Factorization of Fully Augmented Links via Belted Sum Decomposition

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August 20, 2018

Abstract

In his paper *Thrice Punctured Spheres in Hyperbolic 3-Manifolds* Colin Adams introduces the operation of belted summing two 3-manifolds. The goal of this paper is to study the conditions and restrictions under which Fully Augmented Links (as defined by Jessica Purcell) admit unique prime factorization under the inverse operation of the belted sum of two 3-manifolds - termed the "belted sum decomposition." In this paper an operation called "pseudo trivial belted sum decomposition" is defined, and it is shown that when one allows for this operation to be performed during the belted sum decomposition of a Fully Augmented Link, then this yields a unique factorization into equivalence classes of Fully Augmented Links and Whitehead Links, which cannot factor further.

1 Belted Sum Decomposition

A Fully Augmented Link (FAL), as defined by Jessica Purcell in her paper *An Introduction to Fully Augmented Links* is a link which has had crossing discs placed around its twist regions, and Dehn Fillings performed to twist reduce these regions to having either half or no twists [**?**]. In the same paper, Purcell outlines the process by which a given Fully Augmented Link complement is lifted to a labeled fundamental domain in $\mathbb{H}^3$, which from this point forward will be referred to as a Fully Augmented Domain (FAD). It is the case that there is a bijection mapping FALs to the FADs which they correspond to in $\mathbb{H}^3$.

Since the goal of this paper is to study the factorization of FALs via belted sum decomposition, it must first be understand precisely what it means to take the belted sum of two 3-manifolds, and thereby what it means to perform a belted sum decomposition on a 3-manifold. In his paper *Thrice Punctured Spheres in Hyperbolic 3-Manifolds*, Colin Adams describes the process of belt summing two manifolds M_1 and M_2 as follows:

"Let S_1 and S_2 be incompressible thrice-punctured spheres properly embedded in compact orientable 3-manifolds M_1 and M_2, respectively. Let $M_i' = M_i - N(S_i)$. Let S_i^0 and S_i^1 be the two copies of S_i in ∂M.

"Let $\lambda_0 : S_1^0 \to S_2^0$ and $\lambda_1 : S_1^1 \to S_2^1$ be any two homeomorphisms that either both preserve orientations or both reverse orientations. Let M be the 3-manifold obtained from M_1 and M_2 by identifying S_1^0 and S_2^0 using λ_0 and identifying S_1^1 and S_2^1 using λ_1."

Note that the belted sum operation can only be performed on orientable 3-manifolds, and can only result in an orientable 3-manifold. The inverse process of belted summing two manifolds can now be defined.

Definition 1.1. Let S^0 and S^1 be thrice punctured spheres in an orientable 3-manifold M such that cutting along S^0 and S^1 separates M into two regions M_1 and M_2. The *belted sum decomposition* of M using S^0 and S^1 would be performed by cutting along S^0 and S^1, and then regluing the copy of S^0 in M_1 to the copy of S^1 in M_1 and doing the same for S^0 and S^1 in M_2.

Using this definiton, several lemmas can be shown which will be useful in recognizing thrice punctured spheres which form valid belted sum decompositions.

Lemma 1.1. *Two thrice punctured spheres S^0 and S^1 in a 3-manifold M form a valid belted sum decomposition if and only if slicing along each thrice punctured sphere separates M.*

Proof. This follows immediately from the definition of a belted sum decompotion. □

Lemma 1.2. *Whenever two thrice punctured spheres form a valid belted sum decomposition, they must share at least one puncture.*

Proof. It is the case that a single thrice punctured sphere does not itself separate a 3-manifold. Thus the union of two disjoint thrice punctured spheres cannot separate a 3-manifold. So if two thrice punctured spheres form a valid belted sum decomposition, they cannot be disjoint by Lemma 1.1. By the way that the belted sum operation is defined, the thrice punctured spheres in question cannot intersect in any non-puncture region. Hence they must share between one and three of their punctures. □

We can now assert one of the guiding ideas of the paper.

Theorem 1.3. *The belted sum decomposition of an FAL F into F_1 and F_2 corresponds directly to the belted sum decomposition of the FAD F' into FADs F_1' and F_2', where the link complements of F, F_1, and F_2 lift to F', F_1', and F_2' respectively.*

Proof: It follows from the process which Purcell describes to lift a given FAL complement to its corresponding FAD that S^0 is a thrice punctured sphere in the FAL complement if and only if it lifts to a thrice punctured sphere in the FAD. It also follows that two thrice punctured sphere separate the FAL complement if and only if the thrice punctured spheres which they lift to separate the FAD. Thus, two thrice puctured spheres form a valid belted sum decomposition in an FAL if and only if they lift to thrice punctured spheres which form a valid belted sum decomposition in the corresponding FAD. Because the projection from the FAD to its identification space is continuous, we see that the pieces $F_1^{'}$ and $F_2^{'}$ correspond to F_1 and F_2.

