# Volume Bounds of T-links

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**Abstract**

We prove that all left nested $(p, q)$ torus links are octahedral, and determine the number of octahedra that make up left nested and complete nested $(p, q)$ torus links. Using these results, we improve known volume bounds for T-links that have nested $(p, q)$ torus link parents. Finally, we find lower bounds on volumes for this class of links.

## 1 Introduction

Guckenheimer and Williams introduced the Lorenz template, which is an embedded branched surface in $\mathbb{R}^3$ with a semi-flow. We define the Lorenz braid to be the braid on the Lorenz template as shown below (Figure taken from [2]).

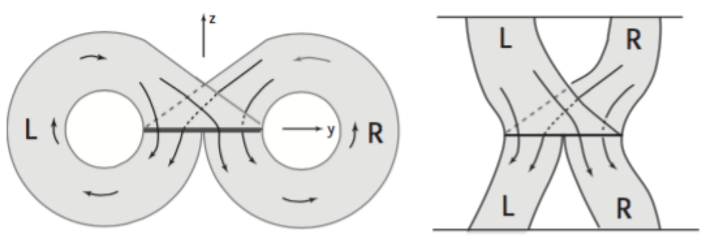



Figure 1: (a)The Lorenz template. (b)The Lorenz braid template

The $(p, q)$ Lorenz braid is the braid existing on the Lorenz template with $p$ strands crossing over from the left to the right, and $q$ strands crossing under from the right to the left, as seen in Figure 1. The left nested $(p, q)$ torus link (Figure 2) is the link on the Lorenz template with the crossing disks on the left lobe, with the innermost crossing disk twice punctured by the two left most strands and the remaining crossing circles following consecutively, all being twice punctured. The complete nested $(p, q)$ torus link (Figure 3) is the link on the Lorenz template with the innermost crossing disk being twice punctured by the two left strands in the Lorenz braid, and the remaining crossing disks following consecutively, being twice punctured.

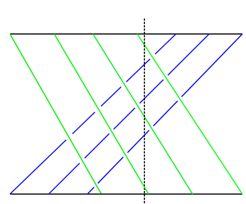


Figure 1: $(p, q)$ Lorenz braid

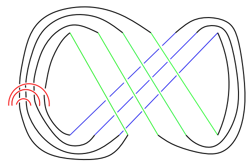


Figure 2: Left nesting

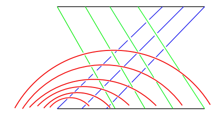


Figure 3: Complete nesting

The process of augmenting a link is useful in the study of hyperbolic links. Augmenting a link starts with a twist-reduced link diagram and adds a trivial component around twist regions [1]. The complement of the result is homeomorphic to the link obtained by removing all full twists [9]. Purcell generalized this process to add trivial components around many strands [10] and Harnois et al introduced the idea of nesting trivial components around multiple strands [8].

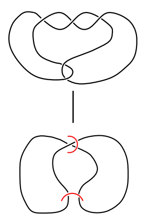


Figure 4: Augmented link

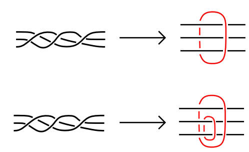


Figure 5: Generalized and nested augmented link

In this paper we will use nested links to obtain volume bounds for an infinite class of links through Dehn filling. Dehn filling is the process of gluing a solid torus to a component of the link, where the meridian of the solid torus glues to the $(1, n)$ curve, where $n$ is the number of times going around the longitude. The link obtained after Dehn filling is know as the *offspring* and the link before Dehn filling is known as the *parent*. It is beneficial to us to study different nestings when Dehn filling because certain nestings give us information on whether a link is hyperbolic or octahedral.

