# Covariant Derivative Lengths in Curvature Homogeneous Manifolds

Gabrielle Konyndyk

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**Abstract**

This research finds new families of pseudo-Riemannian manifolds that are curvature homogeneous and not locally homogeneous. The examples demonstrated look specifically at the mapping of the Levi-Civita connection of the coordinate vector fields. The coordinate vector fields in the examples covariantly differentiate to coordinate vector fields of a higher level. This is done in a way that creates path lengths larger than any previously studied curvature homogenous manifolds. We exhibit a four-dimensional manifold with a path length of four and a five-dimensional generalized plane wave manifold with a path length of four in the covariant derivative of the coordinate vector fields.

## Introduction

The main purpose of this research is to explore new examples of curvature homogeneous manifolds satisfying certain structural requirements. This paper will define two families of curvature homogeneous manifolds with covariant derivative lengths of four on the coordinate vector fields. More specifically, it will look at the sequential structure to the Levi-Civita connection of these manifolds and how it takes certain coordinate vector fields and covariantly differentiates them to other groups of coordinate vector fields. The inspiration for the manifolds of this structure arose from previous examples of generalized plane wave manifolds. In observation of the generalized plane wave manifolds, it became clear that manifolds of this type have limited lengths in the mappings of the covariant derivative of the coordinate vector fields.

While we will be more spacific later, generalized plane wave manifolds with balanced signatures of $(p, p)$ $(p \geq 2)$ demonstrated path lengths of only two in the covariant derivative of the coordinate vector fields. They are constructed in a way that takes one group of vector fields and differentiates them to the other group of vector fields.

Generalized plane wave manifolds with unbalanced signatures of $(2s, s)$ and $(k+1+a, k+1+b)$, such that $a+b=k$, had path lengths of three: these manifolds have three groups of coordinate vector fields covariantly differentiating to each other. The construction takes the first group of vector fields, differentiates them to the second group of vector fields, and then the second group differentiates to the third group of vector fields.

Within the examples constructed in this paper, there were set objectives that I worked to obtain. The main goal was for the manifolds to have a path of length four in the covariant derivatives, meaning that there needed to be four groups of vector fields differentiating to each other. Other aspects I aimed to include were for the manifolds to be generalized plane wave curvature homogeneous manifolds, not locally homogeneous, and to be of smaller dimensions relative to the intended lengths. The last characteristic we attempted to include is an unbalancing of the signature. We produce a family of four-dimensional manifolds that have a path length of four, are curvature homogeneous, and not locally homogeneous. We also produce a family of five-dimensional manifolds that have a path length of four, are curvature homogeneous, and not locally homogeneous, has an unbalanced signature, and is a generalized plane wave manifold.

# Preliminaries

**Definition 0.1.** A **Euclidean** $n$**-space**, $\mathbb{R}^n$, is the set of all n-tuples $p = (p_1, \ldots, p_n)$ of real numbers where $p$ is a point.

A **manifold**, $\mathcal{M}$, is Hausdorff topological space that is covered by coordinate systems and locally resembles a Euclidean space. Knowing specific information about the manifold allows us to classify surfaces and curvature according to the local properties [5].

Manifolds carry the structure of a metric space. A **metric**, $g$, allows us to work to grasp what a manifold looks like in a specified dimension. It is a way of measuring lengths of tangent vectors at any given point on the manifold. More simply, it is an inner product on each tangent space of $\mathcal{M}$ and is described as [2]:

$$g_{ij} = g(\partial_{x_i}, \partial_{x_j}).$$

As an example, note that the **inner product** of $\mathbb{R}^n$ is just the dot product $X \cdot Y = \sum x_i y_j$ with $|X| = \sqrt{X \cdot X}$ [5].

Having information on both the metric and the manifold makes it possible to be able to examine certain properties within the manifold. Specifically, this paper will use the information of the manifold and metric to examine the coordinate vector fields and their specific paths. They will also allow us to look at the curvature tensor $R$ to determine curvature homogeneity and local homogeneity. It is useful to look at certain properties at a specific point on the manifold. To do this, let $\mathcal{M}$ be a manifold and let $p \in \mathcal{M}$. We can then describe $g_p$ as the metric at $p$, $R_p$ as the curvature tensor at $p$, and $T_pM$ as the tangent space of $\mathcal{M}$ at $p$.

**Theorem 0.1.** *If $(x_1, \ldots, x_n)$ is a coordinate system in $\mathcal{M}$, then $\{\partial_{x_1}, \ldots, \partial_{x_n}\}$ are local coordinate vector fields and form a basis for $T_pM$ [5] .*

Define $\partial_{x_i}$ as **coordinate vector fields**, where $\partial_{x_i}(f) = \frac{\partial f}{\partial_{x_i}}$ and each $\partial_{x_i}$ sends functions on $\mathcal{M}$ to functions on $\mathcal{M}$.

Using a certain object, $\nabla$, called the Levi-Civita connection, one can compute the covariant derivatives of the vector fields. This will demonstrate the path length while also allowing for future calculation of the curvature tensor.

**Definition 0.2.** On a pseudo-Riemannian manifold $\mathcal{M}$ there is a unique connection $\nabla$ such that

$[X, Y] = \nabla_X Y - \nabla_Y X$ and,
$X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$.

For all $X, Y, Z \in \mathcal{M}$, $\nabla$ is called the **Levi-Civita connection** of $\mathcal{M}$ and is charaterized by the **Koszul formula**

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) + g(Y, [Z, X]) + g(Z, [X, Y]).$$

Using the Koszul formula, we can compute the Christoffel symbols of the connection.

**Definition 0.3.** Let $(x_1, \ldots, x_n)$ be a coordinate system on $\mathcal{M}$, then the **Christoffel symbols of the first kind** are as follows:

$$\nabla_{\partial x_i} \partial_{x_j} = \sum_{k=1}^{n} \Gamma_{ij}^{k} \partial_{x_k}$$

**Definition 0.4.** The **Christoffel symobls of the second kind** are as follows:

$$\Gamma_{ijk} = g(\nabla_{x_i}\partial_{x_j}, \partial_{x_k}) = \frac{1}{2}(g_{jk/i} + g_{ik/j} - g_{ij/k})$$
$$\text{where } g_{ij/_k} = \frac{\partial}{\partial_{x_k}}(g_{ij}).$$

Using the Levi-Civita connection and Christoffel symbols, one can compute the **Riemannian curvature tensor** $R$ on the coordinate vector fields as follows [2]:

$$R(\partial_{x_i}, \partial_{x_j}, \partial_{x_k}, \partial_{x_l}) = g(\nabla_{\partial_{x_i}}\nabla_{\partial_{x_j}}\partial_{x_k} - \nabla_{\partial_{x_j}}\nabla_{\partial_{x_i}}\partial_{x_k}, \partial_{x_l}).$$

With the above information on the curvature tensor $R$ , we can then also calculate the covariant derivative of the curvature tensor $\nabla R$ :

$$\nabla R(\partial_{xi}, \partial_{xj}, \partial_{xk}, \partial_{xl}; \partial_{xm}) = \nabla_{\partial_{x_m}} R(\partial_{xi}, \partial_{xj}, \partial_{xk}, \partial_{xl}) - R(\nabla_{\partial_{x_m}}\partial_{xi}, \partial_{xj}, \partial_{xk}, \partial_{xl}) - R(\partial_{xi}, \nabla_{\partial_{x_m}}\partial_{xj}, \partial_{xk}, \partial_{xl})$$
$$-R(\partial_{xi}, \partial_{xj}, \nabla_{\partial_{x_m}}\partial_{xk}, \partial_{xl}) - R(\partial_{xi}, \partial_{xj}, \partial_{xk}, \nabla_{\partial_{x_m}}\partial_{xl}).$$

It then follows that the second covariant derivative of the curvature tensor $\nabla^2 R$ is as follows:
$$\nabla^2 R(\partial_{xi}, \partial_{xj}, \partial_{xk}, \partial_{xl}; \partial_{xm}, \partial_{xn}) = \nabla_{\partial_{x_n}} R(\partial_{xi}, \partial_{xj}, \partial_{xk}, \partial_{xl}; \partial_{xm}) - R(\nabla_{\partial_{x_n}}\partial_{xi}, \partial_{xj}, \partial_{xk}, \partial_{xl}; \partial_{xm})$$
$$-R(\partial_{xi}, \nabla_{\partial_{x_n}}\partial_{xj}, \partial_{xk}, \partial_{xl}; \partial_{xm}) - R(\partial_{xi}, \partial_{xj}, \nabla_{\partial_{x_n}}\partial_{xk}, \partial_{xl}; \partial_{xm}) - R(\partial_{xi}, \partial_{xj}, \partial_{xk}, \nabla_{\partial_{x_n}}\partial_{xl}; \partial_{xm})$$
$$-R(\partial_{xi}, \partial_{xj}, \partial_{xk}, \partial_{xl}; \nabla_{\partial_{x_n}}\partial_{xm}).$$

**Definition 0.5.** An **algebraic curvature tensor** (ACT) $R$ over a vector space $V$ with dimension $n$ is a function $R : V \times V \times V \times V \to R$ satisfying [3] :

$R(x, y, z, w) = -R(y, x, z, w) = R(z, w, x, y)$, and
$R(x, y, z, w) + R(z, x, y, w) + R(y, z, x, w) = 0.$

**Definition 0.6.** Let $x = (x_1, \ldots, x_m)$ be the usual coordinates on $\mathbb{R}^m$. A pseudo-Riemannian manifold $\mathcal{M} := (\mathbb{R}^m, g)$ is said to be a **generalized plane wave manifold** if its Levi-Civita connection is of the form [4]

$$\nabla_{\partial_{x_i}}\partial_{x_j} = \sum_{k > max(i,j)} \Gamma_{ij}^k(x_1, \ldots, x_{k-1})\partial_{x_k}.$$

**Definition 0.7.** $(M, g)$ is said to be **curvature homogeneous** if given any two points $P, Q \in \mathcal{M}$ , there exists a linear isomorphism $\psi : T_P M \to T_Q M$ such that $\psi^* R_Q(\psi_* x, \psi_* y, \psi_* z, \psi_* w) = R_P(x, y, z, w)$. More simply, if there exists a change of basis of the tangent space that makes the metric values some specified constant and curvature entries some specified constants, then $\mathcal{M}$ is said to curvature homogenous.