Because of this, factoriation of FALs via belted sum decomposition can be understood by studying the factorization of FADs via belted sum decomposition; this will be the strategy employed in the remainder of the paper.

2 Locating Thrice Punctured Spheres in an FAD

In order to be able to understand what belted sum decompositions are possible in an FAD, it is helpful to be able to know precisely where the thrice punctured spheres lie in the space based on the FADs circle packing on the ideal plane. The aim of this section will be to narrow down the possible locations of thrice punctured spheres with a puncture at infinity in the FAD, and in the next section we will use this information to show explicitly where such thrice punctured spheres must lie.

Definition 2.1. In $\mathbb{H}^3$, Geodesic surfaces with a point at infinity will be referred to as *p-geodesics*, and geodesic surfaces with all points in the ideal plane will be referred to as *d-geodesics*.

This terminology is inspired by the fact that geodesic surfaces with a point at infinity in $\mathbb{H}^3$ look like *p*lains in Euclidean space, and geodesic surfaces with all points in the ideal plane look like *d*omes. Lemma 2.1 consists of information shown in Purcell's paper and thus requires no further proof.

Lemma 2.1. *The following are true of thrice punctured spheres in an FAD:*
i) They are totally geodesic surfaces.
ii) They are orthogonal to all 2-cells in the FAD (including other thrice punctured spheres).
iii) They have area 2π.

In order to capitalize on iii) from lemma 2.1, we must not the following fact which also requires no proof.

Lemma 2.2. *For an n-gon P in hyperbolic space, the area formula for P is $A(P) = (\sum_i^n \epsilon_i) - 2\pi$ where each ϵ_i is an external angle of the n-gon.*

Using lemma 2.1 and 2.2, a classification of possible locations for p-geodesics in an FAD can be performed. P-geodesics have been chosen as they are slightly more intuitive to work with, and since d-geodesics can be viewed as p-geodesics simply by sending one of their ideal points to infinity by isometry. Because of this choice, we will primarily be considering representations of FADs with an ideal point sent to infinity, and their corresponding circle packings.

Theorem 2.3. *If a surface in an FAD (represented with an ideal point send to infinity) is a thrice punctured sphere with a puncture at infinity, then it will appear if one of the following four types of locations:*

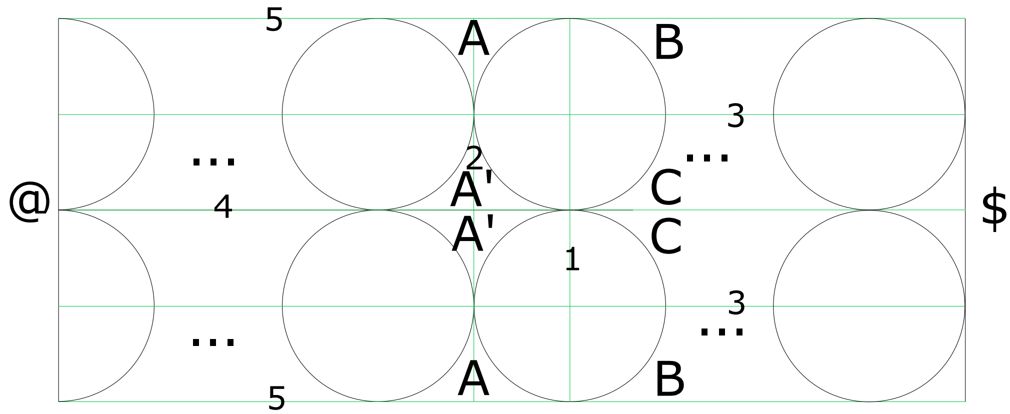

Figure 1: All possible p-geodesic thrice punctured sphere locations in an FAD.

Before continuing with the proof, the information in the picture will be unpacked and interpreted. First, we will discuss the construction choices in the circle packing. The vertical ends to the circle packing can either look like that one the left side (labeled @) or like the one on the right side (labeled $). It will be shown later that this is the case, and that an FAD can have both ends look like @, or like $, or a mix as in the figure above. The "..." sections allow for any configuration of circles which bound triangles to be filled in. The rest of the figure must rigidly be the way it is depicted for the theorem to apply.

Now, let us discuss the p-geodesic regions labeled 1, 2, and 3. Region 1 represents the location of any p-geodesic which traces a vertical line through the circle packing which crosses from the top of P^+ to the bottom of P^- through exactly two crossing circles and nothing else, orthogonal to the top and bottom edges of the FAD. Region 2 represents the location of a p-geodesic which traces a vertical line through the circle packing which crosses from the top of P^+ to the bottom of P^- through exactly all four regions of a single crossing disc and nothing else, again orthogonal to the top and bottom edges of the FAD.

