nerve is simple.

**Connected:** Figure 14 shows a Hamiltonian cycle in the case 2 general cell decomposition. Therefore, the case 2  $N(L_{(P,Q)})$  nerve is connected.

**Planar:** The case 2 general cell decomposition lies on a plane, thus the case 2  $N(L_{(P,Q)})$  nerve is planar.

We have shown that our case 2 nerve is always simple, connected, and planar. Therefore, by the Circle Packing Theorem and Theorem 3.2, case 2  $N(L_{(P,O)})$  links are hyperbolic [7][3].



Figure 13: Case 2  $N(L_{(P,Q)})$  General Cell Decomposition



Figure 14: Case 2  $N(L_{(P,Q)})$  Hamiltonian Cycle

Case 3:  $P - Q \mapsto P, P + Q \mapsto P - 1$ 

If P - Q = 1 then  $N(L_{(P,Q)})$  follows the  $\pi_*(i)$  mapping and is octahedral [4]. Therefore, we restrict case 3 to  $N(L_{(P,Q)})$  such that P - Q > 1.

Case 3  $N(L_{(P,Q)})$  links always result in a cell decomposition shown in figure 15. Similar to those in figure 10, the numbered strands in figure 15 represent a fixed connection that exists in all case 3 links regardless of triangle arrangement (with respect to the case 3 limitations). Strands 1 and 3 result from the case 3 mapping. Strand 2 results from the same Lorenz template requirement as in figure 10. The arbitrary region for case 3 is denoted by '?' in figure 15.

**Simple:** Figure 15 is very similar to figure 10 from case 1. Comparing the general cell decomposition of case 3 to that of case 1, we note that the only difference is the position of the rightmost triangle's vertex.



Figure 15: Case 3  $N(L_{(P,Q)})$  General Cell Decomposition

In case 3, the rightmost triangle's vertex is pointing down instead of up. This change does not introduce any multiple adjacencies between affected regions. Furthermore, the reasoning for no multiple adjacencies between regions bordering the arbitrary region used in case 1 holds for case 3. Therefore, using the same method that was used in case 1, we conclude that the case 3  $N(L_{(P,Q)})$  nerve is simple.

**Connected**: Figure 16 shows a Hamiltonian cycle in the case 3 general cell decomposition. Therefore, the case 3  $N(L_{(P,Q)})$  nerve is connected.

**Planar:** The case 3 general cell decomposition lies on a plane, thus the case 3  $N(L_{(P,Q)})$  nerve is planar.

We have shown that our case 3 nerve is always simple, connected, and planar. Therefore, by the Circle Packing Theorem and Theorem 3.2, case 3  $N(L_{(P,Q)})$  links are hyperbolic [7][3]. Therefore,  $N(L_{(P,Q)})$  is not hyperbolic if and only if  $P + Q - 1 \mapsto P - 1$ ,  $P + Q \mapsto P$ .

After identifying a set of  $N(L_{(P,Q)})$  that are hyperbolic, we proceed by identifying a subset of these links that are octahedral. As mentioned in the preliminary section, octahedral links have more simpler geometries that allow for calculations such as volume estimation to be done more easily and accurately.

**Corollary 2.3.** Let  $N(L_{(P,2)})$  be a left nested Lorenz link satisfying the following: a) The Lorenz template contains no R.R. strands. b)  $P \neq Q$ If  $N(L_{(P,2)})$  is hyperbolic, then  $N(L_{(P,2)})$  is octahedral.

### Proof:

Case 1:  $P - 2 \mapsto P - 1, P + 2 \mapsto P$ 

From the case 1 general cell decomposition found in figure 10, we obtain the general cell decomposition for case 1  $N(L_{(P,2)})$  links shown in figure 17. The nerve of this cell decomposition is shown in figure 18. If we delete the vertex  $X_1$ ,  $X_2$  will then have degree 3. By repeating this pattern and consecutively deleting vertices  $X_2$  through  $X_{P-3}$ , the remaining vertices form  $K_4$ . Thus, the case 1  $N(L_{(P,2)})$  nerve is obtained by centrally subdividing  $K_4$ . Therefore, case 1  $N(L_{(P,2)})$  links are octahedral [6].