**Definition 0.8.** (M,g) is said to be **locally homogeneous** if given any two points $P$ and $Q$, there are neighborhoods $U_P$ and $U_Q$ of $P$ and $Q$ respectively, and an isometry $\psi : U_P \to U_Q$ such that $\psi P = Q$. Taking $\psi := \psi_*$ shows that locally homogeneous manifolds are curvature homogeneous manifolds, but the converse of this fails.

**Definition 0.9.** A **model space** $\mathscr{M}$ is defined by a collection of a vector space, a metric, and an ACT:

$$(V, \langle\cdot,\cdot\rangle, R).$$

It is common to search for a nonconstant isometry invariant to determine when a curvature homogeneous manifold is not locally homogeneous. In previous studies, the Weyl scalar invariants usually provided such an invariant, but in the Riemannian setting, if they do not, the manifold is locally homogeneous. In the case of the manifolds exhibited in this paper, the Weyl invariants are zero and we needed to look further. This made the computation of the structure group important.

Let $\alpha_1, \ldots, \alpha_s$ be a collection of contravariant tensors on $V$. Consider the tuple $\mathscr{M} = (V, \alpha_1, \ldots, \alpha_s)$. There is a natural action of the general linear group $Gl(V)$ on any contravariant tensor:

$$(A^*\alpha)(x_1, \ldots, x_s) = \alpha(A_{x_1}, \ldots, A_{x_s}).$$

We can then define a **structure group** $G_{\mathscr{M}}$ of a model space $\mathscr{M}$ as

$$G_{\mathscr{M}} = \{A \in Gl(V) | A^*\alpha_i = \alpha_i \text{ for } i = 1, \ldots, n\}.$$

On curvature homogeneous manifolds, models $\mathscr{M}_p$, defined $(T_pM, g_p, R_p)$, are all isomorphic to some given $\mathscr{M}$, the structure group of a curvature homogeneous manifold is the structure group of any of the models $\mathscr{M}_p$. In this paper, we will be able to compute covariant derivatives of $R$ on a specialized basis for each tangent space and produce a quantity that is invariant under the action of the group [5] . In calculation of the structure group, it will also be important to understand the kernel.

**Definition 0.10.** Define the **kernel of R** as follows [3] :

$$\ker R := \{v \in V | R(v, v_1, v_2, v_3) = 0, \forall v_1, v_2, v_3 \in V\} \neq 0.$$

This will then allow us to prove the manifolds are not locally homogeneous. Outside of curvature homogeneity, we will want to study whether or not the manifolds are generalized plane wave manifolds. We will determine this based on the following definition.

The following is a brief outline of the paper. Section 1 will begin by defining a four-dimensional family of manifolds. It will describe the covariant derivative length four of the coordinate vector fields and then prove the manifold is curvature homogeneous. Lastly, we will show that the manifold is not locally homogeneous. Section 2 will begin by defining a family of five-dimensional generalized plane wave manifolds. Similar to the four-dimensional manifold, it will describe the covariant derivative length four of the coordinate vector fields and prove the manifold is curvature homogeneous. Lastly, we will show that the manifold is not locally homogeneous.

# 1 Four-Dimensional Manifold

This section will explore a four-dimensional manifold of signature $(2, 2)$ that is curvature homogeneous and not locally homogenous. The most interesting aspect of the manifold is the path length of four of the covariant derivative of the coordinate vector fields (see Theorem 1.1).

**Definition 1.1.** Define the model space $\mathscr{M}$ to be: $V = \text{span}\{X, Y, Z, W\}, \langle X, W\rangle = \langle Y, Z\rangle = 1$, and $R(X, Y, Y, X) = R(X, Y, Z, X) = 1$.

## 1.1 Curvature Homogeneity

**Definition 1.2.** Put coordinates $(x, y, z, w)$ on Euclidean space $M := \mathbb{R}^4$ and let $\{\partial_x, \partial_y, \partial_z, \partial_w\}$ be coordinate vector fields on $\mathcal{M}$ . Define the function $p = p(y)$ such that $p'(y) \neq 0$ , $f = f(x)$, and all nonzero entries of metric, $g$, are as follows:

$$g(\partial x, \partial x) = -2zp(y), \quad g_(\partial y, \partial y) = -2f(x), \quad g(\partial y, \partial z) = 1,$$
$$g(\partial x, \partial z) = -2f(x), \quad g(\partial x, \partial w) = 1.$$

If $\mathcal{M} := (\mathbb{R}^4, g)$ , then $\mathcal{M}$ is a pseudo-Riemannian manifold with signature (2,2).

**Theorem 1.1.** *The nonzero covariant derivatives of the coordinate vector fields are as follows:*

$$\begin{array}{rclll}
\nabla_{\partial x}\partial x & = & -2(f'(x)+p(y))\partial y & +(zp'(y)f(x))\partial z & +(xzp'(y)f(x)-8f'(x)(f(x))^2+4f(x)p(y))\partial w, \\
\nabla_{\partial x}\partial y & = & & (-f'(x))\partial z & -(2f(x)f'(x)+zp'(y))\partial w \\
\nabla_{\partial x}\partial z & = & & & -p(y)\partial w, \\
\nabla_{\partial y}\partial y & = & & & f'(x)\partial w.
\end{array}$$

*Proof.* The nonzero Christoffel symbols of the second kind are

$$\begin{aligned}
\Gamma_{xxy} = g(\nabla_{\partial_x}\partial_x, \partial_y) &= \frac{1}{2}(2\partial_x g(\partial_x, \partial_y) - \partial_y g(\partial_x, \partial_x)) \\
&= \frac{1}{2}(2zp'(y)) = zp'(y), \\
\Gamma_{xxz} = g(\nabla_{\partial_x}\partial_x, \partial_z) &= \frac{1}{2}(2\partial_x g(\partial_x, \partial_z) + \partial_z g(\partial_x, \partial_x)) \\
&= \frac{1}{2}(-2f'(x) - 2f'(x) + 2p'(y)) = -2f'(x) + p(y), \\
\Gamma_{xyx} = g(\nabla_{\partial_x}\partial_y, \partial_x) &= \frac{1}{2}(\partial_y g(\partial_x, \partial_x)) \\
&= \frac{1}{2}(-2zp'(y)) = -zp'(y), \\
\Gamma_{xyy} = g(\nabla_{\partial_x}\partial_y, \partial_y) &= \frac{1}{2}(\partial_x g(\partial_y, \partial_y)) \\
&= \frac{1}{2}(-2f'(x)) = -f'(x), \\
\Gamma_{xzx} = g(\nabla_{\partial_x}\partial_z, \partial_x) &= \frac{1}{2}\partial_z g(\partial_x, \partial_x) \\
&= \frac{1}{2}(-2p(y)) = -p(y), \\
\Gamma_{yyx} = g(\nabla_{\partial_y}\partial_y, \partial_x) &= \frac{1}{2}(2\partial_y g(\partial_y, \partial_x) - \partial_x g(\partial_y, \partial_y) \\
&= \frac{1}{2}(2f'(x)) = f'(x).
\end{aligned}$$

Note: Any covariant derivatives with $\partial_w$ will be zero since all metric entries containing $\partial_w$ will always differentiate to zero. □

The construction of the metric creates a mapping of length four in which coordinate vector fields are covariently differentiating to coordinate vector fields of strictly higher levels. More explicitly, in this example, it creates a path that can be best described as $\nabla : \partial_x \to \partial_y \to \partial_z \to \partial_w \to 0$ . Meaning that any covariant derivative involving $\partial_x$ can only differentiate to $\partial_y, \partial_z, \partial_w$ . Similarly, any covariant derivative involving $\partial_y$ can only differentiate to $\partial_z, \partial_w$. Anything with $\partial_z$ can only differentiate to $\partial_w$. Since $\partial_w$ is in the last link of the chain, any combination involving it must differentiate to zero. Looking at just the covariant derivative of the coordinate vector fields, it is clear that this metric follows the conditions for specified mapping.

While it may be possible $\mathcal{M}$ is a generalized plane wave manifold, the $\nabla_{\partial_x}\partial_x$ entry prevents us from being able to characterize this manifold as a generalized plane wave manifold.

**Theorem 1.2.** *The nonzero entries of the curvature tensor $R$ (up to usual symmetries) are:*

$R(\partial x, \partial y, \partial y, \partial x) = f''(x) + zp''(y)$,
$R(\partial x, \partial y, \partial z, \partial y) = p'(y)$.

