An Upper Bound on $\eta(n)$

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Abstract

Algebraic curvature tensors are tools in differential geometry which we use to determine the curvature of a manifold. Some of these curvature tensors can be constructed from antisymmetric bilinear forms. The number of algebraic curvature tensors built from antisymmetric forms in a vector space of dimension n is denoted $\eta(n)$. In this paper we improve the upper bound on $\eta(n)$ when $n \geq 5$.

1 Introduction

1.1 The Algebraic Curvature Tensor

Let V be a real vector space such that $dim(V) = n$ and a basis for V is given by $\{e_1, e_2, \ldots, e_m, e_n\}$, where $m = n - 1$. Let $\mathcal{A}(V)$ denote the set of algebraic curvature tensors on V. For any $R \in \mathcal{A}(V)$ and vectors $x, y, z, w \in V$, $R : V \times V \times V \times V \to \mathbb{R}$ has the following properties:

1. R is multilinear.
2. $R(x, y, z, w) = -R(y, x, z, w)$
3. $R(x, y, z, w) = R(z, w, x, y)$
4. $R(x, y, z, w) + R(x, w, y, z) + R(x, z, w, y) = 0$

The last of these properties is called the Bianchi identity.

Since algebraic curvature tensors are multilinear, an algebraic curvature tensor is completely determined by how it acts on a basis. For ease of notation, we will denote $R(e_i, e_j, e_k, e_l)$ with R_{ijkl}, indicies not necessarily distinct.

From the properties of algebraic curvature tensors , we know that $R_{iiii} = R_{iiij} = 0$. In fact, due to the three defining properties of an algebraic curvature tensor , the only nonzero entries of R are those of the form R_{ijji}, R_{ijki}, or R_{ijkl}, with distinct letters denoting distinct indicies.

It is known that if the $dim(V) = n$, then $dim(\mathcal{A}(V)) = \frac{n^2(n^2-1)}{12}$. In fact, the independent entries of R can be organized in a column vector like so:

$$R := \begin{bmatrix} R_{1221} \\ R_{1331} \\ \vdots \\ R_{mnnm} \\ ----\\ R_{3123} \\ R_{4124} \\ \vdots \\ R_{n12n} \\ R_{2132} \\ \vdots \\ R_{imni} \\ ----\\ R_{1234} \\ R_{1423} \\ \vdots \\ R_{klmn} \\ R_{knml} \end{bmatrix},$$

The column has length $dim(\mathcal{A}(V))$. This specific arrangement for the entries of R is incredibly useful for their study and is worth detailing.

The first section, above the first dotted line, contains the R_{ijji} entries of the algebraic curvature tensor R, and are arranged with $i < j$, and j cycles through all the other indicies that are not i, and then incrementing i and continuing the process. Note that by the second defining property of algebraic curvature tensors , $R_{ijji} = R_{jiij} = -R_{ijij}$, so it is redundant to list all of them.

The next section, between the two dotted lines, contains the R_{ijki} entries of R, where $j < k$. Since $R_{ijki} = R_{ikji}$, we need not list both in our list. These entries are organized in "clumps" of $n-2$ sorted by entries of the form R_{iabi}, where a and b are fixed and i cycles through the indicies that are not a or b. Once the cycle is complete, we increment the fixed indices j, k and repeat the process.

The last section, below the second dotted line, contains the R_{ijkl} entries, those with four distinct vectors input into the algebraic curvature tensor R. Note that by the Bianchi identity,

$$R_{iklj} = -R_{ijkl} - R_{iljk}, \tag{1}$$

which then implies that R_{ijlj} is dependent upon the first two listed in that equality, and as such is not needed in the column vector.