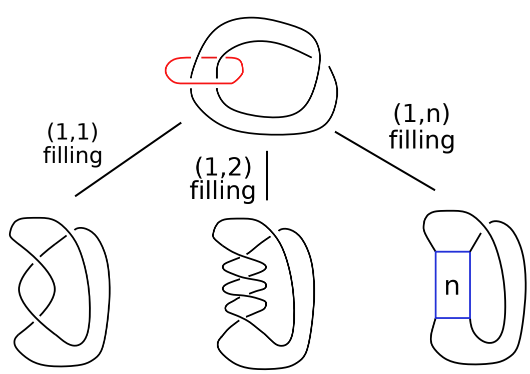



Figure 6: Dehn filling

The filled crossing circle becomes erased and full twists are added. The rectangle under the $(1, n)$ filling represents n full twists. We know that Dehn filling reduces volume [11]; thus, we can bound the volume of offspring by adding full twists through Dehn filling. In a hyperbolic link complement M, a neighborhood of each component of the link corresponds to a *cusp* of M. A cusp is the horospherical neighborhood of an ideal vertex, and a *horoball* is the inside of the horosphere. Longitudes are parallel to the intersection of shaded faces with the cusp boundary and meridians are parallel to the intersection of white faces with the cusp boundary [9].

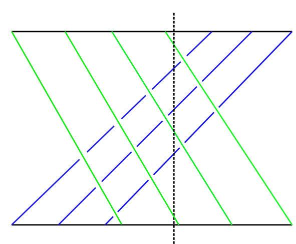
Let $M$ be a complete, finite-volume hyperbolic manifold with cusps. Let $s_1$,...,$s_k$ be the slopes of disjoint horoball neighborhoods of a subset of the cusps. We know we can do Dehn fillings on these cusps. The resulting manifold after the Dehn filling is denoted, $M(s_1, ..., s_k))$. Denote the minimal slope length of the cusps by $l_{min}$. The main tool for our lower bound is the following:

**Theorem 1.1** ([7]). *Given that the above hypothese are met, then*

$$vol(M(s_1, ..., s_k)) \geq \left(1 - \left(\frac{2\pi}{l_{min}}\right)^2\right)^{\frac{3}{2}} vol(M)$$

## 2 Octahedral Links

Since our goal is to bound volume on a class of links that are octahedral, we will begin by proving which specific class of links are known to be octahedral. Looking at the Lorenz template, label the top and bottom strings $1, 2, ..., n$ where $p + q = n$. We will proceed by following the mapping where $i \mapsto i + q$ for all $i \in (p, q)$, which we know determines a unique $(p, q)$ [3] as shown below.



**Lemma 2.1** ([6]). *All left nested (p,q) torus links following the mapping described above and where $p \neq q$ and $p + q \geq 4$ are octahedral.*

*Proof.* To show this is octahedral, we will use a result of Purcell which says that if the nerve of a circle packing is a central subdivision of $K_4$, the complete graph on four vertices, then the polyhedra corresponding to the circle packing is octahderal [9, Proposition 3.8]. We will prove this through induction. Assume this is true for some left nested $(p, q)$, then the general cell decomposition is shown in Figure 7.

The upper regions, denoted by $x_i$, are made by the strands on the Lorenz template that stay in the left lobe, known as left-left strands. There will be $p - q$ of these regions. The lower regions, denoted by $y_i$, are made by the strands on the Lorenz template that cross from the left lobe to the right lobe and that cross from the right lobe to the left lobe, known as left-right and right-left strands. There are $q$ of these regions. Thus, the generalized circle packing and nerve of the left nested $(p, q)$ are shown in Figure 8 and in Figure 9.

We note that the set of vertices $w$, $x_1$, $x_2$, and $y_1$ forms $K_4$. We also note that $deg(y_q) = 3$, $deg(y_{q-1}) = deg(y_{q-2}) = ... = deg(y_2) = 4$, and $deg(x_{p-q}) = 3$, $deg(x_{p-q-1}) = deg(x_{p-q-2}) = ... = deg(x_2) = 4$. We can delete $y_q$ from the nerve, resulting in $deg(y_{q-1}) = 3$. We can continue this pattern to delete vertices $y_q$ through $y_2$. Similarly, we can delete vertices $x_{p-q}$ through $x_3$. Thus, the remaining vertices, $w$, $x_1$, $x_2$, and $y_1$ form $K_4$. Reversing this process shows that the nerve is a central subdivision of $K_4$.