*Proof.* We may use the above calculations of the covariant derivatives to see that

$$\begin{aligned}
R(\partial_x,\partial_y,\partial_y,\partial_x) &= g(\nabla_{\partial_x}\nabla_{\partial_y}\partial_y,\partial_x) - g(\nabla_{\partial_y}\nabla_{\partial_x}\partial_y,\partial_x)\\
&= g(\nabla_{\partial_x}(f'(x)\partial_w) - \nabla_{\partial_y}(-f'(x)\partial_z - 2f(x)f'(x) - zp'(y)\partial_w),\partial_x)\\
&= g(f''(x)\partial_w + zp''(y)\partial_w,\partial_x)\\
&= f''(x) + zp''(y),\\
R(\partial_x,\partial_y,\partial_z,\partial_x) &= g(\nabla_{\partial_x}\nabla_{\partial_y}\partial_z,\partial_x) - g(\nabla_{\partial_y}\nabla_{\partial_x}\partial_z,\partial_x)\\
&= g(-\nabla_{\partial_y}(-p(y)\partial_w),\partial_x)\\
&= g(-(-p'(y))\partial_w,\partial_x) = p'(y).
\end{aligned}$$

□

**Theorem 1.3.** *The nonzero entries of the covariant derivative tensor $\nabla R$ (up to the usual symmetries) are:*
$\nabla R(\partial x,\partial y,\partial y,\partial x;\partial x) = 2f'(x)p'(y) + f'''(x)$,
$\nabla R(\partial x,\partial y,\partial y,\partial x;\partial y) = zp'''(y)$,
$\nabla R(\partial x,\partial y,\partial y,\partial x;\partial z) = p''(y)$,
$\nabla R(\partial x,\partial y,\partial z,\partial x;\partial y) = p''(y)$.

*Proof.* Note: If $X,Y,Z,T$ are any coordinate vector fields, then $\nabla R(X,Y,Y,X;T) = \nabla_T R(X,Y,Y,X) - 2R(\nabla_T X,Y,Y,X) - 2R(X,\nabla_T Y,Y,X)$.

Using this and the result from the curvature tensor, the covariant derivative of $R$ is given by:

$$\begin{aligned}
\nabla R(\partial_x,\partial_y,\partial_y,\partial_x;\partial_x) &= \nabla_{\partial x} R(\partial_x,\partial_y,\partial_y,\partial_x) - 2R(\nabla_{\partial x}\partial_x,\partial_y,\partial_y,\partial_x) - 2R(\partial_x,\nabla_{\partial x}\partial_y,\partial_y,\partial_x)\\
&= \nabla_{\partial y}(f''(x) + zp''(y)) - 2R(\partial_x,(-f'(x)\partial_z),\partial_y,\partial_x)\\
&= f'''(x) + 2f'(x)p'(y),\\
\nabla R(\partial_x,\partial_y,\partial_y,\partial_x;\partial_y) &= \nabla_{\partial y} R(\partial_x,\partial_y,\partial_y,\partial_x) - 2R(\nabla_{\partial y}\partial_x,\partial_y,\partial_y,\partial_x) - 2R(\partial_x,\nabla_{\partial y}\partial_y,\partial_y,\partial_x)\\
&= \nabla_{\partial x}(f''(x) + zp''(y))\\
&= zp'''(y),\\
\nabla R(\partial_x,\partial_y,\partial_y,\partial_x;\partial_z) &= \nabla_{\partial z} R(\partial_x,\partial_y,\partial_y,\partial_x) - 2R(\nabla_{\partial z}\partial_x,\partial_y,\partial_y,\partial_x) - 2R(\partial_x,\nabla_{\partial z}\partial_y,\partial_y,\partial_x)\\
&= \nabla_{\partial z}(f'' + zp''(y))\\
&= p''(y),\\
\nabla R(\partial_x,\partial_y,\partial_z,\partial_x;\partial_y) &= \nabla_{\partial y} R(\partial_x,\partial_y,\partial_z,\partial_x) - 2R(\nabla_{\partial y}\partial_x,\partial_y,\partial_z,\partial_x) - R(\partial_x,\nabla_{\partial y}\partial_y,\partial_z,\partial_x)\\
&\quad - R(\partial_x,\partial_y,\nabla_{\partial y}\partial_z,\partial_x)\\
&= \nabla_{\partial y}(p'(y))\\
&= p''(y).
\end{aligned}$$

□

This information will be important in calculation the isometry invariant to prove $\mathcal{M}$ is not locally homogeneous.

**Theorem 1.4.** *$\mathcal{M}$ is curvature homogeneous with the model in definition 1.1.*

*Proof.* Begin by defining the basis on each tangent space to be the following:

$X = \sqrt{\alpha}(\partial_x + zp(y)\partial_w)$,

$Y = \sqrt{\lambda}(\partial_y + f(x)\partial_z + 2f(x)^2\partial_w)$,
$Z = \frac{1}{\sqrt{\lambda}}(\partial_z + 2f(x)\partial_w)$,
$W = \frac{1}{\sqrt{\alpha}}\partial_w$.

where $\alpha$ and $\lambda$ will be defined presently and the nonzero inner products that follow are $\langle X, W\rangle = \langle Y, Z\rangle = 1$.

The potentially nonzero curvature entries are :

$$\begin{aligned}
R(X,Y,Y,X) &= \alpha\lambda R(\partial_x, \partial_y + f(x)\partial_z, \partial_y + f(x)\partial_z, \partial_x)\\
&= \alpha\lambda[R(\partial_x, \partial_y, \partial_y, \partial_x) + 2f(x)R(\partial_x, \partial_y, \partial_z, \partial_x)]\\
&= \alpha\lambda(f''(x) + zp''(y) + 2f(x)p'(y)),\\
R(X,Y,Z,X) &= \alpha(R(\partial_x, \partial_y + f(x)\partial_z, \partial_z, \partial_x)\\
&= \alpha(R(\partial_x, \partial_y, \partial_z, \partial_x) + f(x)R(\partial_x, \partial_z, \partial_z, \partial_x))\\
&= \alpha(p'(y)).
\end{aligned}$$

To complete the proof, we must set
$\alpha = \frac{1}{p'(y)}$
$\lambda = \frac{1}{f''(x)+zp''(y)+2f(x)p'(y)}$.

By assumption upon the metric, $p'(y) \neq 0$. We also know $f''(x) + zp''(y) + 2f(x)p'(y) \neq 0$. Thus, $\alpha$ and $\lambda$ are certain to be nonzero, defined functions. We then obtain the constant curvature entries $R(X, Y, Y, X) = 1$ and $R(X, Y, Z, X) = 1$, and thus $\mathcal{M}$ is curvature homogeneous. □

**Theorem 1.5.** *The nonzero entries of the covariant derivative of $R$ on $\{X, Y, Z, W\}$ are as follows:*

$\nabla R(X,Y,Y,X;X) = \alpha\lambda\sqrt{\alpha}(2f'(y)p'(y) + f''(x))$,
$\nabla R(X,Y,Y,X;Y) = \alpha\lambda\sqrt{\lambda}(zp'''(y)) + 3\alpha\lambda\sqrt{\lambda}(f(x)p''(y))$,
$\nabla R(X,Y,Y,X;Z) = \alpha\sqrt{\lambda}(p''(y))$,
$\nabla R(X,Y,Z,X;Y) = \alpha\sqrt{\lambda}(p''(y))$.

*Proof.*

$$\begin{aligned}
\nabla R(X,Y,Y,X;X) &= \nabla R(\sqrt{\alpha}\partial_x, \sqrt{\lambda}\partial_y + \sqrt{\lambda}f(x)\partial_z, \sqrt{\lambda}\partial_y + \sqrt{\lambda}f(x)\partial_z, \sqrt{\alpha}\partial_x; \sqrt{\alpha}\partial_x)\\
&= \alpha\lambda\sqrt{\lambda}(\nabla R(\partial_x, \partial_y, \partial_y\partial_x; \partial_x) + 2\nabla R(\partial_x, \partial_y, \partial_z, \partial_x; \partial_x))\\
&= \alpha\lambda\sqrt{\lambda}\nabla R(\partial_x, \partial_y, \partial_y\partial_x; \partial_x)\\
&= \alpha\lambda\sqrt{\lambda}(2f'(x)p'(y) + f''(x)),\\
\nabla R(X,Y,Y,X;Y) &= \nabla R(\sqrt{\alpha}\partial_x, \sqrt{\lambda}\partial_y + \sqrt{\lambda}f(x)\partial_z, \sqrt{\lambda}\partial_y + \sqrt{\lambda}f(x)\partial_z, \sqrt{\alpha}\partial_x; \sqrt{\lambda}\partial_y + \sqrt{\lambda}f(x)\partial_z)\\
&= \alpha\lambda\sqrt{\lambda}\nabla R(\partial_x, \partial_y, \partial_y, \partial_x; \partial_y) + \alpha\lambda\sqrt{\lambda}f(x)R(\partial_x, \partial_y, \partial_y, \partial_x; \partial_z)\\
&+ 2\alpha\lambda\sqrt{\lambda}f(x)\nabla R(\partial_x, \partial_y, \partial_z, \partial_x; \partial_y) + 2\alpha\lambda\sqrt{\lambda}f(x)\nabla R(\partial_x, \partial_y, \partial_z, \partial_x; \partial_z)\\
&= \alpha\lambda\sqrt{\lambda}(zp'''(y)) + \alpha\lambda\sqrt{\lambda}f(x)(p''(y)) + 2\alpha\lambda\sqrt{\lambda}(f(x)p''(y)),\\
\nabla R(X,Y,Y,X;Z) &= \nabla R(\sqrt{\alpha}\partial_x, \sqrt{\lambda}\partial_y + \sqrt{\lambda}f(x)\partial_z, \sqrt{\lambda}\partial_y + \sqrt{\lambda}f(x)\partial_z, \sqrt{\alpha}\partial_x; \frac{1}{\sqrt{\lambda}}\partial_z)\\
&= \alpha\sqrt{\lambda}\nabla R(\partial_x, \partial_y, \partial_y, \partial_x; \partial_z)\\
&= \alpha\sqrt{\lambda}(p''(y)),
\end{aligned}$$

$$
\begin{aligned}
\nabla R(X,Y,Z,X;Y) &= \nabla R(\sqrt{\alpha}\partial_x, \sqrt{\lambda}\partial_y + \sqrt{\lambda}f(x)\partial_z, \frac{1}{\sqrt{\lambda}}\partial_z, \sqrt{\alpha}\partial_x; \sqrt{\lambda}\partial_y) \\
&= \alpha\sqrt{\lambda}\nabla R(\partial_x, \partial_y, \partial_z, \partial_x; \partial_y) \\
&= \alpha\sqrt{\lambda}(p''(y)).
\end{aligned}
$$

□

These calculations will be needed for the computation the isometry invariant.