1.2 Antisymmetric Forms and Curvature Tensors

An antisymmetric form, ψ, is a bilinear mapping $V \times V \to \mathbb{R}$ such that for any vectors $x, y \in V$,

$$\psi(x, y) = -\psi(y, x).$$

Conveniently, these forms can be represented by antisymmetric matricies whose (i, j) entry is $\psi(e_i, e_j)$. That is, if an antisymmetric bilinear form is represented by a matrix, ψ, then $\psi^T = -\psi$. It is known that antisymmetric forms like these can be used to construct algebraic curvature tensors. Any desired entry of an algebraic curvature tensor R can be determined like so:

$$(R_\psi)_{ijkl} = \psi(e_i, e_l)\psi(e_j, e_k) - \psi(e_i, e_k)\psi(e_j, e_l) - 2\psi(e_k, e_l) \tag{2}$$

$$\Leftrightarrow (R_\psi)_{ijkl} = \psi_{il}\,\psi_{jk} - \psi_{ik}\,\psi_{jl} - 2\,\psi_{ij}\,\psi_{kl}.$$

We denote as $\mathcal{A}_\Lambda(V)$ the set of algebraic curvature tensors constructed in this way.

Similarly, if ϕ is a symmetric bilinear form on V, symbolically denoted $\phi \in S^2(V)$, then ϕ can be used to create another algebraic curvature tensor using the following formula:

$$(R_\phi)_{ijkl} = \phi(e_i, e_l)\phi(e_j, e_k) - \phi(e_i, e_k)\phi(e_j, e_l)$$

$$\Leftrightarrow (R_\phi)_{ijkl} = \phi_{il}\,\phi_{jk} - \phi_{ik}\,\phi_{jl}.$$

We say that algebraic curvature tensors constructed in this way belong to the set $\mathcal{A}_S(V)$. The following theorem is a result from Gilkey [1].

Theorem 1. *$span\{\mathcal{A}_\Lambda(V)\} = span\{\mathcal{A}_S(V)\} = \mathcal{A}(V)$.*

Since this theorem tells us that the algebraic curvature tensors on antisymmetric forms span $\mathcal{A}(V)$, the question arises as to how many of these algebraic curvature tensors from $\mathcal{A}_\Lambda(V)$ are necessary to linearly combine into a desired algebraic curvature tensor R.

Definition. *The minimum number of algebraic curvature tensors from $\mathcal{A}_\Lambda(V)$ needed to linearly combine into a given algebraic curvature tensor R is denoted $\eta(R)$.*
The minimum number of algebraic curvature tensors from $\mathcal{A}_\Lambda(V)$ needed to linearly combine into any arbitrary algebraic curvature tensor in a vector space of dimension n is denoted $\eta(n)$.

Using this language, we can phrase our question like so: What is $\eta(n)$? An analogous question is asked for algebraic curvature tensors constructed from symmetric bilinear forms and has been studied much more than its antisymmetric counterpart.
In fact, very little is known about $\eta(n)$. Other authors have shown that $\eta(3) = 3$ and $\eta(4) \leq 11$[3][4]. However, the only upper bound that was known up to now for the general n dimensional case was that

$$\eta(n) \leq dim(\mathcal{A}(V)) = \frac{n^2(n^2-1)}{12}.$$

In this paper we aim to improve on this upper bound.

2 Methods

We will introduce some background that lays the foundation for an algorithm which will construct any desired algebraic curvature tensor as a linear combination of algebraic curvature tensors from $\mathcal{A}_\Lambda(V)$.

2.1 The Ricci Decomposition

The following theorem gives us a useful way to decompose an arbitrary algebraic curvature tensor into simpler parts [2].

Theorem 2 (The Ricci Decomposition)**.** *If $R \in \mathcal{A}(V)$, then*

$$R = S + E + C,$$

where S is the scalar *part, E is the* semi-traceless *part, and C is* the traceless Weyl tensor*.*

These tensors are defined below.

2.1.1 The Scalar Curvature Component

Choose $\langle \cdot, \cdot \rangle$ to be a positive definite metric on our vector space V, and let the scalar curvature of V be denoted by τ. Then the scalar part of the Ricci decomposition above can be expressed as

$$S = \frac{\tau}{n(n-1)} R_{\langle \cdot, \cdot \rangle}$$

Theorem 3. *If $\langle \cdot, \cdot \rangle$ is a positive definite metric on a vector space V and $\phi \in S^2(V)$, then there exists some orthonormal basis $\mathcal{B} = \{b_1, b_2, \ldots, b_n\}$ such that*

$$\phi(b_i, b_j) = \begin{cases} \lambda_i & , i = j \\ 0 & , i \neq j \end{cases}$$

Lemma 1. *Let S be the scalar part of a Ricci decomposed algebraic curvature tensor. Then the only nonzero entries of S are of the form S_{ijji}.*