Lastly, and with the most restrictions, region 3 represents a p-geodesic which traces two lines through the circle packing (this still represents a single geodesic surface, under the FAD's gluing maps), the first of which lies in the P^+ section of the FAD, the second of which lies in the P^- section of the FAD. Each of the lines lies orthogonal to @ and $, as well as orthogonal or tangent to each knot circle which it touches. Note that this means that region 3 cannot cross through a knot circle non-orthogonally in the "..." regions of the diagram. We are now ready to move on to the proof of the theorem.

Proof. By lemma 2.1, thrice punctured spheres in the FAD are orthogonal to other 2-cells and thrice punctured spheres in the FAD. Because of this, we can restrict our attention to locations in the FAD which trace lines through the circle packing that are orthogonal to any other circle packing lines they cross. Because the perimeter of the circle packing with a point at infinity is a rectangle, this orthogonality condition divides the possible locations into being vertical or horizontal lines, excluding diagonal lines.

Let us first consider the vertical lines. Because the vertical lines must be orthogonal to all knot circles in the circle packing, the correct location is restricted to those depicted by 1 and 2 - at this point without the specificity of the labeling surrounding 2. In order to obtain this specificity, we must consider area. By lemma 2.1 a thrice punctured sphere be a geodesic surface which attains an area of exactly 2π. By Purcell, all 2-cells in the FAD are orthogonal. Thus in the circle packing, when a line crosses a vertex this represents an external angle of π, and when a line crosses another line (besides the heuristic line dividing P^+ from P^-) this represents an external angle of $\pi/2$. Now consider if region 2 did not cross over all four regions of the same crossing disc, but rather over triangular regions from two or more crossing discs. Since regions of the same crossing disc glue to one another, this would extend the p-geodesic over such a region to cross over other parts of the circle packing. However, both regions 1 and 2 as depicted have an area of exactly π already; thus if the line is traced by a thrice punctured sphere, this cannot be the case, and hence region 2 is forces to cross over all regions of the same crossing disc as claimed.

Now, let us consider horizontal lines. By orthogonality restrictions, a thrice punctured sphere can immediately be restricted to existing in regions of type 3, 4, and 5. By area considerations, a p-geodesic in one of these regions is only a thrice punctured sphere when the horizontal length of the FAD is precisely right. □

The goal of the classification is ultimately to ensure that it is known where thrice punctured spheres are located to be paired together and used in belted sum decompositions. Since this is the ends, we can pare down the regions under consideration even further, by concluding that thrice punctured spheres in regions 4 and 5 cannot form belted sums with any other thrice punctured spheres. In order to show that this is the case, the following lemma will be useful.

Lemma 2.4. *In a triangular region with a circle packing which bounds triangles, an odd number of triangles are bounded.*

This figure will aid in visualizing the proof.

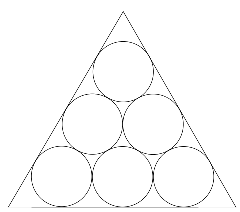

Figure 2: An example of circle packing in a triangular region as described by Lemma 2.4.

Proof. One can draw a graph in such a circle packing where vertices correspond to triangles bounded by the circle packing, and edges lie between triangles which share points. Note that by this construction, ever vertex would have valence three except for the vertices corresponding to the three triangles at the vertices of the triangular region bounding the circle packing, which have valence two. Let there be n triangles bounded by circles in the circle packing. Then in the graph that was constructed, the sum of the valences is $3n - 3$, since assuming each vertex has valence three over counts by three. However, the sum of the valences of a graph must be even, as it is twice the number of edges in the graph. So $3n - 3 = 2e$, where e is the number of edges in the graph. This forces n to be odd. □

Theorem 2.5. *Thrice punctured spheres located in regions 4 and 5 of* $Figure 1$ *will never form belted sum decompositions with any other thrice punctured sphere.*

The proof will refer to the following diagrams.

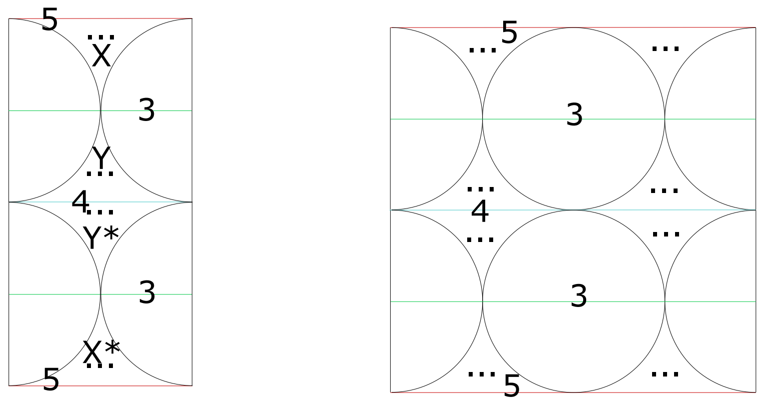

Figure 3: Arbitrary FADs shortened horizontally to the proper length for regions 3 and 5 to represent thrice punctured spheres (right) and for region 4 to represent a thrice punctured sphere (left).