## 1.2 Not Locally Homogeneous

**Theorem 1.6.** *For any $a_{ij}$, $A$ is a structure group on $\mathcal{M}$ and is defined as the following:*

$$
\begin{array}{rclllll}
AX = \overline{X} & = & \varepsilon_1 X & +a_{12}Y & +a_{13}Z & +a_{14}W, \\
AY = \overline{Y} & = & & \varepsilon_2 Y & & +a_{24}W, \\
AZ = \overline{Z} & = & & & \varepsilon_2 Z & +a_{34}W, \\
AW = \overline{W} & = & & & & \varepsilon_1 W,
\end{array}
$$

*where $\varepsilon_1 = \pm 1$ and $\varepsilon_2 = \pm 1$.*

*Proof.* We must have $\langle \overline{X}, \overline{W} \rangle = \langle \overline{Y}, \overline{Z} \rangle = 1$ as the only nonzero inner products. Suppose A statisfies:

$AX = \overline{X} = a_{11}X + a_{12}Y + a_{13}Z + a_{14}W,$

$AY = \overline{Y} = a_{21}X + a_{22}Y + a_{23}Z + a_{24}W,$

$AZ = \overline{Z} = a_{31}X + a_{32}Y + a_{33}Z + a_{34}W,$

$AW = \overline{W} = a_{41}X + a_{42}Y + a_{43}Z + a_{44}W.$

Claim (1): kerR= span $W$
*Proof of Claim (1).*
Assume $\ker R = \{v \in V : R(v, v_1, v_2, v_3) = 0, (\forall v_1, v_2, v_3 \in V)\}$
($\supseteq$) Let$\beta W \in \operatorname{span} W$
Then $R(\beta W, v_1, v_2, v_3) = \beta R(W, v_1, v_2, v_3) = 0$
($\subseteq$) Let $v \in \ker R$, to show $v \in \operatorname{span} W$
Suppose $v = aX + bY + cZ + dW$ , to show $a = b = c = 0$
$R(v, Y, Y, X) = R((aX + bY + cZ + dW), Y, Y, X) = 0$
$\Rightarrow aR(X, Y, Y, X) = 0 \Rightarrow a = 0$
$R(X, v, Z, X) = R(X, (bY + cZ + dW), Z, X)$
$\Rightarrow bR(X, Y, Z, X) = 0 \Rightarrow b = 0$
$R(X, Y, v, X) = R(X, Y, cZ, X) = 0 \Rightarrow c = 0$
and thus $\ker R = \operatorname{span} W$.

Claim (2): If $A \in G_{\mathcal{M}}$, then $A : \ker R \to \ker R$
*Proof of Claim (2).*
Let $W \in \ker R$ , to show $AW \in \ker R$
Let $v_1, v_2, v_3 \in V$
$R(AW, v_1, v_2, v_3) = R(AW, AA^{-1}v_1, AA^{-1}v_2, AA^{-1}v_3)$
$= (A^*R)(W, A^{-1}v_1, A^{-1}v_2, A^{-1}v_3) = R(W, A^{-1}v_1, A^{-1}v_2, A^{-1}v_3) = 0$
And thus $A : \ker R \to \ker R$. [3]

Since $\ker R = \operatorname{span} W$ and $A : \ker R \to \ker R$, it follows that $a_{41}, a_{42}, a_{43} = 0$.

Other simplifications on A can be made through the following steps:

(1) $\langle \overline{W}, \overline{Z} \rangle = 0 \Rightarrow a_{31} = 0$
$\langle \overline{W}, \overline{Y} \rangle = 0 \Rightarrow a_{21} = 0$
$\langle \overline{W}, \overline{X} \rangle = 1 \Rightarrow a_{44}a_{11} = 1$ and $a_{44}, a_{11} \neq 0$.

(2) $\langle \overline{Z}, \overline{Z} \rangle = \langle a_{32}Y + a_{33}Z, a_{32}Y + a_{33}Z \rangle = 0 \Rightarrow 2a_{33}a_{32} = 0$.

(3) $R(\overline{X}, \overline{Z}, \overline{Z}, \overline{X}) = 0 \Rightarrow {a_{11}}^2({a_{32}}^2 + 2a_{32}a33) = 0$
$\Rightarrow a_{32} = 0$. by (2)

(4) $\langle \overline{Y}, \overline{Z} \rangle = \langle a_{22}Y + a_{23}Z, a_{33}Z \rangle = 1$
$\Rightarrow a_{22}a_{23} = 1$.

(5) $\langle \overline{Y}, \overline{Y} \rangle = 0 \Rightarrow 2a_{22}a_{23} = 0 \Rightarrow a_{23} = 0$ by (4).

(6) $R(\overline{X}, \overline{Y}, \overline{Y}, \overline{X}) = 1 \Rightarrow R(a_{11}X, a_{32}Y, a_{32}Y, a_{11}X) = 1$
$\Rightarrow 1 = {a_{11}}^2{a_{22}}^2$.

(7) $R(\overline{X}, \overline{Y}, \overline{Y}, \overline{X}) = 1 \Rightarrow {a_{11}}^2 a_{22}a_{23} = 1$
Since $a_{22}a_{33} = 1$ by (4)$\Rightarrow {a_{11}}^2 = 1 \Rightarrow a_{11} = \pm 1 = \varepsilon_1$
By (1) $a_{44}a_{11} = 1 \Rightarrow a_{44} = \varepsilon_1$
By (6) ${a_{11}}^2{a_{22}}^2 = 1 \Rightarrow {a_{22}}^2 = \pm 1 = \varepsilon_2$
By (4) $\varepsilon_2 a_{33} = 1 \Rightarrow a_{33} = \varepsilon_2$.

And thus by items 1-7 we can define our basis as

$$\begin{array}{rclllll} AX = \overline{X} & = & \varepsilon_1 X & +a_{12}Y & +a_{13}Z & +a_{14}W, \\ AY = \overline{Y} & = & & \varepsilon_2 Y & & +a_{24}W, \\ AZ = \overline{Z} & = & & & \varepsilon_2 Z & +a_{34}W, \\ AW = \overline{W} & = & & & & \varepsilon_1 W. \end{array}$$

□

**Definition 1.3.** For simplification of computations to follow, redefine the covariant derivatives of R on the basis {X, Y, Z, W} to be the following:

$j = \nabla R(X, Y, Y, X; X) = \alpha\lambda\sqrt{\alpha}(2f'(y)p'(y) + f''(x))$,
$k = \nabla R(X, Y, Y, X; Y) = \alpha\lambda\sqrt{\lambda}(zp'''(y)) + 3\alpha\lambda\sqrt{\lambda}(f(x)p''(y))$,
$l = \nabla R(X, Y, Y, X; Z) = \alpha\sqrt{\lambda}(p''(y))$,
$m = \nabla R(X, Y, Z, X; Y) = \alpha\sqrt{\lambda}(p''(y))$.

**Theorem 1.7.** *The nonzero covariant derivatives of R on A (up to the symmetries) are as follows:*

$\nabla R(\overline{X}, \overline{Y}, \overline{Y}, \overline{X}; \overline{X}) = \varepsilon_1 j + a_{13}k + a_{13}l$,
$\nabla R(\overline{X}, \overline{Y}, \overline{Y}, \overline{X}; \overline{Y}) = \varepsilon_2 k$,
$\nabla R(\overline{X}, \overline{Y}, \overline{Y}, \overline{X}; \overline{Z}) = \varepsilon_2 l$,
$\nabla R(\overline{X}, \overline{Y}, \overline{Z}, \overline{X}; \overline{Y}) = \varepsilon_2 m$,
$\nabla R(\overline{X}, \overline{Y}, \overline{Z}, \overline{X}; \overline{X}) = a_{12}m$.

*Proof.*

$$\begin{aligned} \nabla R(\overline{X}, \overline{Y}, \overline{Y}, \overline{X}; \overline{X}) &= \nabla R(\varepsilon_1 X + a_{12}Y + a_{13}Z, \varepsilon_2 Y, \varepsilon_2 Y, \varepsilon_1 X + a_{12}Y, ) \\ &= \nabla R(\varepsilon_1 X, \varepsilon_2 Y, \varepsilon_2 Y, \varepsilon_1 X; \varepsilon_1 X) + \nabla R(\varepsilon_1 X, \varepsilon_2 Y, \varepsilon_2 Y, \varepsilon_1 X; a_{12}Y) + \nabla R(\varepsilon_1 X, \varepsilon_2 Y, \varepsilon_2 Y, \varepsilon_1 X; a_{12}Z) \\ &= {\varepsilon_1}^3{\varepsilon_2}^2 j + {\varepsilon_1}^2{\varepsilon_2}^2 a_{13}k + {\varepsilon_1}^2{\varepsilon_2}^2 a_{13}l \\ &= \varepsilon_1 j + a_{13}k + a_{13}l, \end{aligned}$$