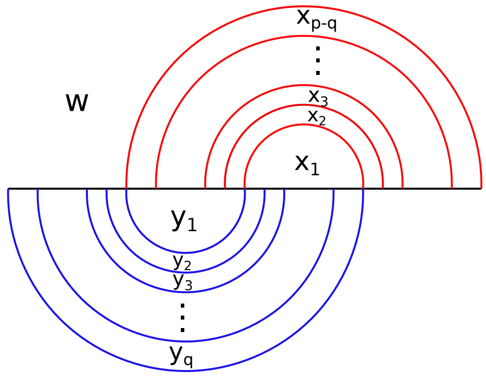



Figure 7: General left nested (p,q) cell decomposition

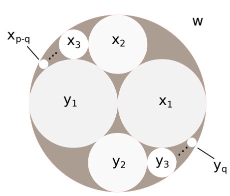



Figure 8: Left nested $(p, q)$ circle packing

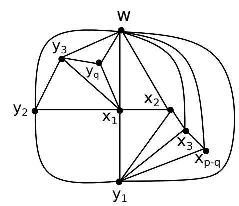



Figure 9: Left nested $(p, q)$ nerve

If we have the left nested $(p + 1, n)$ torus link where $n < p + 1$, we would either have a $y_{q+1}$ or an $x_{p-q+1}$ vertex both with degree 3. Without loss of generality assume there is a $y_{q+1}$ vertex. The vertices can then be deleted in the same pattern as described above with the remaining vertices being $w$, $x_1$, $x_2$, and $y_1$ which results in $K_4$. Therefore, the nerve is a central subdivision of $K_4$, implying that $(p + 1, n)$ is octahderal. □

**Corollary 2.1.1.** *Any left nested $(p, q)$ torus link is made from gluing $2(p - 2)$ octahedra.*

*Proof.* Referring to Figure 8 and Figure 9, the circle packing for the left nested $(p, q)$ torus link consists of $p + 1$ unshaded regions, one of them coming from the outside region, $p - q$ of them coming from the left-left strands, and $q$ of them coming from the left-right and right-left strands. The nerve that results from the circle packing will contain $p + 1$ vertices, each coming from the unshaded faces. We know that each central subdivision of $k_4$ is equivalent to gluing an additional octahedron [9, Proposition 3.8]. Thus, four of these vertices will create one octahedron, and each additional vertex which glue an addition vertex, resulting in $p - 2$ octahedra. These octahedra form one of the regions, $P_+$ that results from the cellular decomposition of the link. Another $p - 2$ octahedra form the second region, $P_-$ of the cellular decomposition. Thus, there are $2(p - 2)$ octahedra. □

We know that nesting a $(p, q)$ torus link on the left gives us an octahedral link. It is of interest to us to study other nesting types, such as complete nesting, to see if links retain the same properties.

**Lemma 2.2** ([5])**.** *All complete nested $(p, q)$ torus links following the mapping described in Lemma 2.1 and where $p \neq q$, and $p + q \geq 4$, are octahedral.*

**Corollary 2.2.1.** *Any complete nested $(p, q)$ torus link is made from gluing $2(p + q - 2)$ octahedra.*

*Proof.* Following the reasoning in the previous corollary, the circle packing for the complete nested $(p, q)$ torus link consists of $p + q + 1$ unshaded regions, one of them coming from the outside regions, $p$ of them coming from the left strands, and $q$ of them coming from the right strands. The nerve thats results from the circle packing will contain $p+q+1$ vertices, each coming from the unshaded faces. Since each central subdivision adds another octahedron, four of these vertices will create one octahedron, and each subsequent vertex is an additional octahedron. This results in $p+q-2$ octahdera for $P_+$ and $P_-$ in the cellular decomposition. Therefore, there are $2(p+q-2)$ octahedra. □

## 2.1 T-links with octahedral parents

We define Lorenz links to be all links on the Lorenz template [2] as shown above in Figure 1. The link defined by the closure of the braid $(\sigma_1 ... \sigma_{r-1})^s$ is a Torus link $T(r,s)$. For $2 \leq r_1 \leq ... \leq r_k$, $0 < s_i$, $i = 1,...,$ $k$, let $T((r_1, s_1), ..., (r_k, s_k))$ be the link defined by the closure of the following braid, all of whose crossing are positive:

$$\mathbb{T} = (\sigma_1 \sigma_2 ... \sigma_{r_1-1})^{s_1} (\sigma_1 \sigma_2 ... \sigma_{r_2-1})^{s_2} ... (\sigma_1 \sigma_2 ... \sigma_{r_k-1})^{s_k}$$

.