Proof. Let us first consider whether or not region 5 could possibly form a belted sum decomposition with any other thrice punctured sphere. As already discussed, this question is equivalent to that of whether there are any other thrice punctured spheres which share a puncture with 5 and jointly separate the FAD. Let us consider whether or not any other p-geodesic thrice punctured spheres can separate the space when paired with 5. As per the left diagram of Figure 3, the only other p-geodesic that exists at the same time as 5 (due to area considerations) without intersecting 5 is 3.

Let us consider whether or not 5 and 3 form a belted sum decomposition. Consider the region X in the above diagram. Whatever circle packing exists in X must have an odd number of crossing disc regions by Lemma 2.4. Since each crossing disc has 2 components in P^+, there must be at least one crossing disc whose counterpart lies in Y or on the left or right boundary of the diagram. In either case, since crossing discs glue to one another, the regions sectioned off when 5 and 3 are cut along would glue together by this crossing disc and thus not be separated. Hence, 5 cannot form a belted sum decomposition with 3.

This means that if 5 forms a belted sum decomposition with another thrice punctured sphere, it must be with a d-geodesic one. If such a d-geodesic thrice punctured sphere were a face of the FAD, then cutting along 5 and the face would not separate the FAD, and thus could not possibly form a valid belted sum decomposition. However, if the d-geodesic wasn't a face of the FAD, then cutting along 5 and this d-geodesic thrice punctured sphere would separate the FAD if and only if cutting along the d-geodesic thrice punctured sphere alone would separate the FAD. Since this cannot be the case, 5 doesn't form a belted sum decomposition with such a d-geodesic. Since 5 cannot form a belted sum decomposition with a p-geodesic or d-geodesic thrice punctured sphere, it forms no belted sum decompositions as desired.

Now let us consider region 4. When the FAD is the proper horizontal length for 4 to have the correct area to be a thrice punctured sphere, no other horizontal p-geodesic is, as can be seen in the right diagram of Figure 3. Then the same argument as was used to show that 5 could not be paired with a d-geodesics to form a belted sum decomposition could be used to show that 4 doesn't form a belted sum decomposition with any other thrice punctured spheres. □

This effectively allows us to only consider regions 1, 2, and 3 from this point forward. The argument used in *theorem* 2.3 doesn't ensure that thrice punctured spheres will always exist in these locations, simply that they can only exist in these locations. To assert something such a more commanding statement, gluing maps must be considered.

3 Regluing Types

When performing a belted sum decomposition, the initial 3-manifold is split into two pieces, each containing two thrice punctured spheres which are glued back together. This section is dedicated to understanding and

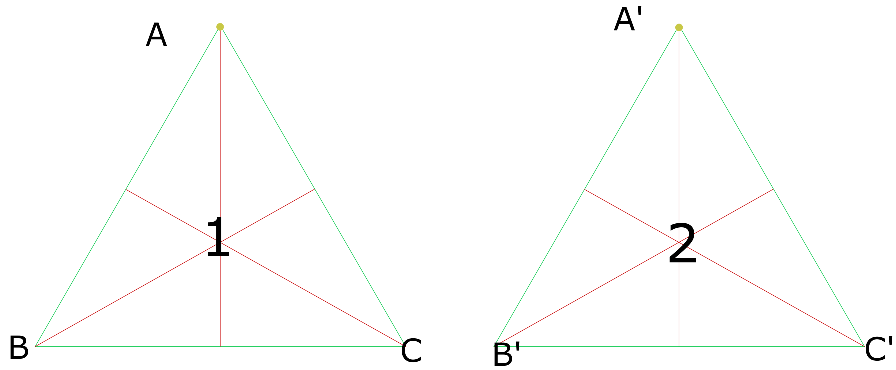

Figure 4: A model of two thrice punctured spheres where the points of each triangle are punctures, A and A' are shared points, the red lines respresent intracusp geodesics, and the green lines represent intercusp geodesics.

classifying the types of regluings which can occur in a manifold resulting from a belted sum decomposition, and then using this classification to explicity describe the locations of p-geodesic thrice punctured spheres in each regluing type.

Theorem 3.1. *Let triangles 1 and 2 in the diagram below represent two thrice punctured spheres in the boundary of a 3-manifold resulting from a belted sum decomposition.*

Then there are only two way to reglue triangle 1 to triangle 2: by gluing B' to B and C' to C, or by rotating by π and gluing C' to B and B' to C.

Proof. First, let us justify the fact that there must be a point shared between the two thrice punctured spheres being reglued. This manifold resulted from a belted sum decomposition, and hence the thrice punctured spheres being reglued formed a valid belted sum decomposition. By Lemma 1.2 they must share at least one puncture.