$$
\begin{aligned}
\nabla R(\overline{X},\overline{Y},\overline{Y},\overline{X};\overline{Y}) &= \nabla R(\varepsilon_1 X + a_{12}Y + a_{13}Z, \varepsilon_2 Y, \varepsilon_2 Y, \varepsilon_1 X + a_{12}Y + a_{13}Z; \varepsilon_2 Y) \\
&= \nabla R(\varepsilon_1 X, \varepsilon_2 Y, \varepsilon_2 Y, \varepsilon_1 X; \varepsilon_2 Y) \\
&= {\varepsilon_1}^2 {\varepsilon_2}^3 k = \varepsilon_2 k, \\
\nabla R(\overline{X},\overline{Y},\overline{Y},\overline{X};\overline{Z}) &= \nabla R(\varepsilon_1 X, \varepsilon_2 Y, \varepsilon_2 Y, \varepsilon_1 X; \varepsilon_2 Z) \\
&= {\varepsilon_1}^2 {\varepsilon_2}^3 l = \varepsilon_2 l, \\
\nabla R(\overline{X},\overline{Y},\overline{Z},\overline{X};\overline{Y}) &= \nabla R(\varepsilon_1 X, \varepsilon_2 Y, \varepsilon_2 Z, \varepsilon_1 X; \varepsilon_2 Y + a_{24}) \\
&= {\varepsilon_1}^2 {\varepsilon_2}^3 m = \varepsilon_2 m, \\
\nabla R(\overline{X},\overline{Y},\overline{Z},\overline{X};\overline{X}) &= \nabla R(\varepsilon_1 X, \varepsilon_2 Y, \varepsilon_2 Z, \varepsilon_1 X; \varepsilon_1 X + a_{12}Y + a_{13}Z + a_{14}W) \\
&= \nabla R(\varepsilon_1 X, \varepsilon_2 Y, \varepsilon_2 Z, \varepsilon_1 X; a_{12}Y) \\
&= {\varepsilon_1}^2 {\varepsilon_2}^2 a_{12} m = a_{12} m.
\end{aligned}
$$

□

**Theorem 1.8.** *Under changes to the covariant derivative of the curvature tensor, $\nabla R$, there exists a nonconstant quantity that is invariant under the action of the structure group, and hence $\mathcal{M}$ is not locally homogeneous.*

*Proof.*

$$
\begin{aligned}
(\nabla R(\overline{X},\overline{Y},\overline{Y},\overline{X};\overline{Z}))^2 = (\nabla R(X,Y,Y,X;Z))^2 &= l^2 \\
&= \alpha^2 \lambda (p''(y))^2 \\
&= \frac{p(y)p''(y)}{p'(y)(f''(x) + zp''(y) + 2f(x)p'(y))}.
\end{aligned}
$$

Since, in general, this quantity is nonconstant, $\mathcal{M}$ is not a locally homogenous. □

# 2 Section 2

This section will explore a five-dimensional generalized plane wave manifold of signature $(2,3)$ that is curvature homogeneous and not locally homogenous. The most interesting aspect of the manifold is the path length of four of the covariant derivative of the coordinate vector fields (see theorem 2.1).

**Definition 2.1.** Define the model space $\mathscr{M}$ to be:
$V = \text{span}\{X, Y, Z, W_1, W_2\}, \langle X, W_1\rangle = \langle Y, W_2\rangle = \langle Z, Z\rangle = 1$, and $R(X,Y,Y,X) = 1, R(X,Y,Z,X) = -1$.

## 2.1 Curvature Homogeneity

**Definition 2.2.** Put coordinates $(x, y, z, w_1, w_2)$ on Euclidean space $M := \mathbb{R}^5$ and let $\{\partial_x, \partial_y, \partial_z, \partial_{w_1}, \partial_{w_2}\}$ be coordinate vector fields on $\mathcal{M}$ . Define $b = b(y)$ such that $b'''(y) \neq 0$ and $c = c(x)$ such that $c''(x) \neq 0$. Define all nonzero entries of the metric $g$ as follows:

$$g(\partial_x, \partial_x) = -2zy, \quad g(\partial_x, \partial_y) = -2z, \quad g(\partial_x, \partial_{w_1}) = 1,$$

$$g(\partial_x, \partial_{w_2}) = -2c(x), \quad g(\partial_y, \partial_{w_2}) = 1,$$

$$g(\partial_y, \partial_y) = -2b(y), \qquad g(\partial_z, \partial_z) = 1.$$

If $\mathcal{M} := (\mathbb{R}^5, g)$ , then $\mathcal{M}$ is a pseudo-Riemannian manifold with signature (2,3).

**Theorem 2.1.** *The nonzero covariant derivatives of the coordinate vector fields are as follows:*

$$\begin{array}{lclllll}
\nabla_{\partial_x}\partial_x & = & -2c'(x)\partial_y & +y\partial_z & -(8c(x)c'(x)b(y)+4c'(x)z-2zc(x))\partial_{w_1} & -(4b(y)c'(x)-z)\partial_{w_2} \\
\nabla_{\partial_x}\partial_y & = & & \partial_z & -z\partial_{w_1} & \\
\nabla_{\partial_x}\partial_z & = & & & (-y-2c(x))\partial_{w_1} & -\partial_{w_2} \\
\nabla_{\partial_y}\partial_y & = & & & -2c(x)b'(y)\partial_{w_1} & -b'(y)\partial_{w_2} \\
\nabla_{\partial_y}\partial_z & = & & & -\partial_{w_1} &
\end{array}$$

*Proof.* The nonzero Christoffel symbols of the second kind are

$$\begin{aligned}
\Gamma_{xxy} = g(\nabla_{\partial_x}\partial_x,\partial_y) &= \frac{1}{2}(2\partial_x g(\partial_x,\partial_y)-\partial_y g(\partial_x,\partial_x)) \\
&= \frac{1}{2}(\partial_y g(\partial_x,\partial_x)) = \frac{1}{2}(2z) = z, \\
\Gamma_{xxz} = g(\nabla_{\partial_x}\partial_x,\partial_z) &= \frac{1}{2}(2\partial_x g(\partial_x,\partial_z)-\partial_z g(\partial_x,\partial_x)) \\
&= \frac{1}{2}(-\partial_z g(\partial_x,\partial_x)) = \frac{1}{2}(2y) = y, \\
\Gamma_{xxw_2} = g(\nabla_{\partial_x}\partial_x,\partial_{w_2}) &= \frac{1}{2}(2\partial_x g(\partial_x,\partial_{w_2})-\partial_{w_2} g(\partial_x,\partial_x)) \\
&= \frac{1}{2}(2\partial_x g(\partial_x,\partial_{w_2})) = -2c'(x), \\
\Gamma_{xyx} = g(\nabla_{\partial_x}\partial_x,\partial_x) &= \frac{1}{2}(\partial_y g(\partial_x,\partial_x)+\partial_x g(\partial_y,\partial_x)-\partial_x g(\partial_y,\partial_x)) \\
&= \frac{1}{2}(\partial_y g(\partial_x,\partial_x)) = \frac{1}{2}(-2z) = -z, \\
\Gamma_{xyz} = g(\nabla_{\partial_x}\partial_y,\partial_z) &= \frac{1}{2}(\partial_y g(\partial_x,\partial_z)+\partial_x g(\partial_y,\partial_z)-\partial_z g(\partial_x,\partial_y)) \\
&= \tfrac{1}{2}(-\partial_x g(\partial_x\partial_y)) = \tfrac{1}{2}(2) = 1, \\
\Gamma_{xzx} = g(\nabla_{\partial_x}\partial_z,\partial_x) &= \frac{1}{2}(\partial_z g(\partial_x,\partial_x)+\partial_x g(\partial_x,\partial_z)-\partial_x g(\partial_x,\partial_z)) \\
&= \tfrac{1}{2}(\partial_z g(\partial_x,\partial_x)) = \tfrac{1}{2}(-2y) = -y, \\
\Gamma_{xzy} = g(\nabla_{\partial_x}\partial_z,\partial_y) &= \frac{1}{2}(\partial_z g(\partial_x,\partial_y)+\partial_x g(\partial_z,\partial_y)-\partial_y g(\partial_x,\partial_z)) \\
&= \tfrac{1}{2}(\partial_z g(\partial_x,\partial_y)) = \tfrac{1}{2}(-2) = -1, \\
\Gamma_{yyy} = g(\nabla_{\partial_y}\partial_y,\partial_y) &= \frac{1}{2}(\partial_y g(\partial_y,\partial_y)+\partial_y g(\partial_y,\partial_y)-\partial_y g(\partial_y,\partial_y)) \\
&= \tfrac{1}{2}(\partial_y g(\partial_y,\partial_y)) = \tfrac{1}{2}(-2b'(y)) = -b'(y), \\
\Gamma_{yzx} = g(\nabla_{\partial_y}\partial_z,\partial_x) &= \frac{1}{2}(\partial_z g(\partial_x,\partial_y)+\partial_y g(\partial_z,\partial_x)-\partial_x g(\partial_y,\partial_z)) \\
&= \tfrac{1}{2}(\partial_z g(\partial_x,\partial_y)) = \tfrac{1}{2}(-2) = -1.
\end{aligned}$$

□

Similar to the four-dimensional manifold, this arrangement demonstrates a mapping of length four in which coordinate vector fields are covariantly differentiating to coordinate vector fields of strictly higher levels. It can be seen explicity as : $\nabla : \partial_x \to \partial_y \to \partial_z \to \{\partial_{w_1}, \partial_{w_2}\} \to 0$. In observation of the covariant derivative of the coordinate vector fields, it can be seen that they follow the specified mapping. It can also be seen in the covariant derivative of the coordinate vector fields that $\mathcal{M}$ is a generalized plane wave manifold.