We call $\mathbb{T}$ a T-braid, and refer to the link T that its closure defines as a T-link [2]. We know that every Lorenz link is a T-link, and every T-link is a Lorenz link [2, Theorem 1]. By Lemma 1.1, we know that all nested $(p,q)$ torus links are octahedral. Thus, we will describe all the T-links that come from the nested $(p,q)$ parent. Since we can study the parent and offspring of a nested $(p,q)$ torus link, we can add full twists to the strands coming through any of the crossing disks as seen in Figure 10. Adding any number of full twists to the strands results in the T-link in Figure 11 where the rectangles contain any multiple of full twists and the largest rectangle can have any number of twists.

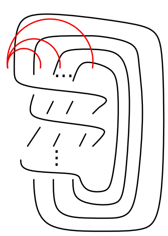


Figure 10: Nested (p,q) Torus Link

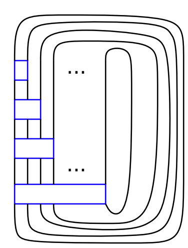


Figure 11: Generalized T-link

Thus, the T-links that are known to have octahedral parents can be written in the form:

$$T((2,2a)(3,3b)(4,4c)...(p,q+pn))$$

where $a, b, c, ...n \in \mathbb{Z}$.

# 3 Upper Volume Bound

Champanerkar et al proved that if $L$ is the T-link $T((r_k, s_k), ...(r_1, s_1), (p,q))$, then $Vol(L) < v_3(\frac{1}{3}r_1^3 + \frac{5}{2}r_1^2 + 5r_1 - 5)$ [4, Theorem 1.7]. We'll refer to this bound as the CFKNP bound in honor of the authors. It's noteworthy that even though it takes many parameters to specify a T-link, it takes a single coordinate to bound the volume.

**Lemma 3.1.** *If $L$ is the T-link $T((r_k, s_k), ...(r_1, s_1), (p, q))$, then*

$$Vol(L) < 2(p-2)v_8, \quad \text{if } L \text{ is left nested}$$
$$Vol(L) < 2(p+q-2)v_8, \quad \text{if } L \text{ is complete nested}$$

*Proof.* By Corollary 2.1.1 and Corollary 2.2.1, $L$ is made up of $2(p-2)$ octahedra if it is left nested and $2(p+q-2)$ octahedra if it is complete nested. Since all ideal octahedra have volume less than the regular ideal octahedra, it follows that $Vol(L) < 2(p-2)v_8$ for the left nesting and $Vol(L) < 2(p+q-2)v_8$ for the complete nesting. □

**Theorem 3.2.** *The bounds of Lemma 3.1 improve on the CFKNP bound when $r_1 \geq 2$, $p \geq 3$, and $\sqrt[3]{6pv_8} < r_1$.*

*Proof.* We know $Vol(L) < (\frac{1}{3}r_1^3 + \frac{5}{2}r_1^2 + 5r_1 - 5)v_3$ [4, Theorem 1.7]. Since $r_1 \geq 2$, we know $\frac{5}{2}r_1^2 + 5r_1 - 5 > 0$ and since $v_3 > 1$, we have $\frac{1}{3}r_1^3 < (\frac{1}{3}r_1^3 + \frac{5}{2}r_1^2 + 5r_1 - 5)v_3$. Now replacing $r_1$ with $\sqrt[3]{6pv_8}$ we get

$$\begin{aligned}(\frac{1}{3}r_1^3 + \frac{5}{2}r_1^2 + 5r_1 - 5)v_3 &> \frac{1}{3}r_1^3 \\ &> \frac{1}{3}6pv_8 \\ &> 2(p-2)v_8\end{aligned}$$

By Lemma 3.1, we know that $Vol(L) < 2(p-2)v_8$. Therefore, $Vol(L) < 2(p-2)v_8 < (\frac{1}{3}r_1^3 + \frac{5}{2}r_1^2 + 5r_1 - 5)v_3$. □

We note that the new volume upper bound from Lemma 3.1 improves on the upper bound from Theorem 1.7 of [4] for $p$ and $r_1$ closer together. Once $r_1$ becomes small relative to $p$, the upper bound from Theorem 1.7 becomes the more precise bound.