Because the gluing is an orientation preserving homeomorphism which fixes at least one point, the only options available are the ones described - any type of reflection mapping 2 to 1 would be orientation reversing, making the resulting 3-manifold non-orientable. $\square$

Definition 3.1. If the gluing maps of a geodesic surface in $\mathbb{H}^3$ which forms a thrice punctured sphere are those on the left of the figure below, this will be referred to as a Type 1 thrice punctured sphere; if they are those on the right of the figure below, this will be referred to as a Type 2 thrice punctured sphere.

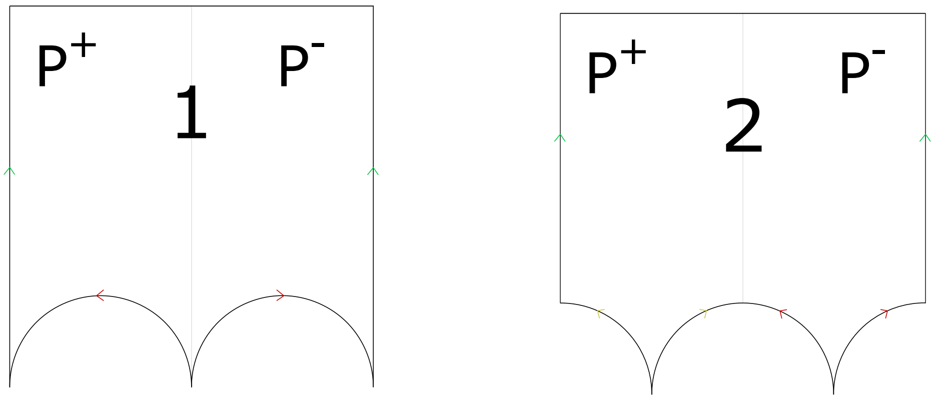

Figure 5: Type 1 thrice punctured sphere (left) and Type 2thrice punctured sphere (right).

Theorem 3.2. *Whenever a belted sum decomposition is completed, the thrice punctured spheres in the resulting manifold which are glued back togehter are either both type 1, both type 2, or one of each - such manifolds will be referred to as Type 1-1, Type 2-2, and Type 1-2 respectively.*

Proof. Consider an arbitrary belted sum decomposition. By lemma 1.2, the thrice punctured spheres in question share a puncture. Because they also cannot intersect, when this shared puncture is sent to infinity, they both must trace vertical lines in the circle packing, and thus either be in location 2 or 4 from the previous section. The thrice punctured spheres which result from this will be called type 1 and type 2 respectively. Thus a regluing can occur between two type 1s, two type 2s, or a type 1 and type 2 mix. □

Lemma 3.3. *Twisting when regluing the right and left ends of the FAD is equivalent to sheering the FAD in its tesselation of* $\mathbb{H}^3$.

Proof. This sheering represents a homeomorphism from one possible orientation preserving gluing of the right and left end pieces to another. By Theorem 3.1 there are only two such orientation preserving regluings, and so sheering must take us from one to the other. □

We are now in a position to classify the regluing types of FADs resulting from a belted sum decomposition. From the previous work in this section, we it is exhaustive to study Type 1-1 untwisted and twisted, Type 2-2 untwisted and twisted, and Type 1-2 untwisted and twisted regluings. From Theorem 3.3, sheering the tesselation of $\mathbb{H}^3$ by the FAD will be a useful tool in seeing what the twist gluing must look like.

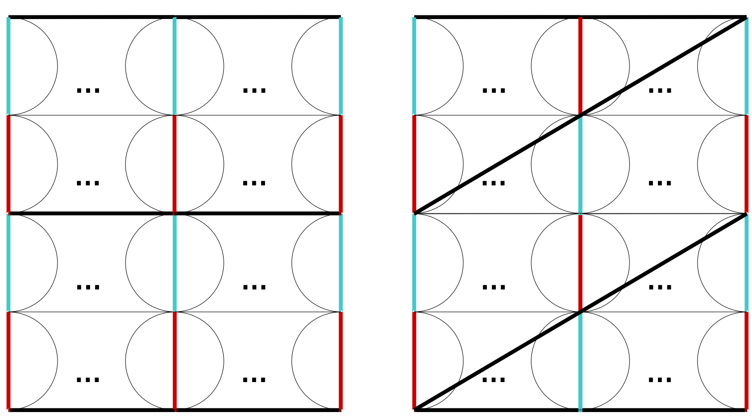

Figure 6: Type 1-1 tessellation of $\mathbb{H}^3$ with and without a sheer representing a twist. The bold lines border fundamental domains of the space.