**Theorem 2.2.** *The nonzero entries of the curvature tensor $R$ (up to usual symmetries) are:*

$R(\partial_x, \partial_y, \partial_y, \partial_x) = -2c'(x)b'(y) + 1,$
$R(\partial_x, \partial_y, \partial_z, \partial_x) = -1.$

*Proof.* We may use the calculations of the covariant derivatives to see that

$$\begin{aligned}
R(\partial_x, \partial_y, \partial_y, \partial_x) &= g(\nabla_{\partial_x}\nabla_{\partial_y}\partial_y, \partial_x) - g(\nabla_{\partial_y}\nabla_{\partial_x}\partial_y, \partial_x),\\
&= g(\nabla_{\partial_x}(-2c(x)b'\partial_{w_1} - b'(y)\partial_{w_2}) - \nabla_{\partial_y}(\partial_z - z\partial_{w_1}), \partial_x)\\
&= g(-2c'(x)b'(y)\partial_{w_1} + \partial_{w_1}, \partial_x)\\
&= -2c'(x)b'(y) + 1,\\
R(\partial_x, \partial_y, \partial_z, \partial_x) &= g(\nabla_{\partial_x}\nabla_{\partial_y}\partial_z, \partial_x) - g(\nabla_{\partial_y}\nabla_{\partial_x}\partial_z, \partial_x)\\
&= g(\nabla_{\partial_x}(-\partial_{w_1}) - \nabla_{py}((-y - 2c(x))\partial_{w_1} - \partial_{w_2}), \partial_x)\\
&= g(-\partial_{w_1} - \partial_{w_2}, \partial_x) = -1.
\end{aligned}$$

□

**Theorem 2.3.** *The nonzero entries of the covariant derivative tensor $\nabla R$ (up to the usual symmetries) are:*

$\nabla R(\partial_x, \partial_y, \partial_y, \partial_x; \partial_x) = -2c''(x)b'(y) - 2,$
$\nabla R(\partial_x, \partial_y, \partial_y, \partial_x; \partial_y) = -2c'(x)b''(y).$

*Proof.* Note: If $X, Y, Z, T$ are any coordinate vector fields, then $\nabla R(X, Y, Y, X; T) =$
$\nabla_T R(X, Y, Y, X) - 2R(\nabla_T X, Y, Y, X) - 2R(X, \nabla_T Y, Y, X).$
Using this and the results from the curvature tensor, the covariant derivative of $R$ is given by:

$$\begin{aligned}
\nabla R(\partial_x, \partial_y, \partial_y, \partial_x; \partial_x) &= \nabla_{\partial x} R(\partial_x, \partial_y, \partial_y, \partial_x) - 2R(\nabla_{\partial x}\partial_x, \partial_y, \partial_y, \partial_x) - 2R(\partial_x, \nabla_{\partial x}\partial_y, \partial_y, \partial_x)\\
&= \nabla_{\partial_x}(-2c'(x)b'(y) + 1) - 2R(\partial_x, \partial_z, \partial_y, \partial_x)\\
&= -2c''(x)b'(y) - 2,\\
\nabla R(\partial_x, \partial_y, \partial_y, \partial_x; \partial_y) &= \nabla_{\partial y} R(\partial_x, \partial_y, \partial_y, \partial_x) - 2R(\nabla_{\partial y}\partial_x, \partial_y, \partial_y, \partial_x) - 2R(\partial_x, \nabla_{\partial y}\partial_y, \partial_y, \partial_x)\\
&= \nabla_{\partial_y}(-2c'b'(y) + 1)\\
&= -2c'(x)b''(y).
\end{aligned}$$

□

**Theorem 2.4.** *The nonzero entries of the second covariant derivative tensor $\nabla^2 R$ (up to the usual symmetries) are:*
$\nabla^2 R(\partial_x, \partial_y, \partial_y, \partial_x; \partial_x, \partial_x) = -4c'(x)^2 b''(y) - 2c'''(x)b'(y),$
$\nabla^2 R(\partial_x, \partial_y, \partial_y, \partial_x; \partial_x, \partial_y) = -2c''(x)b''(y),$
$\nabla^2 R(\partial_x, \partial_y, \partial_y, \partial_x; \partial_y, \partial_y) = -2c'(x)b'''(y).$

*Proof.*

$$\begin{aligned}
\nabla^2 R(\partial_x, \partial_y, \partial_y, \partial_x; \partial_x, \partial_x) &= \nabla_{\partial_x} R(\partial_x, \partial_y, \partial_y, \partial_x; \partial_x) - 2R(\nabla_{\partial_x}\partial_x, \partial_y, \partial_y, \partial_x; \partial_x) - 2R(\partial_x, \nabla_{\partial_x}\partial_y, \partial_y, \partial_x; \partial_x)\\
&\quad - R(\partial_x, \partial_y, \partial_y, \partial_x; \nabla_{\partial_x}\partial_x)\\
&= \nabla_{\partial_x}(-2c''(x)b'(y) - 2) - R(\partial_x, \partial_y, \partial_y, \partial_x; (-2c'(x)\partial_y + y\partial_z))\\
&= -2c'''(x)b'(y) - (-2c'(x))(-2c'(x)b''(y))\\
&= -2c'''(x)b'(y) - 4c'(x)^2 b''(y)
\end{aligned}$$

$$
\begin{aligned}
\nabla^2 R(\partial_x,\partial_y,\partial_y,\partial_x;\partial_x,\partial_y) &= \nabla_{\partial_y} R(\partial_x,\partial_y,\partial_y,\partial_x;\partial_x) - 2R(\nabla_{\partial_y}\partial_x,\partial_y,\partial_y,\partial_x;\partial_x) - 2R(\partial_x,\nabla_{\partial_y}\partial_y,\partial_y,\partial_x;\partial_x)\\
&\quad - R(\partial_x,\partial_y,\partial_y,\partial_x;\nabla_{\partial_y}\partial_x)\\
&= \nabla_{\partial_y} R(\partial_x,\partial_y,\partial_y,\partial_x;\partial_x,\partial_y) = \nabla_{\partial_y}(-2c''(x)b'(y)-2)\\
&= -2c''(x)b''(y)\\
\nabla^2 R(\partial_x,\partial_y,\partial_y,\partial_x;\partial_y,\partial_y) &= \nabla_{\partial_y} R(\partial_x,\partial_y,\partial_y,\partial_x;\partial_y) - 2R(\nabla_{\partial_y}\partial_x,\partial_y,\partial_y,\partial_x;\partial_y) - 2R(\partial_x,\nabla_{\partial_y}\partial_y,\partial_y,\partial_x;\partial_y)\\
&\quad - R(\partial_x,\partial_y,\partial_y,\partial_x;\nabla_{\partial_y}\partial_y)\\
&= \nabla_{\partial_y} R(\partial_x,\partial_y,\partial_y,\partial_x;\partial_y) = \nabla_{\partial_y}(-2c'(x)b''(y))\\
&= -2c'(x)b'''(y)
\end{aligned}
$$

□

Calculations of the first and second covariant derivatives of the curvature tensor will be important in the calculation of the isometry invariant for proving the manifold is not locally homogeneous.

**Theorem 2.5.** *$\mathcal{M}$ is curvature homogeneous.*

*Proof.* Begin by defining a basis, $\{X, Y, Z, W_1, W_2\}$, on each tangent space such that

$$
\begin{aligned}
&X = \tfrac{1}{\sqrt{\alpha}}(\partial_x + zy\partial_{w_1}),\\
&Y = \alpha(\partial_y + (2z + 2b(y)c(x))\partial_{w_1} + b(y)\partial_{w_2},\\
&Z = \partial_z,\\
&W_1 = \sqrt{\alpha}\partial_{w_1},\\
&W_2 = \tfrac{1}{\alpha}(2c(x)\partial_{w_1} + \partial_{w_2}).
\end{aligned}
$$

where $\alpha$ will be defined presently and the nonzero inner products that follow are
$\langle X, W_1\rangle = \langle Y, W_2\rangle = \langle Z, Z\rangle = 1$.

The potentially nonzero curvature entries are as follows:

$$
\begin{aligned}
R(X,Y,Y,X) =& R\left(\frac{1}{\sqrt{\alpha}}\partial_x, \alpha\partial_y, \alpha\partial_y, \frac{1}{\sqrt{\alpha}}\partial_x\right)\\
&= \alpha R(\partial_x,\partial_y,\partial_y,\partial_x)\\
&= \alpha(-2c'(x)b'(y) + 1),\\
R(X,Y,Z,X) =& R\left(\frac{1}{\sqrt{\alpha}}\partial_x, \alpha\partial_y, \partial_z, \frac{1}{\sqrt{\alpha}}\partial_x\right)\\
&= R(X,Y,Z,X) = -1.
\end{aligned}
$$

To complete the proof, set $\alpha = \frac{1}{-2c'(x)b'(y)+1}$.
By assumption upon the metric, $b'''(y) \neq 0$ and $c''(x) \neq 0$. Also assume, $\frac{1}{-2c'(x)b'(y)+1} \neq 0$ and thus we obtain that $\alpha$ is defined. We then obtain the nonzero, constant curvature entries $R(X,Y,Y,X) = 1$ and $R(X,Y,Z,X) = -1$ proving that $\mathcal{M}$ is curvature homogeneous. □

**Theorem 2.6.** *The nonzero entries of the covariant derivative on $\{X, Y, Z, W_1, W_2\}$ of $R$ are as follows:*

$$
\begin{aligned}
&\nabla R(X,Y,Y,X;X) = \sqrt{\alpha}(-2c''(x)b'(y) - 2)),\\
&\nabla R(X,Y,Y,X;Y) = \alpha^2(-2c'(x)b''(y)).
\end{aligned}
$$

*Proof.*

$$
\begin{aligned}
\nabla R(X,Y,Y,X;X) &= \nabla R\left(\frac{1}{\sqrt{\alpha}}\partial_x, \alpha\partial_y, \alpha\partial_y, \frac{1}{\sqrt{\alpha}}; \frac{1}{\sqrt{\alpha}}\partial_x\right)\\
&= \sqrt{\alpha}\nabla R(\partial_x, \partial_y, \partial_y, \partial_x; \partial_x)\\
&= \sqrt{\alpha}(-2c''(x)b'(y) - 2)),\\
\nabla R(X,Y,Y,X;Y) &= \nabla R\left(\frac{1}{\sqrt{\alpha}}\partial_x, \alpha\partial_y, \alpha\partial_y, \frac{1}{\sqrt{\alpha}}; \sqrt{\alpha}\partial_x\right)\\
&= \alpha^2\nabla R(\partial_x, \partial_y, \partial_y, \partial_x; \partial_y)\\
&= \alpha^2(-2c'(x)b''(y)).
\end{aligned}
$$