Proof. Since the metric $\langle \cdot, \cdot \rangle$ belongs to the set $S^2(V)$, converting to the basis which is guaranteed by Theorem 3, gives us that

$$S_{ijkl} = \frac{\tau}{n(n-1)} (\langle e_i, e_l \rangle \langle e_j, e_k \rangle - \langle e_i, e_k \rangle \langle e_j, e_l \rangle) \neq 0 \Rightarrow i = l \text{ and } j = k \text{ OR } i = k \text{ and } j = l.$$

$\square$

2.1.2 The Semi-Traceless Component

The semitraceless component of the Ricci decomposition, E, is defined as a multiple of the Kulkarni-Nomizu product of the inner product of a vector space and the Ricci tensor of an algebraic curvature tensor. As such, the entries of E are given as follows:

$$\begin{aligned} E_{ijkl} = [&\langle e_i, e_k\rangle\rho(e_j, e_l) \\ &+ \langle e_j, e_k\rangle\rho(e_i, e_k) \\ &- \langle e_i, e_l\rangle\rho(e_j, e_k) \\ &- \langle e_j, e_k\rangle\rho(e_i, e_l)] \end{aligned}$$

Using similar logic to the reasoning behind 1, we arrive at the following result:

Lemma 2. *Let E be the scalar part of a Ricci decomposed algebraic curvature tensor. Then the only nonzero entries of E are of the form E_{ijji} on a certain basis.*

Proof. Let V be a real vector space, R be an algebraic curvature tensor on V, and E be the semi-traceless part of the Ricci decomposed R. Choose the basis guaranteed by Theorem 3 to be the basis for V.
If an entry of E, say E_{ijkl}, with indicies not necessarily distinct, was nonzero, then at least one term from

$$\begin{aligned} E_{ijkl} = [&\langle e_i, e_k\rangle\rho(e_j, e_l) \\ &+ \langle e_j, e_k\rangle\rho(e_i, e_k) \\ &- \langle e_i, e_l\rangle\rho(e_j, e_k) \\ &- \langle e_j, e_k\rangle\rho(e_i, e_l)] \end{aligned}$$

must be nonzero. Without loss of generality, assume the first term,

$$\langle e_i, e_k\rangle\rho(e_j, e_l) \neq 0$$

$$\Rightarrow \langle e_i, e_k\rangle \neq 0 \text{ and } \rho(e_j, e_l) \neq 0$$
$$i = k and j = l$$

.
Choosing other terms to be nonzero gives similar results. Thus, if an entry of E is nonzero, it must take only two distinct inputs, meaning the nonzero entries must all be of the form E_{ijji}. ∎

2.1.3 The Weyl Tensor Component

Following directly from 1 and 2, we arrive at the following result:

Lemma 3. *Let R be an algebraic curvature tensor with Ricci decomposed parts S, E, and C, and let the vector space they reside in be diagonalized with respect to a positive definite inner product on V. Then the only nonzero entries of R of the form R_{ijki} or R_{ijkl} are those contributed by the Weyl component C.*

Proof. Let V be a vector space and R be an algebraic curvature tensor on V with Ricci decomposition $R = S + E + C$. Also, let the orthonormal basis described in Theorem 3 be our basis for V.
By Lemma 1 and Lemma 2, we know that the only nonzero entries of S and E are of the form R_{ijji}. Thus, if the original tensor has nonzero entries of th form R_{ijki} or R_{ijkl}, then they must be from the Weyl component, C. □

Apart from being the only component of R that contributes nonzero R_{ijki} and R_{ijkl} entries upon diagonalization with respect to an inner product, the Weyl tensor C also has the useful property of being *Ricci flat*. That is, for all vectors $x, y \in V$,

$$\rho_C(x, y) = \sum_{i=1}^{n} C(e_i, x, y, e_i) = 0.$$

Choosing x and y to be basis vectors, e_a, e_b ($a < b$), we get:

$$\rho_C(e_a, e_b) = \sum_{i=1}^{n} C(e_i, e_a, e_b, e_i) = 0.$$

Expanded out, this sum looks like

$$C_{1ab1} + C_{2ab2} + \cdots + (C_{aaba}) + \cdots + C_{nabn} = 0 \tag{3}$$

From equation 3, we may extract the following dependence relation among the C_{ijki} entries.

$$C_{nabn} = -C_{1ab1} - C_{2ab2} - ...$$

Note that while the C_{aaba} and C_{babb} terms are present in the sum, they will always be equal to zero by the defining properties of an algebraic curvature tensor and may thus be ignored.