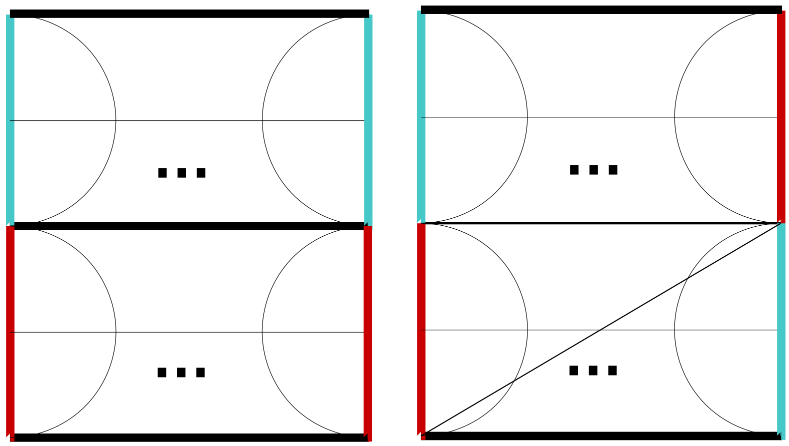

Figure 7: Type 1-1 fundamental domain with no twist (left) and with a twist (right), as can be seen in Figure 6.

Figures 6 and 7 illustrate the process of sheering the FAD to understand what the twist and untwist gluing maps on the side edges of a Type 1-1 fundamental domain must look like. The same process will now be deomstrated for Type 2-2 and Type 1-2 FADs.

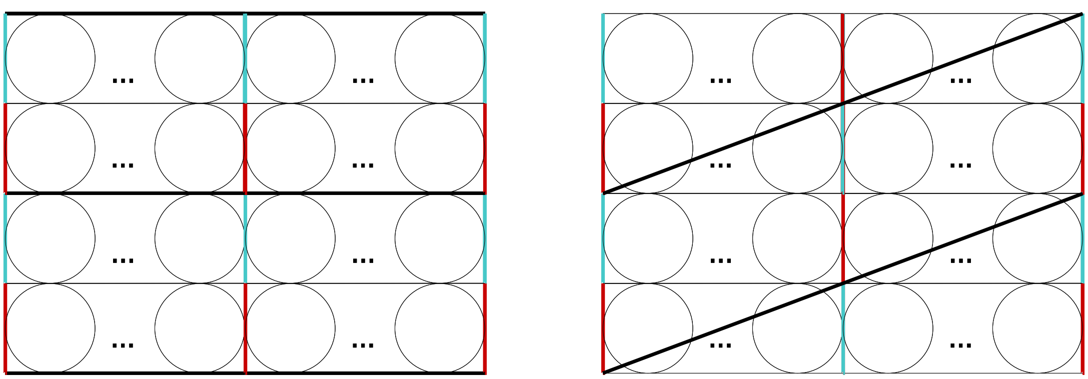

Figure 8: Type 1-1 tessellation of $\mathbb{H}^3$ with and without a sheer representing a twist. The bold lines border fundamental domains of the space.

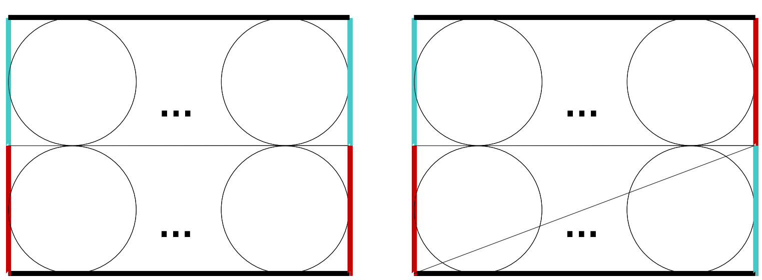

Figure 9: Type 2-2 fundamental domain with no twist (left) and with a twist (right), as can be seen in Figure 8.

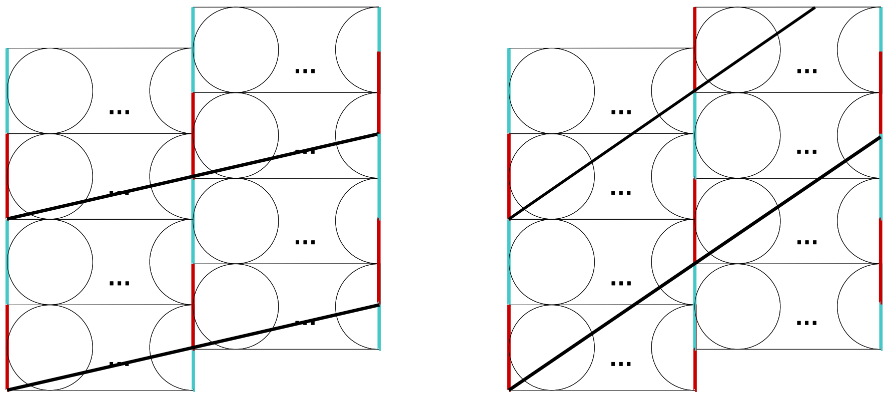

Figure 10: Type 1-2 tessellation of $\mathbb{H}^3$ with and without a sheer representing a twist. The bold lines border fundamental domains of the space.