We will proceed by giving an upper bound on the complete nested $(p, q)$ torus link. Since it is possible for the complete nesting to contain more crossing circles than the left nesting, this upper bound will be larger than the upper bound from Lemma 3.1.

**Theorem 3.3.** *The bounds of Lemma 3.1 improve on the CFKNP bound when $p \geq 2$, $q \leq \frac{p^3}{6v_8} - p$, $\sqrt[3]{6(p+q)v_8} \leq r_1$ and $p < r_1 \leq p + |q|$.*

*Proof.* Following the same reasoning as in the previous proof, but now replacing $r_1$ with $\sqrt[3]{6(p+q)v_8}$ we get

$$\begin{aligned}(\frac{1}{3}r_1^3 + \frac{5}{2}r_1^2 + 5r_1 - 5)v_3 &> \frac{1}{3}r_1^3 \\ &> \frac{1}{3}6(p+q)v_8 \\ &> 2(p+q-2)v_8\end{aligned}$$

By hypothesis, $q \leq \frac{p^3}{6v_8} - p \Rightarrow \sqrt[3]{6(p+q)v_8} \leq p$, and since $p < r_1$, it holds that $\sqrt[3]{6(p+q)v_8} \leq r_1$, meaning $(\frac{1}{3}r_1^3 + \frac{5}{2}r_1^2 + 5r_1 - 5)v_3 > 2(p+q-2)v_8$. If $q > \frac{p^3}{6v_8} - p$, it is still possible that $(\frac{1}{3}r_1^3 + \frac{5}{2}r_1^2 + 5r_1 - 5)v_3 > 2(p+q-2)v_8$ given that $r_1$ is sufficiently large. By Lemma 3.1, we know that $Vol(L) < 2(p+q-2)v_8$.

Therefore, $Vol(L) < 2(p+q-2)v_8 < (\frac{1}{3}r_1^3 + \frac{5}{2}r_1^2 + 5r_1 - 5)v_3$.

Since $q > 0$ the complete nested $(p, q)$ torus link will have larger volume than the left nesting, thus the left nesting would provide a better upper bound for volume.

□

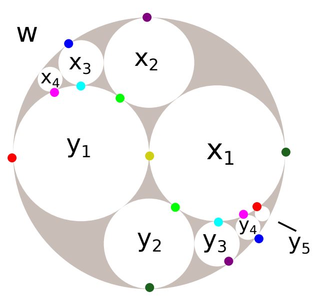
# 4 Lower Volume Bound

We will give a lower bound to the offspring of the left nested $(p, q)$ torus link, which we know to be octahedral, through Dehn filling. Since the circle packing for each cusp, when sent to infinity, have different lengths, we determine our lower bound by calculating the shortest length.

**Theorem 4.1.** *Let $n > 3$ and let $n$ be constant for each of the crossing circles. Define $M' = M(s_1, s_2, ..., s_k)$. Then by using Theorem 1.1 we can give lower bound*

$$Vol(M') \geq \left(1 - \frac{\pi^2}{1+n^2}\right)^{\frac{3}{2}} Vol(M)$$

*Proof.* To prove this lower volume bound specifically for octahedral links, we will show that $l \geq 2\sqrt{1+n^2}$, by looking at the left nested $(9, 5)$ torus link and then generalizing. The circle packing along with corresponding cusps are shown below. The outer crossing circle corresponds to the cusp between $y_1$ and $x_1$, resulting in two rectangles for the circle packing. The remaining crossing circles will correspond to two cusps, resulting in four rectangles for the circle packing.




We will begin with the outer most crossing disk, which corresponds to the cusp between $y_1$ and $x_1$. This will result in the following circle packing, as shown below, in Figure 12.