Figure 16: Case 3 ${\cal N}(L_{(P,Q)})$ Hamiltonian Cycle



Figure 17: Case 1 $N({\cal L}_{(P,2)})$ General Cell Decomposition



Figure 18: Case 1 ${\cal N}(L_{(P,2)})$ Nerve

Case 2:  $P - 3 \mapsto P - 1, P - 2 \mapsto P$ 

From the case 2 general cell decomposition found in figure 13, we obtain the general cell decomposition for case 2  $N(L_{(P,2)})$  links shown in figure 19. The nerve of this cell decomposition is shown in figure 20. By consecutively deleting vertices  $X_1$  through  $X_{n-1}$  and  $X_{P-2}$  through  $X_{n+1}$ , the remaining vertices form  $K_4$ . Thus, the case 2  $N(L_{(P,2)})$  nerve is obtained by centrally subdividing  $K_4$ . Therefore, case 2  $N(L_{(P,2)})$  left nested Lorenz links are octahedral [6].



Figure 19: Case 2  $N(L_{(P,2)})$  General Cell Decomposition

#### Case 3: $P - 2 \mapsto P, P + 2 \mapsto P - 1$

From the case 3 general cell decomposition found in figure 15, we obtain the general cell decomposition for case 3  $N(L_{(P,2)})$  links shown in figure 21. The nerve of this cell decomposition is shown in figure 22. By consecutively deleting vertices  $X_1$  through  $X_{P-3}$ , the remaining vertices form  $K_4$ . Thus, the case 3  $N(L_{(P,2)})$  nerve is obtained by centrally subdividing  $K_4$ . Therefore, case 3  $N(L_{(P,2)})$  links are octahedral [6].

Within the case 1  $N(L_{(P,Q)})$  links, there exists a subset of hyperbolic links that are never octahedral. If a case 1 link satisfies the following:

a) Q > 2

b)  $P - Q \ge 2$ 

c)  $P + i \mapsto i$  (where  $i \leq P + Q - 1$ )

then the link has a general cell decomposition shown in figure 23. This cell decomposition is very similar to the (P,Q) left nested torus links mentioned in [4]. The only difference is that the  $Y_Q$  region is adjacent to every  $X_j$  region (where  $j \leq P - Q$ ) as opposed to only being adjacent to  $X_1$ . This slight change in the mapping of strands on the Lorenz template generates a subset of case 1  $N(L_{(P,Q)})$  links that are never octahedral.

**Corollary 2.4.** If a case 1  $N(L_{(P,Q)})$  link satisfies the following: a) Q > 2b)  $P - Q \ge 2$ c)  $P + i \mapsto i$  (where  $i \le P + Q - 1$ ) then  $N(L_{(P,Q)})$  is not octahedral.

### **Proof:**

From the general cell decomposition found in figure 23, we obtain the nerve for the subset of case 1  $N(L_{(P,Q)})$ 



Figure 21: Case 3  $N(L_{(P,2)})$  General Cell Decomposition

hyperbolic links as shown in figure 24. The vertices of our nerve have the following adjacencies:  $Y_2$  through  $Y_{Q-1}$  are adjacent to W,  $X_1$ ,  $Y_{i+1}$ , and  $Y_{i-1}$  (where  $2 \le i \le Q-1$ ).  $X_2$  through  $X_{P-Q}$  are adjacent to  $Y_1$ ,  $Y_Q$ ,  $X_{j-1}$ , and  $X_{j+1}$  (or W if j = P - Q) (where  $2 \le j \le P - Q$ ).  $X_1$  is adjacent to  $Y_1$  through  $Y_Q$ , and  $X_2$ .  $Y_1$  is adjacent to  $X_1$  through  $X_{P-Q}$ , W, and  $Y_2$ .  $Y_Q$  is adjacent to  $X_1$  through  $X_{P-Q}$ , W, and  $Y_{Q-1}$ . W is adjacent to  $Y_1$  through  $Y_Q$  and  $X_{P-Q}$ .