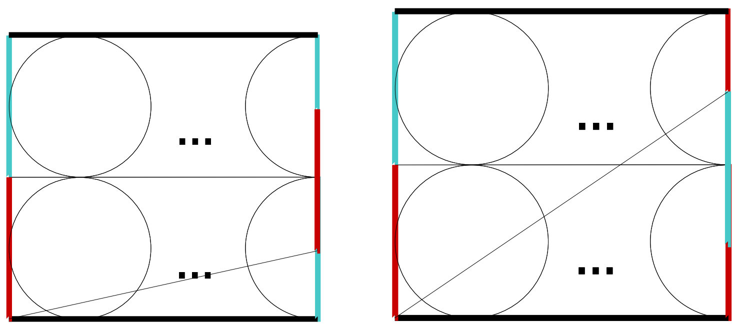

Figure 11: Type 2-2 fundamental domain with no twist (left) and with a twist (right), as can be seen in Figure 8.

It must be noted that Type 1-2 FADs do not actually correspond to FALs due to the extra bit of sheer/twist which they have. This will be resolved in section 4. Now that we can visualize the regluings for any possible manifold resulting from a belted sum decomposition, it can be studied where specifically in each regluing type one can find thrice punctured spheres, completing the prievious sections classification.

Theorem 3.4. *In any possible regluing type as outline above, no thrice punctured spheres are created, and in twisted gluing types, horizontal type 3 thrice punctured spheres are destroyed.*

Proof. In order to show the result, we will examine the FAD of each gluing type equipped with the thrice punctured sphere regions explored in section 2 to understand whether or not they truly house thrice punctured spheres. The results from each gluing type will be discussed after the figures are displayed, which is warranted by the amount of similarity in the results between figures.

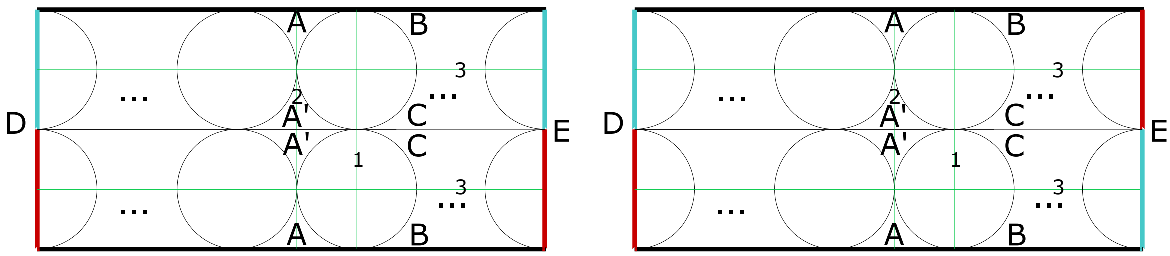

Figure 12: Possible thrice punctured sphere regions in Type 1-1 FADs with no twist (top) and with a twist (bottom).

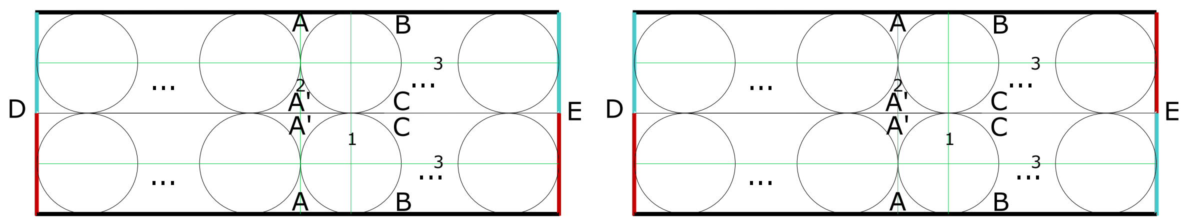

Figure 13: Type 2-2 fundamental domain with no twist (left) and with a twist (right).

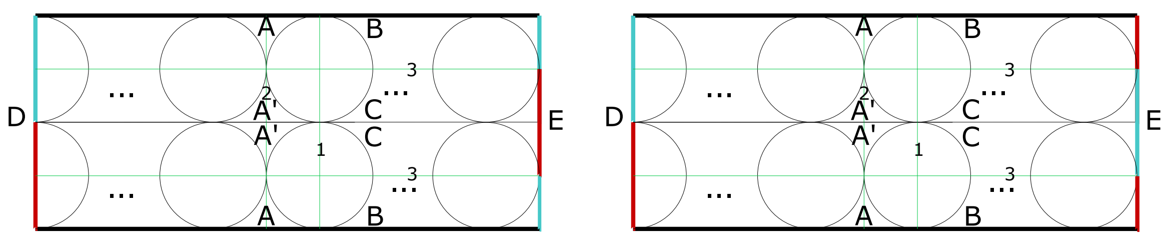

Figure 14: Type 1-2 fundamental domain with no twist (left) and with a twist (right).