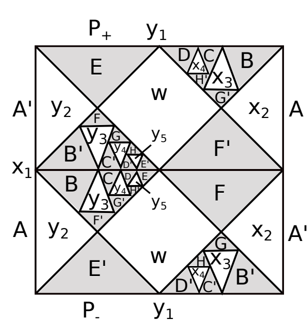



Figure 12: Circle packing with cusp sent to infinity

Sending this cusp to infinity results in the $A'$ and $A$ shaded faces becoming vertical and the $y_1$ and $x_1$ unshaded faces to become horizontal. The rest of the unshaded and shaded faces can be filled in according to their position relative to the other regions, with remaining unshaded faces being tucked into the shaded triangular faces in the outer corners. To get $P_-$ reflect across $x_1$, the plane of projection, and change the shaded faces from prime to not prime and vice versa. Since there is a single cusp for this crossing disk, in order to obtain a fundamental region, instead of gluing laterally side to side, we must glue on a diagonal as seen below.

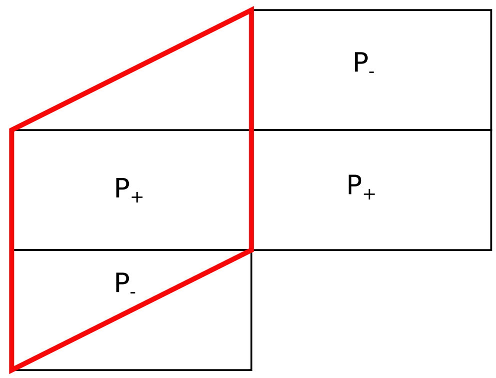



Figure 13: Fundamental Region

with $P_+$ gluing to $P_-$. The resulting fundamental region would be a parallelogram rather than a rectangle. We know that the horoballs can expand about the cusps so that the midpoint of every edge is a point of tangency [9, Theorem 3.4]. In our case, the midpoints will meet, so that the radius is $\frac{1}{2}$, thus, the length of each tile will be 1 unit. Each time we Dehn fill, we attach another fundamental region vertically in order to go across $n$ longitudes, our vertical sides. Performing a $(1, n)$ Dehn filling would require finding the distance between $(0, 0)$ to $(1, 2n + 1)$, which is $\sqrt{4n^2 + 4n + 2}$.

We can send the other cusps to infinity and by following this same process find the rest of the circle packings, which will have varying lengths. Next, we will look at the inner most crossing disk. If we send the inner crossing disk to infinity, corresponding to the cusps tangent to $w$ and $x_3$ and to $w$ and $y_4$ we get the following circle packing, as shown below in FIgure 14.

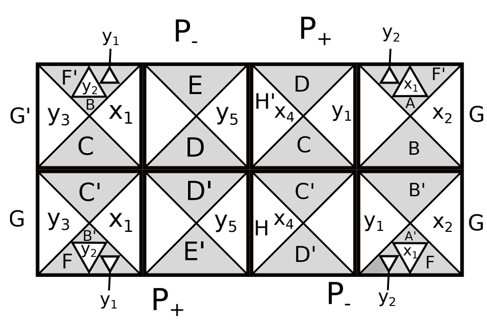
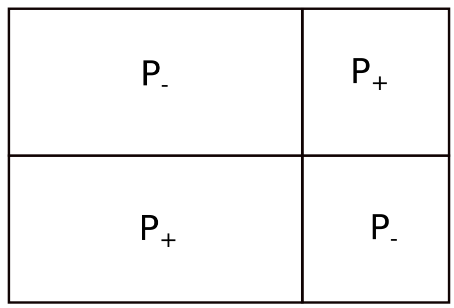



Figure 14: Circle packing with a cusp sent to infinity

Sending the upper cusp to infinity results in the $H'$ and $G'$ shaded faces becoming vertical and the $W$ and $X_3$ white faces to become horizontal. The rest of the unshaded and shaded faces can be filled in according to their position relative to the other regions, with remaining unshaded faces being tucked into the shaded triangular faces in the outer corners. To get $P_{-}$ reflect across $x_3$, the plane of projection, and change the shaded faces from prime to not prime and vice versa. This process gets repeated for the second lower cusp, and the two regions get glued together such that $H$ and $H'$ match up with themselves by rotating the second circle packing.