Figure 22: Case 3  $N(L_{(P,2)})$  Nerve



Figure 23: Case 1 $N({\cal L}_{(P,Q)})$  Subset General Cell Decomposition

Since Q > 2 and  $P - Q \ge 2$ , the minimum degree possible for our nerve is 4. Thus, the nerve is not obtained by centrally subdividing  $K_4$ . Therefore,  $N(L_{(P,Q)})$  is not octahedral [6].

Although this proves that there is a fairly large subset of case 1 links that are not octahedral, there is also a subset of links that have the same cell decomposition as shown in figure 23 that are octahedral.

**Corollary 2.5.** Case 1  $N(L_{(P,Q)})$  links such that P - Q = 1 (where  $Q \ge 2$ ) are octahedral.

#### **Proof:**

On the Lorenz template for this link, the rightmost Q vertices in the upper P-section map to all of the vertices in the lower Q section. Since P - Q = 1, only the leftmost vertex in the upper P-section remains to be mapped. Furthermore, since  $N(L_{(P,Q)})$  is a case 1 link,  $1 \mapsto P - 1$  and  $P + Q \mapsto P$ . As a result, the



Figure 24: Case 1  $N(L_{(P,Q)})$  Subset Nerve

remaining vertices in the upper Q-section follow the  $\pi_*(i)$  mapping. The cell decomposition for this type of link is similar to that of figure 23, the only difference being that there is only one X region in the cell decomposition when P - Q = 1. The nerve for this cell decomposition is shown in figure 25. Since  $Y_Q$  has degree 3 and  $Y_{Q-1}$  through  $Y_3$  have degree 4, we can consecutively delete these vertices. The resulting graph is  $K_4$ . Therefore, our nerve is obtained by centrally subdividing  $K_4$  and  $N(L_{(P,Q)})$  is octahedral [6].



Figure 25: General Nerve for Case 1  $N(L_{(P,Q)})$  when P - Q = 1

# 3 Complete Nested Lorenz Links

As we have done with left nested Lorenz links, we can also topologically manipulate the cell decomposition of a complete nested Lorenz link. In doing so, we create an isotopic cell decomposition that is more straightforward and beneficial. We begin with a complete nested cell decomposition as shown in figure 26. Proceeding topologically, we straighten out the spine into a horizontal line. Then, we rotate all the triangles that are connected to the left lobe strands 180 degrees. We label the regions created by the left lobe strands  $Y_i$  and the regions created by the right lobe strands  $X_j$ . The region of the plane  $\hat{C}$  is denoted by W. The resulting cell decomposition is shown in figure 27. This cell decomposition is isotopic to the original and has some noticeable properties that can be useful. Because of the location of our nesting, there will always be a P number of  $Y_i$  regions and a Q number of  $X_j$  regions. Furthermore, there are always 2(P+Q-1) triangles in our cell decomposition. This method for topologically manipulating the cell decomposition holds for any  $C(L_{(P,Q)})$  link and will be used for the remainder of the section.



Figure 26: A  $C(L_{(P,Q)})$  Link with Crossing Disk Triangles



Figure 27: Flattened Spine and Rotating Left Lobe Triangles

**Theorem 3.1.** All  $C(L_{(P,Q)})$  are hyperbolic.

#### **Proof:**

There are two cases for the general cell decomposition of complete nested Lorenz links. For both cases of complete nested cell decompositions, by the Lorenz template restrictions  $P \mapsto P + Q$  always occurs [1]. Because of this fact and the positioning of our crossing disks for complete nesting, the strand puncturing

crossing disk A will always be the outermost strand of the left lobe. Similarly, the strand puncturing crossing disk A' will always be the innermost strand of the right lobe. As a result, triangle A will always be vertex down and triangle A' will always be vertex up. Furthermore, by the Lorenz template restrictions  $P + 1 \rightarrow 1$  will always occur [1]. Thus, the left strand puncturing the innermost crossing disk will always be the outermost strand on the right lobe. As a result, the leftmost vertex in the cell decomposition will always have a strand connecting to the rightmost vertex up triangle. Since  $P + 1 \rightarrow 1$  always occurs, the two cases for complete nesting cell decompositions result from the strand that connects to 2 in the lower P-section of the Lorenz template. Either  $P + 2 \rightarrow 2$ , or  $1 \rightarrow 2$ .