□

**Theorem 2.7.** *The nonzero entries of the second covariant derivative of $R$ on $\{X, Y, Z, W_1, W_2\}$ of $R$ are as follows:*

$\nabla^2 R(X,Y,Y,X;X,X) = -4c'(x)^2 b''(y) - 2c'''(x)b'(y),$
$\nabla^2 R(X,YY,X;X,Y) = \alpha\sqrt{\alpha}(-2c''(x)b''(y),$
$\nabla^2 R(X,YY,X;Y,Y) = \alpha^3(-2c'(x)b'''(y)).$

*Proof.*

$$
\begin{aligned}
\nabla^2 R(X,Y,Y,X;X,X) &= \nabla^2 R\left(\frac{1}{\sqrt{\lambda}}\partial_x, \alpha\partial_y, \alpha, \frac{1}{\sqrt{\lambda}}; \frac{1}{\sqrt{\lambda}}\partial_x, \frac{1}{\sqrt{\lambda}}\partial_x\right)\\
&= \nabla^2 R(\partial_x, \partial_y, \partial_y, \partial_y; \partial_x, \partial_x) = -4c'(x)^2 b''(y) - 2c'''(x)b'(y),\\
\nabla^2 R(X,Y,Y,X;X,Y) &= \nabla^2 R\left(\frac{1}{\sqrt{\lambda}}\partial_x, \alpha\partial_y, \alpha, \frac{1}{\sqrt{\lambda}}; \frac{1}{\sqrt{\lambda}}\partial_x, \alpha\partial_y\right)\\
&= \alpha\sqrt{\alpha}\nabla^2 R(\partial_x, \partial_y, \partial_y, \partial_y; \partial_x, \partial_y) = \alpha\sqrt{\alpha}(-2c''(x)b''(y),\\
\nabla^2 R(X,Y,Y,X;Y,Y) &= \nabla^2 R\left(\frac{1}{\sqrt{\lambda}}\partial_x, \alpha\partial_y, \alpha, \frac{1}{\sqrt{\lambda}}; \alpha\partial_y, \alpha\partial_y\right)\\
&= \alpha^3\nabla^2 R(\partial_x, \partial_y, \partial_y, \partial_y; \partial_y, \partial_y) = \alpha^3(-2c'(x)b'''(y)).
\end{aligned}
$$

□

Calculations of the covariant derivatives of $R$ will be useful in proving $\mathcal{M}$ is not locally homogeneous.

## 2.2 Not Locally Homogeneous

**Theorem 2.8.** *For any $a_{ij}$, $A$ is a structure group on $\mathcal{M}$ and is as follows:*

$$
\begin{array}{rclllll}
AX = \overline{X} & = & a_{11}X & +a_{12}Y & +a_{13}Z & +a_{14}W_1 & +a_{15}W_2\\
AY = \overline{Y} & = & & a_{22}Y & +a_{23}Z & +a_{24}W_1 & +a_{25}W_2\\
AZ = \overline{Z} & = & & & \varepsilon_3 Z & +a_{34}W_1 & +a_{35}W_2\\
AW_1 = \overline{W}_1 & = & & & & a_{44}W_1 & \\
AW_2 = \overline{W}_2 & = & & & & a_{54}W_1 & +a_{55}W_2
\end{array}
$$

*Proof.* Claim (1): $\ker R = \text{span}\{W_1, W_2\}$
*Proof of Claim (1).*
$(\supseteq)\{\beta W_1 + \mu W_2\} \in \text{span}\{W_1, W_2\}$
Then $R(\beta W_1 + \mu W_2, v_1, v_2, v_3) = \beta R(W_1, v_1, v_2, v_3) + \mu R(W_2, v_1, v_2, v_3) = 0$
$(\subseteq)$ Let $v \in \ker R$, to show $v \in \text{span}\{W_1, W_2\}$

Suppose $v = aX + bY + cZ + dW_1, +eW_2$, to show $a = b = c = 0$
$R(v, Y, Y, X) = R(aX, +bY + cZ + dW_1, +eW_2, Y, Y, X) = 0$
$\Rightarrow R(aX, Y, Y, X) = 0 \Rightarrow a = 0$
$R(aX, Y, Y, X) = R(X, bY + cZ + dW_1 + eW_2, ZX) = 0$
$\Rightarrow R(X, bY, Z, X) = 0 \Rightarrow b = 0$
$R(X, V, Y, X) = R(X, cZ + dW_1 + eW_2, Y, X) = 0$
$\Rightarrow R(X, cZ, Y, X) = 0 \Rightarrow c = 0$
Thus $\ker R = \operatorname{span}\{W_1, W_2\}$

Claim (2): If $A \in G_{\mathcal{M}}$, then $A : \ker R \to \ker R$ .
*Proof of Claim (2).*
Let $\{W_1, W_2\} \in \ker R$, to show $\{AW_1, AW_2\} \in \ker R$
Let $v_1, v_2, v_3 \in V$, then
$R(aW_1 + aW_2, v_1, v_2, v_3) = R(AW_1, AA^{-1}v_1, AA^{-1}v_2, AA^{-1}v_3) + R(AW_2, AA^{-1}v_1, AA^{-1}v_2, AA^{-1}v_3)$
$= (A^*R)(W_1, A^{-1}v_1, A^{-1}v_2, A^{-1}v_3) + (A^*R)(W_2, A^{-1}v_1, A^{-1}v_2, A^{-1}v_3)$
$= R(W_1, A^{-1}v_1, A^{-1}v_2, A^{-1}v_3) + R(W_2, A^{-1}v_1, A^{-1}v_2, A^{-1}v_3) = 0$
Thus $A : \ker R \to \ker R$

By claims (1) and (2), $a_{41} = a_{42} = a_{43} = 0$ and $a_{51} = a_{52} = a_{53} = 0$

Claim (3): $\ker R^\perp = \operatorname{span}\{Z, W_1, W_2\}$
*Proof of Claim (3).*
($\supseteq$) Let $aZ, bW_1, cW_2 \in \operatorname{span}\{W_1, Z, W_2\}$ and $dW_1, eW_2 \in \operatorname{span}\{W_1, W_2\}$
Then $\langle aZ + bW_1 + cW_2, dW_1 + eW_2\rangle = 0$
And $\operatorname{span}\{Z, W_1, W_2\} \in \ker R^\perp$
($\subseteq$) Let $v = aX + bY + cZ + dW_1 + eW_2 \in \ker R^\perp$
To show $a = b = 0$
$\langle v, W_1\rangle = \langle aX, W_1\rangle = 0 \Rightarrow a = 0$
$\langle v, W_2\rangle = \langle bY, W_2\rangle = 0 \Rightarrow b = 0$
Thus $\ker R^\perp = \operatorname{span}\{Z, W_1, W_2\}$

Claim (4): $A : \ker R^\perp \to \ker R^\perp$
*Proof of Claim (4).*
Let $v \in \ker R^\perp$, to show $Av \in \ker R^\perp$
To show $\langle Av, w\rangle = 0, (\forall w \in \ker R)$
Define $\tilde{w} := A^{-1}w \iff A\tilde{w} = w$
$A \in G_{\mathcal{M}} \Rightarrow A^{-1} \in G_{\mathcal{M}}$, and $A^{-1} : \ker R \to \ker R$
$w \in \ker R \Rightarrow \tilde{w} \in \ker R,\ A^{-1}w \in \ker R,\ A^{-1}w = \tilde{w} \in \ker R$
$\langle Av, w\rangle = \langle Av, A\tilde{w}\rangle = \langle v, \tilde{w}\rangle = 0$ since $v \in \ker R$ and $\tilde{w} \in \ker R^\perp$
Thus $A : \ker R^\perp \to \ker R^\perp$

By claims (3) and (4), $a_{31} = a_{32} = 0$.

Other simplifications on $A$ can be made through the following steps:

(1) $\langle \overline{Z}, \overline{Z}\rangle = 1 \Rightarrow \langle a_{33}Z, a_{33}Z\rangle = 1 \Rightarrow {a_{33}}^2$ and $a_{33} = \pm 1 = \varepsilon_3$.

(2) $R(\overline{X}, \overline{Y}, \overline{Z}, \overline{X}) = R(a_{11}X + a_{22}Y, a_{12}X + a_{22}Y, \varepsilon_3 Z, a_{11}X + a_{12}Y) = 1$
$\Rightarrow \varepsilon_3 a_{11} R(a_{11}X + a_{12}Y, a_{21}X + a_{22}Y, Z, X) = 1$
$\Rightarrow \varepsilon_3 a_{11}(a_{11}a_{22} - a_{12}a_{21}) = 1 \Rightarrow a_{11} \neq 0.$

(3)$R(\overline{X}, \overline{Y}, \overline{Z}, \overline{Y}) = R(a_{21}X + a_{22}Y + a_{23}Z, a_{11}X + a_{12}Y + a_{13}Z, \varepsilon_3 Z, a_{21}X + a_{22}Y + a_{23}Z) = 0$

$\Rightarrow R(a_{12}X, a_{12}Y, \varepsilon_3 Z, a_{21}X) + R(a_{22}Y, a_{11}X, \varepsilon_3 Z, a_{21}X) = 0$
$\Rightarrow a_{21}(a_{21}a_{12} - A_{11} - A_{22}) = 0 \Rightarrow a_{21} = 0$ by (2).