The terms on the left side of 3 give us a special set of entries that we will discuss in the next section.

2.2 MAD Families

Equation 3 gives us a set of curvature entries that sum to zero, and thus a dependence relation between them. Their mutually destructive nature has earned them the following characterization.

Definition. *Let basis vectors e_a, e_b be fixed, with $a < b$, and let R be an algebraic curvature tensor . We define the* abMAD family *to be the set of curvature entries $\{R_{iabi} \mid a \neq i \neq b\}$.*

Note. *If we are working in a vector space of dimension n and on an algebraic curvature tensor R, then the following properties regarding MAD families hold:*

- *All the R_{ijki} entries are partitioned into MAD families.*
- *There are $\binom{n}{2}$ such MAD families.*
- *Each MAD family consists of $n-2$ entries of R.*

As an example, consider the entries of an algebraic curvature tensor in dimension 4 sorted by MAD family.

$$\begin{bmatrix} R_{ijji} \\ ---- \\ R_{3123} \\ R_{4124} \\ \mathbf{R}_{2132} \\ \mathbf{R}_{4134} \\ R_{2142} \\ R_{3143} \\ \mathbf{R}_{1231} \\ \mathbf{R}_{4234} \\ R_{1241} \\ R_{3243} \\ \mathbf{R}_{1341} \\ \mathbf{R}_{2342} \\ ---- \\ R_{ijkl} \end{bmatrix}$$

As seen above, there are $\binom{4}{2} = 6$ MAD families, each with $4-2=2$ entries.

We are now ready to begin the algorithm for linearly combining algebraic curvature tensors from $\mathcal{A}_\Lambda(V)$ into any arbitrary algebraic curvature tensor R.

3 The Algorithm

We will now describe an algorithm that will construct any arbitrary algebraic curvature tensor as a linear combination of algebraic curvature tensors from $\mathcal{A}_\Lambda(V)$ using less than $dim(\mathcal{A}(V))$ algebraic curvature tensors from $\mathcal{A}_\Lambda(V)$ to do so. The algorithm starts with a "blank slate" zero tensor which we call $\bar{R}$. We aim to fix the entries of $\bar{R}$ by using a strategic choice of antisymmetric forms and scalars to match the entries of R and $\bar{R}$.

The outline of the process is as follows:
Step 0) Define $\bar{R}$ to be the zero tensor. We will append algebraic curvature tensors from $\mathcal{A}_\Lambda(V)$ to $\bar{R}$ as we move forward.

Step 1) Fix the entries of each abMAD family where neither a or b is n. This will create extraneous values in the inMAD family for some other index i.
Step 2) Fix the extraneous entries in the inMAD family one at a time with one algebraic curvature tensor from $\mathcal{A}_\Lambda(v)$.
Step 3) Fix the remaining entries of the form R_{ijkl}.
Step 4) Fix the remaining entries of the form R_{ijji}.

3.1 Step 0

We begin by setting $\bar{R}$ to be the zero tensor.

$$\bar{R} = \begin{bmatrix} \\ \bar{R}_{ijji} \\ --- \\ \\ \\ \\ \\ \bar{R}_{ijki} \\ \\ \\ \\ \\ --- \\ \bar{R}_{ijkl} \\ \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ --- \\ \vdots \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ --- \\ \vdots \\ 0 \end{bmatrix}$$

3.2 Step 1

We proceed by appending algebraic curvature tensors to $\bar{R}$ in such a way that a single MAD family can be targeted and fixed, but that creates an unwanted byproduct in the process. We do so by providing an antisymmetric form A that produces an algebraic curvature tensor with an entry that can be scaled to match, say, the R_{pabp} entry, along with its negative in the R_{nabn} entry, as desired by Lemma 3. For these to be true, we must have the following be true:

$$(R_A)_{pabp} = A(p,p)A(a,b) - A(p,b)A(a,p) - 2A(p,a)A(b,p) = 3A(p,a)A(p,b). \tag{4}$$