Case 1:  $P + 2 \mapsto 2$ 

The case 1 general cell decomposition is shown in figure 28. Like the Lorenz Mapping Theorem, we must prove that the case 1 nerve is simple, connected, and planar.

**Simple:** Each region's adjacency information for case 1 is shown in figure 29. It is clear that the regions unaffected by the arbitrary region do not have multiple adjacencies with another region. It remains to be proven that the regions bordering the arbitrary region do not have multiple adjacencies. If there is a region bordering the arbitrary region that has multiple adjacencies, there must be at least one triangle that does not have a strand connected to its vertex. By construction, all of the triangle vertices are either ideal points or connected to a strand. Thus, there are no multiple adjacencies between regions bordering the arbitrary region. Therefore, our case 1  $C(L_{(P,Q)})$  nerve is simple.



Figure 28: Case 1  $C(L_{(P,Q)})$  General Cell Decomposition

**Connected:** A Hamiltonian cycle for the case 1 general cell decomposition is shown in figure 30. Therefore, our case 1  $C(L_{(P,Q)})$  nerve is connected.

**Planar:** Since our cell decomposition lies on a plane, our case 1  $C(L_{(P,Q)})$  nerve is planar. Therefore, by the Circle Packing Theorem and Theorem 3.2, case 1  $C(L_{(P,Q)})$  links are hyperbolic [7][3].

Case 2:  $1 \mapsto 2$ 

The case 2  $C(L_{(P,Q)})$  general cell decomposition is shown in figure 31.

**Simple:** Each region's adjacency information for case 2 is shown in figure 32. Proceeding in the same manner as in case 1, it is clear that there are no multiple adjacencies in our cell decomposition. Therefore,



Figure 29: Case 1  $C(L_{(P,Q)})$  Adjacent Regions



Figure 30: Case 1  $C(L_{(P,Q)})$  Hamiltonian Cycle

our case 2  $C(L_{(P,Q)})$  nerve is simple.

**Connected:** A Hamiltonian Cycle for the case 2 general cell decomposition is shown in figure 33. Therefore, our case 2  $C(L_{(P,Q)})$  nerve is connected.

**Planar:** Since our cell decomposition lies on a plane, our case 2  $C(L_{(P,Q)})$  nerve is planar. Therefore, by the Circle Packing Theorem and Theorem 3.2, case 2  $C(L_{(P,Q)})$  links are hyperbolic [7][3].

Corollary 3.2. All (P,Q) complete nested torus links are octahedral.

#### **Proof:**

A general (P,Q) Torus link is shown in figure 34. After complete nesting our torus link, we proceed to topologically manipulate the cell decomposition. The result is a general cell decomposition for complete nested (P,Q) torus links as shown in figure 35. The nerve of this general cell decomposition is shown in 36. Consecutively deleting  $X_Q$  through  $X_2$  and  $Y_P$  through  $Y_3$  results in the graph  $K_4$ . Thus, our nerve is obtained by centrally subdividing  $K_4$ . Therefore, all (P,Q) complete nested torus links are octahedral.



Figure 31: Case 2 ${\cal C}({\cal L}_{(P,Q)})$ General Cell Decomposition



Figure 32: Case 2 ${\cal C}({\cal L}_{(P,Q)})$  Adjacent Regions



Figure 33: Case 2 Hamiltonian Cycle



Figure 34: General Torus Link on the Lorenz Template



Figure 35: General Cell Decomposition of (P,Q) Torus Links



Figure 36: Complete Nested Torus Link Nerve

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