(4) $\varepsilon_3 {a_{11}}^2 a_{22} = 1 \Rightarrow a_{11}, a_{22} \neq 0$
$\Rightarrow \varepsilon_3 a_{11}^2 a_{22} = 1$ by $(2, 3)$.

(5) $\langle \overline{Y}, \bar{W_1} \rangle = \langle a_{22}Y, a_{45}W_2 \rangle = 0$
$\Rightarrow a_{22}a_{45} = 0$ and $a_{45} = 0$ by (4).

(6) $\langle \overline{X}, \bar{W_1} \rangle = 1 \Rightarrow a_{11}a_{44} = 1 \Rightarrow a_{44} \neq 0$.

(7) $\langle \overline{Y}, \bar{W_2} \rangle = 1 \Rightarrow a_{22}a_{55} = 1 \Rightarrow a_{55} \neq 0$.

And thus by items 1-7 we can define our basis as:

$$\begin{array}{rclllll}
AX = \overline{X} & = & a_{11}X & +a_{12}Y & +a_{13}Z & +a_{14}W_1 & +a_{15}W_2, \\
AY = \overline{Y} & = & & a_{22}Y & +a_{23}Z & +a_{24}W_1 & +a_{25}W_2, \\
AZ = \overline{Z} & = & & & \varepsilon_3 Z & +a_{34}W_1 & +a_{35}W_2, \\
AW_1 = \overline{W}_1 & = & & & & a_{44}W_1, & \\
AW_2 = \overline{W}_2 & = & & & & a_{54}W_1 & +a_{55}W_2.
\end{array}$$

□

**Definition 2.3.** For simplification of computations to follow, redefine the covariant derivatives of $R$ on the basis $\{X, Y, Z, W_1, W_2\}$ to be the following:

$h = \nabla R(X, Y, Y, X; X) = \sqrt{\alpha}(-2c''(x)b'(y) - 2)),$
$j = \nabla R(X, Y, Y, X; Y) = \alpha^2(-2c'(x)b''(y)).$

**Theorem 2.9.** *The nonzero covariant derivatives of $R$ on $A$ (up to the symmetries) are as follows:*

$\nabla R(\overline{X}, \overline{Y}, \overline{Y}, \overline{X}; \overline{X}) = \varepsilon_3 a_{11}a_{22}h + \varepsilon_3 {a_{22}}^2 j,$
$\nabla R(\overline{X}, \overline{Y}, \overline{Y}, \overline{X}; \overline{Y}) = \varepsilon_3 {a_{22}}^2 j.$

*Proof.*

$$\begin{aligned}
\nabla R(\overline{X}, \overline{Y}, \overline{Y}, \overline{X}; \overline{X}) &= \nabla R(a_{11}X, a_{22}Y, a_{22}Y, a_{11}X; a_{11}X + a_{12}Y) \\
&= \nabla R(a_{11}X, a_{22}Y, a_{22}Y, a_{11}X; a_{11}X) + \nabla R(a_{11}X, a_{22}Y, a_{22}Y, a_{11}X; a_{12}) \\
&= {a_{11}}^3 {a_{22}}^2 h + {a_{11}}^2 {a_{22}}^3 j = \varepsilon_3 a_{11}a_{22}h + \varepsilon_3 {a_{22}}^2 j, \\
\nabla R(\overline{X}, \overline{Y}, \overline{Y}, \overline{X}; \overline{Y}) &= \nabla R(a_{11}X, a_{22}Y, a_{22}Y, a_{11}X; a_{22}Y) \\
&= {a_{11}}^2 {a_{22}}^3 j = \varepsilon_3 {a_{22}}^2 j.
\end{aligned}$$

□

**Definition 2.4.** For simplifications, also redefine the second covariant derivatives of $R$ on the basis $\{X, Y, Z, W_1, W_2\}$ to be the following:

$t = \nabla^2 R(X, Y, Y, X; X, X) = -4c'(x)^2 b''(y) - 2c'''(x)b'(y),$
$v = \nabla^2 R(X, YY, X; X, Y) = \alpha\sqrt{\alpha}(-2c''(x)b''(y),$
$u = \nabla^2 R(X, YY, X; Y, Y) = \alpha^3(-2c'(x)b'''(y)).$

**Theorem 2.10.** *The nonzero second covariant derivatives of $R$ on $A$ (up to the symmetries) are as follows:*

$\nabla^2 R(\overline{X}, \overline{Y}, \overline{Y}, \overline{X}; \overline{X}, \overline{X}) = \varepsilon_3 t + 2\varepsilon_3 a_{11}a_{22}a_{12}v + a_{22}{a_{12}}^2 u,$
$\nabla^2 R(\overline{X}, \overline{Y}, \overline{Y}, \overline{X}; \overline{X}, \overline{Y}) = \varepsilon_3 a_{11}a_{22}v + \varepsilon_3 {a_{22}}^2 a_{12}u,$
$\nabla^2 R(\overline{X}, \overline{Y}, \overline{Y}, \overline{X}; \overline{Y}, \overline{Y}) = {a_{11}}^2 {a_{22}}^4 u.$

*Proof.*

$$\begin{aligned}
\nabla^2 R(\overline{X},\overline{Y},\overline{Y},\overline{X};\overline{X},\overline{X}) &= \nabla^2 R(a_{11}X, a_{22}Y, a_{22}Y, a_{11}X; a_{11}X + a_{12}Y, a_{11}X + a_{12}Y) \\
&= \nabla^2 R(a_{11}X, a_{22}Y, a_{22}Y, a_{11}X; a_{11}X, a_{11}X) + 2\nabla^2 R(a_{11}X, a_{22}Y, a_{22}Y, a_{11}X; a_{11}X, a_{12}Y) \\
&+ \nabla^2 R(a_{11}X, a_{22}Y, a_{22}Y, a_{11}X; a_{12}Y, a_{12}Y) \\
&= {a_{11}}^4 {a_{22}}^2 t + 2{a_{11}}^3 {a_{22}}^2 a_{12} V + {a_{11}}^2 {a_{22}}^2 {a_{12}}^2 u \\
&= \varepsilon_3 t + 2\varepsilon_3 a_{11} a_{22} a_{12} v + a_{22} {a_{12}}^2 u, \\
\nabla^2 R(\overline{X},\overline{Y},\overline{Y},\overline{X};\overline{X},\overline{Y}) &= \nabla^2 R(a_{11}X, a_{22}Y, a_{22}Y, a_{11}X; a_{11}X + a_{12}Y, a_{22}Y) \\
&= \nabla^2 R(a_{11}X, a_{22}Y, a_{22}Y, a_{11}X; a_{11}X, a_{22}Y) + \nabla^2 R(a_{11}X, a_{22}Y, a_{22}Y, a_{11}X; a_{12}Y, a_{22}Y) \\
&= {a_{11}}^3 {a_{22}}^3 v + {a_{11}}^2 {a_{22}}^3 a_{12} u \\
&= \varepsilon_3 a_{11} a_{22} v + \varepsilon_3 {a_{22}}^2 a_{12} u, \\
\nabla^2 R(\overline{X},\overline{Y},\overline{Y},\overline{X};\overline{Y},\overline{Y}) &= \nabla^2 R(a_{11}X, a_{22}Y, a_{22}Y, a_{11}X; a_{22}Y, a_{22}Y) \\
&= {a_{11}}^2 {a_{22}}^4 u.
\end{aligned}$$

□

**Theorem 2.11.** *Under changes to the covariant derivative of the curvature tensor, $\nabla R$, there exists a nonconstant quantity that is invariant under an action of the structure group, and hence $\mathcal{M}$ is not locally homogeneous.*

*Proof.* On the structure group we can compute:

$$\frac{\nabla R\left(\overline{X},\overline{Y},\overline{Y},\overline{X};\overline{Y}\right)^6}{\nabla^2 R\left(\overline{X},\overline{Y},\overline{Y},\overline{X};\overline{Y},\overline{Y}\right)^4} = \frac{(\varepsilon_3 {a_{22}}^2 j)^6}{({a_{11}}^2 {a_{22}}^4 u)^4} = \frac{{a_{22}}^8 j^6}{{a_{22}}^8 u^4} = \frac{j^6}{u^4} = \frac{\left(\alpha^2(2c'(x)b''(y))\right)^6}{(2c'(x)b'''(y))^4} = \frac{(2c'(x)b''(y))^6}{(2c'(x)b'(y)-1)(2c'(x)b'''(y))^4}.$$

Following the same computations, on the basis $\{X, Y, Z, W, W_1, W_2\}$, we obtain the following:

$$\frac{\nabla R\left(X,Y,Y,X;Y\right)^6}{\nabla^2 R\left(X,Y,Y,X;Y,Y\right)^4} = \frac{j^6}{u^4} = \frac{\left(\alpha^2(2c'(x)b''(y))\right)^6}{(2c'(x)b'''(y))^4} = \frac{(2c'(x)b''(y))^6}{(2c'(x)b'(y)-1)(2c'(x)b'''(y))^4}.$$

Since this is generally nonconstant, $\mathcal{M}$ is not locally homogeneous. □

# Open Questions

1. This research on the manifolds was limited to curvature homogeneity. There are a lot of other properties that have yet to be explored on these specific manifolds. What other properties exist within these manifolds? For example, are they indecomposable or complete? Are they weak curvature homogeneous and homothety curvature homogeneous?

2. These manifolds are similar in terms of the metric, curvature tensors, and formation of the structure groups. Is it possible to continue finding examples of curvature homogeneous manifolds with these specific conditions for the mapping of the covariant derivative and could you work to generalize the results? What happens when you try to expand the path length beyond four?

3. Can you compute the structure groups of common model spaces arising from generalized plane wave manifolds and compute invariants of these assumed curvature homogeneous manifolds?

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