Now is a good time to note that the section to the right of the second equal sign is a simpler computation when calculating R_{ijki} entries of an algebraic curvature tensor built from an antisymmetric form. Similarly,

$$(R_A)_{nabn} = 3A(n,a)A(n,b). \tag{5}$$

The simplest way to achieve the desired results from equations 4 and 5 is to build a tensor from an antisymmetric form with the entries $A_{pa} = 1, A_{pb} = 1, A_{na} = 1$, and $A_{nb} = -1$, along with the corresponding entries needed to ensure that A is antisymmetric. The antisymmetric form that produces this tensor looks like this:

$$\begin{bmatrix} 0 & & & \cdots & 0 & \cdots & & & 0 \\ \vdots & \ddots & & & & & & & \vdots \\ & & 0 & \cdots & A_{pa} & \cdots & A_{pb} & \cdots & 0 \\ \vdots & & \vdots & 0 & & & & & \vdots \\ 0 & & A_{ap} & & 0 & & & & A_{an} \\ \vdots & & \vdots & & & 0 & & & \vdots \\ & & A_{bp} & & & & 0 & & A_{bn} \\ \vdots & & \vdots & & & & & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & A_{na} & \cdots & A_{nb} & \cdots & 0 \end{bmatrix} = \begin{bmatrix} 0 & & & \cdots & 0 & \cdots & & & 0 \\ \vdots & \ddots & & & & & & & \vdots \\ & & 0 & \cdots & 1 & \cdots & 1 & \cdots & 0 \\ \vdots & & \vdots & 0 & & & & & \vdots \\ 0 & & -1 & & 0 & & & & -1 \\ \vdots & & \vdots & & & 0 & & & \vdots \\ & & -1 & & & & 0 & & 1 \\ \vdots & & \vdots & & & & & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 1 & \cdots & -1 & \cdots & 0 \end{bmatrix}$$

With antisymmetric forms constructed in this way, we will achieve the desired result that $R_{pabp} = 3 = -R_{nabn}$. Cycling p through the elements of $\{1, 2, \ldots, m\}$, putting the outputs into a column vector, and choosing the right coefficients, we also get the following linear combination:

$$\begin{bmatrix} \bar{R}_{1ab1} \\ \bar{R}_{2ab2} \\ \bar{R}_{3ab3} \\ \vdots \\ \bar{R}_{mabm} \\ \bar{R}_{nabn} \end{bmatrix} = \frac{R_{1ab1}}{3} \begin{bmatrix} 3 \\ 0 \\ 0 \\ \vdots \\ 0 \\ -3 \end{bmatrix} + \frac{R_{2ab2}}{3} \begin{bmatrix} 0 \\ 3 \\ 0 \\ \vdots \\ 0 \\ -3 \end{bmatrix} + \cdots + \frac{R_{mabm}}{3} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 3 \\ -3 \end{bmatrix} = \begin{bmatrix} R_{1ab1} \\ R_{2ab2} \\ R_{3ab3} \\ \vdots \\ R_{mabm} \\ -R_{1ab1} - R_{2ab2} - \cdots - R_{mabm} \end{bmatrix}$$

At first, it may seem that we have up to now linearly combined $m = n - 1$ algebraic curvature tensors to fix all but one of the members of the abMAD family. However, recall that R_{aaba} and R_{baba} are always zero and thus not listed in the MAD family. Since those entries are excluded, we see that there are only $n - 3$ entries in this part of the combination. Also, by the dependence relation derived from equation 3, we see that the last entry is in fact R_{nabn}, thus making the entries of $\bar{R}$ equal to those of an arbitrary algebraic curvature tensor R in the abMAD family. Thus, we have successfully used $n - 3$ algebraic curvature tensors built from antisymmetric forms to fix the $n - 2$ entries in all the abMAD families where neither a nor b is equal to n, of which there are $\binom{n-1}{2}$, saving us a grand total of $\binom{n-1}{2}$ algebraic curvature tensors on antisymmetric forms needed to linearly combine into any arbitrary algebraic curvature tensor R.

The next steps show how to fix the remaining entries of $\bar{R}$.