One can compute the lengths of the $(1, n)$ slopes for all the crossing circle cusps. For $q \neq p-1$, the crossing disk that will correspond to the circle packing with the shortest length will be the $(p-q)^{th}$ innermost crossing disk. For this case, the shortest length will come from the $4^{th}$ innermost crossing disk. This corresponds to the cusps between $x_4$ and $y_1$ and between $x_1$ and $y_4$. The fundamental region would have the following configuration:




The minimal length would be $\sqrt{9+4n^2}$. Although this is Euclidean length, we know that since the horospheres at a height of $z = 1$ the hyperbolic length will equal the Euclidean length [7]. We use this minimal length in the volume bound proven by [7]. We note that the lengths of all the cusps are of the same degree, so as $n$ gets larger, the lengths of each of the cusps will get closer together.

This process holds for the complete nested $(p, q)$ torus link, since we know that the cell decomposition with this nesting follows the same configuration as the cell decomposition for the left nesting. The complete nesting results in $q$ more circles in the circle packing for $P_{+}$. We proceed to find the shortest length to calculate the volume given by [7]. For example, looking at the complete nested $(5, 4)$ torus link, after the cellular decomposition, we have the following circle packing shown in Figure 15. We note that the circle packing for the complete nested $(5, 4)$ torus link is the same as the circle packing for the left nested $(p, q)$ torus link.

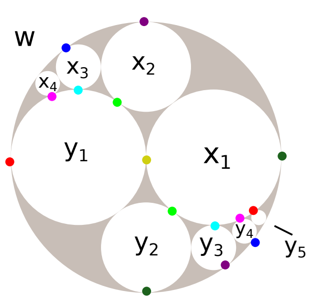



Figure 15: Complete nested $(5, 4)$ torus link

Even though the complete nesting has more circles than the left nesting, once we send any of the cusps to infinity, the additional circles, the unshaded faces, will get tucked into a single shaded face, as seen in the corners of Figure 14. Thus, the length of the tiles does not increase. Comparing the left and complete nesting for the nested $(p, q)$ torus link, the minimum length for the left nesting is $\sqrt{4n^2 + 4n + 2}$ while the minimum length for the complete nesting is $\sqrt{4n^2 + 9}$. We can see that for $n > 1$, $l_{min}$ for the complete nesting will be less than $l_{min}$ for the left nesting.

We observe that the complete nesting will have the smaller $l_{min}$ compared to the left nesting for all $(p, q)$ except for the $(2p, p)$ case. For example, we will look at $(6, 3)$.

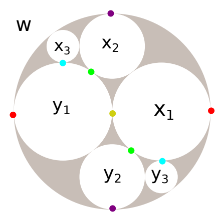



Figure 16: Left nested $(6, 3)$ circle packing

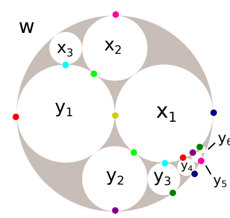



Figure 17: Complete nested $(6, 3)$ circle packing

The left nesting will have a minimum length of $\sqrt{4n^2 + 4}$ while the left nesting will have a minimum length of $\sqrt{4n^2 + 9}$. The result of this is that for most cases, the complete nesting will give a better lower volume bound compared to the left nesting. We contrast this with the upper volume bound, where using the left nesting gives a better bound than the complete nesting.

We note that the minimal length from the previous example will be the minimal length for any cusp in any $(p, q)$ torus link. This is due to the fact that the cusps with the shortest occur on the smallest circles on the upper left and lower right. In order to get the minimal length, these two cusps must correspond to the same crossing disk, which occurs in the $(2p, p)$ torus link. Every other case will result in lengths larger than this. Thus, $l_{min} \geq \sqrt{4n^2 + 4}$ and plugging this into the lower volume equation will give us the new lower bound.

$\square$

The previous upper and lower bounds were given for general torus links. By looking at a specific class of links, left and complete nested torus links, which are known to be octahedral, we were able to improve on the volume bound. This gives an infinite class of links that these volume bounds apply to.

# 5 Open Questions

Questions for further research:

- Are different nestings for $(p, q)$ torus links still octahedral?
- Can the current lower bound be improved on for octahedral torus links?
- What bounds can be achieved for non-octahedral nested Lorenz links?

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