3.3 Step 2

However, the antisymmetric forms that we use to build these tensors produce some extra, less desirable entries outside the abMAD families. There are extra nonzero entries that could be generated of the form R_{ijji} and R_{ijkl}, but those will be addressed in Step 3 and Step 4. The extraneous entries that we must fix next are of the form R_{ijki}. Specifically, if we construct an antisymmetric form to fix R_{pabp} and R_{nabn}, the pnMAD family will be disrupted.
Fortunately, for any member of the pnMAD family, say R_{qpnq} (or any MAD family, for that matter), there exists an antisymmetric form that will produce a nonzero R_{qpnq} entry that can be scaled to match any arbitrary curvature entry one could desire while leaving all other MAD families unperturbed. In particular, since $(R_A)_{qpnq} = 3A_{qp}A_{qn}$, choosing A such that $A_{qp} = A_{qn} = 1 = -A_{pq} = -A_{nq}$, else zero, achieves this. In the column vector representation, this looks like this:

$$\begin{bmatrix} \bar{R}_{1pn1} \\ \bar{R}_{2pn2} \\ \bar{R}_{3pn3} \\ \vdots \\ \bar{R}_{mpnm} \end{bmatrix} = \alpha_c \begin{bmatrix} 3 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \alpha_{c+1} \begin{bmatrix} 0 \\ 3 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \alpha_{c+2} \begin{bmatrix} 0 \\ 0 \\ 3 \\ \vdots \\ 0 \end{bmatrix} + \cdots + \alpha_{c+n-2} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 3 \end{bmatrix}.$$

We can choose the scalars α to produce whatever value we desire for each respective entry of the algebraic curvature tensor R.
With all the MAD families now fixed, we have now fixed all entries of $\bar{R}$ of the form R_{ijki}.

3.4 Step 3

In Step 3 we will fix the entries of the form R_{ijkl}. In an algebraic curvature tensor from a vector space of dimension n, there are $2\binom{n}{4}$ entries of this form. This comes from the fact that out of the n total basis vectors of V, we must choose 4 distinct vectors, are for each choice of 4 vectors, there are 2 independent entries using those four by the dependence relation described in Equation 1. We will colloquially refer to the entries which take the same four basis vectors as entries as *Bianchi surrogates.* We proceed by producing algebraic curvature tensors that do not disturb any R_{ijki} entries but form a linearly independent spanning set in the R_{ijkl} entries.

We begin by using 2 algebraic curvature tensors on antisymmetric forms for each pair of Bianchi surrogates. For this pair, say R_{abcd} and R_{adbc}, we will produce tensors that can linearly combined to make those desired entries whatever we desire, while simultaneously leaving all the other R_{ijkl} entries and all of the R_{ijki} entries undisturbed. Some byproducts will be produced in the R_{ijji} entries, but those will be taken care of in Step 4.

From Equation 2, we see that if A is some antisymmetric form represented by matrix, then

$$(R_A)_{abcd} = A_{ad}A_{bc} - A_{ac}A_{bd} - 2A_{ab}A_{cd}, \text{ and}$$

$$(R_A)_{adbc} = A_{ac}A_{db} - A_{ab}A_{dc} - 2A_{ad}A_{bc}.$$

For the first antisymmetric form, say ψ, we may construct ψ such that $\psi_{ab} = 1$ and $\psi_{cd} = -1$. For the second antisymmetric form, say ψ^*, we may construct ψ^* such that $\psi^*_{ac} = 1$ and $\psi^*_{bd} = -1$. In both of these, we always add the extra entries needed to ensure that ψ and ψ^* are antisymmetric.
Returning to our vector representation of the algebraic curvature tensors these antisymmetric forms produce, we get the following result:

$$\begin{bmatrix} \bar{R}_{abcd} \\ \bar{R}_{adbc} \end{bmatrix} = \alpha_x \begin{bmatrix} 2 \\ -1 \end{bmatrix} + \alpha_{x+1} \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

Since the 2×2 matrix comprised of those two "columns" has nonzero determinant, we know that these columns span all of $\mathbb{R}^2$, so we may choose the scalars so that even though they were tampered with in Step 1 and Step 2, we may achieve any desired value in each respective entry.

Repeating this process for all $\binom{n}{4}$ pairs of Bianchi surrogates will allow us to ensure that thee entries may be linearly combined into any arbitrary value.

3.5 Step 4

At this point, all entries of our linear combination of algebraic curvature tensors , $\bar{R}$ of the form $\bar{R}_{ijki}$ and $\bar{R}_{ijkl}$ have been made to match the corresponding entry of an arbitrary algebraic curvature tensor R. In Step 4, we will finish matching the remaining entries of $\bar{R}$, all of which are of the for $\bar{R}_{ijji}$.

In any algebraic curvature tensor , we know that there are always $\binom{n}{2}$ entries of the form R_{ijji}. That being said, we will use the same number of algebraic curvature tensors on antisymmetric forms to fix these entries. We do so by individually targeting each entry, much akin to the process from Step 2.

The antisymmetric forms that we will use to target the R_{ijji} entries are simpler than those in previous steps. In order to fix a given entry, say R_{abba}, we can use an antisymmetric form ψ such that $\psi_{ab} = 1 = -\psi_{ba}$. The algebraic curvature tensor that ψ produces a single nonzero entry, $(R_\psi)_{abba}$. This entry can be scaled to achieve any arbitrary value, despite the entries of this form being tampered with in every preceding step of the algorithm. In vector form, this step looks like:

$$\begin{bmatrix} \bar{R}_{1221} \\ \bar{R}_{1331} \\ \bar{R}_{1441} \\ \vdots \\ \bar{R}_{1nn1} \\ \bar{R}_{2332} \\ \vdots \\ \bar{R}_{mnnm} \end{bmatrix} = \alpha_y \begin{bmatrix} 3 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \alpha_{y+1} \begin{bmatrix} 0 \\ 3 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \alpha_{y+2} \begin{bmatrix} 0 \\ 0 \\ 3 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \cdots + \alpha_{y+\binom{n}{2}-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 3 \end{bmatrix}$$

This step finishes off the last unfixed entries of $\hat{R}$, and thus the algorithm is complete.

4 Conclusion

4.1 Results

The question remains as to what $\eta(n)$ is for any dimension n. The previous upper bound was the dimension of the entire space of algebraic curvature tensors , which is $\frac{n^2(n^2-1)}{12}$. What we have done in this paper is describe an algorithm that will linearly combine algebraic curvature tensors built on antisymmetric forms using less than $dim(\mathcal{A}(V))$ of them, therefore improving the upper bound. This leaves us with the following result.

Theorem 4. *Let V be a real vector space such that $dim(V) = n$. Then:*

$$\eta(n) \leq \frac{n^2(n^2-1)}{12} - \binom{n-1}{2} = \frac{(n-1)(n^3+n^2-6n+12)}{12}$$

The table below shows some of the previous bounds versus the bounds given by the algorithm presented here.

Dimension	Previous Upper Bound	New Upper Bound	Savings
3	3*	5	-2
4	11	17	-6
5	50	**44**	**6**
6	105	**95**	**10**
7	196	**181**	**15**
8	336	**315**	**21**
9	540	**512**	**28**

4.2 Future Work

Further study of this subject could focus on the following open problems.

- Can we improve the upper bounds for $\eta(n)$?
- Can we find a meaningful lower bound for $\eta(n)$?
- Can we find better bounds (or the exact value) of $\eta(n)$ for specific values of n?

Improvement upon this new upper bound is likely, since better results are known for dimension 3 and 4.

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References

1. Gilkey, Peter B. The Geometry of Curvature Homogeneous Pseudo-Riemannian Manifolds. Imperial College Press; Distributed by World Scientific Pub, 2007

2. Huisken, Gerhard. Ricci deformation of the metric on a Riemannian manifold. J. Differential Geom. 21 (1985), no. 1, 47–62. doi:10.4310/jdg/1214439463. https://projecteuclid.org/euclid.jdg/1214439463

3. Thomas, Joseph. An Upper Bound on Antisymmertic Algebraic curvature Tensors. https://www.math.csusb.edu/reu/studentwork.html. Accessed 8 Aug. 2018.

4. Treadway, Blake. Algebraic Curvature Tensors and Antisymmetric Forms. https://www.math.csusb.edu/reu/studentwork.html. Accessed 8 Aug. 2018.