In all gluing types, all 1 and 2 regions are thrice punctured spheres, as their gluings match those of type 1 and type 2 thrice punctured spheres, respectively. This includes the right and left ends of the FAD. Type 3 regions are thrice punctured spheres in the untwisted versions of each Type, but become non-orientable in the twisted versions and thus cannot represent thrice punctured spheres in these settings. They are, however, sent to themselves set-wise under the twist regluings. This serves to classify what p-geodesic surfaces look like in the different gluing types.

For each d-geodesic surface, twisting and untwisting of the right and left end does not affect their orientability, nor does it send them anywhere else set-wise. Thus the considerations for whether d-geodesic surfaces are thrice punctured spheres are independent of right and left end gluing considerations. □

To conclude this section, let us summarize the results. A classification of all possible regluing types of FADs resulting from a belted sum decomposition was provided. Using this classification, it was shown using that in untwisted FAD types, all regions outline in the previous section truly do house thrice punctured spheres. It was also shown that in twisted FAD types, horizontal type 3 thrice punctured sphere regions became non-orientable, and thus could not possibly house thrice punctured spheres, but are preserved set-wise. Thus, moving from an untwisted to a twisted version of the same FAD destroys type 3 thrice punctured spheres, and moving the opposite way creates them. This gives us a classification of exactly which thrice punctured spheres exist in any gluing type, and exactly how moving from the twisted to untwisted gluing type of the same FAD and vice versa affects thrice punctured spheres in the space. It must again be noted that Type 1-2 FADs do not correspond to FALs, but this will be resolved in the next section.

4 Order of Decomposition

Now that the preliminaries have been set in place, the goal of this section will be to establish uniqueness of factorization. Primeness will be concluded in the next section. This ordering was chosen because the uniqueness argument relies heavily on the work done in section 3, while primeness does not. In order to establish uniqueness, it must be shown that the order of decomposition does not matter. More precisely stated, it must be shown that any two pairs of belted sum decompositions can be performed in any order - that performing one does not destroy the option to perform the other. It must also be shown that if performing any belted sum decomposition creates any new options for belted sum decomposition that this could happen at any point in the process, and thus order of factorization still doesn't matter. These results all hold if we restrict our attention to untwisted FADs, but not for twisted FADs. Thus it turns out to be the case that factorization via belted sum decomposition is not unique. However, we can introduce a new operation called "pseudo-trivail belted sum decomposition," which is very similar to belted sum decomposition and allows us to conclude uniqueness of factorization for all FADs and intermediary regluings during factorization

Theorem 4.1. *When excluding any twisted regluings, any pair of belted sum decompositions can be performed in any order.*

Proof. This will be shown using a case analysis of all types of pairs of belted sum decompositions. Note that the first two can always be considered to both be p-geodesics, as their shared point can be lifted to infinity before the comparison. Note also that two belted sum decompositions can either use four or three thrice punctured spheres in total, depending on whether or not one of the thrice punctured spheres is used by both belted sum decompositions. It must also be noted that these pairs of thrice punctured spheres can be disjoint or intersecting (if one of the thrice punctured spheres is shared, we will consider them to be intersecting if the non-shared thrice punctured spheres intersect). Our notation for each type will be as follows: 4pdD will denote the case with four thrice punctured spheres between the two belted sum decompositions, where the first pair consists of two p-geodesics (as will always be the case during this analysis), the second pair consists of a p-geodesic and a d-geodesic, and the pairs are disjoing. Likewise, 3dI will denote the case where three thrice punctured sphere are being used between the two belted sum decompositions, where again the first pair are p-geodesics, the remaining thrice punctured sphere is a d-geodesic, and they intersect. The notation will follow this style. In the following figures, pairs of thrice punctured spheres which belted sum decompose together will be colored the same way. Without further ado, let us begin the analysis.

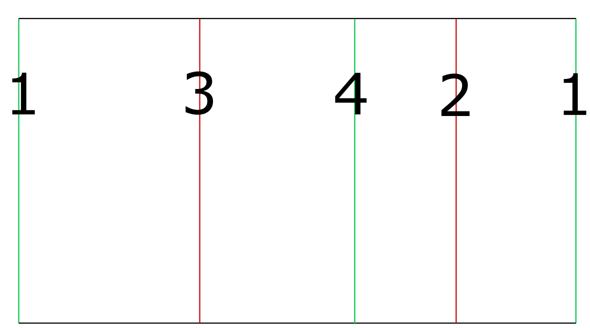

Figure 15: Two ways which case 4pdD could look, eithe with or without tangency.

In case 4ppD whenever one belted sum decomposition is performed, the other is available still in the resulting FADs in the fact that both cuts can still be made, which will still separate the space, using the prior thrice punctured spheres (which are not identified in the resulting FADs) to perform the decompostion.

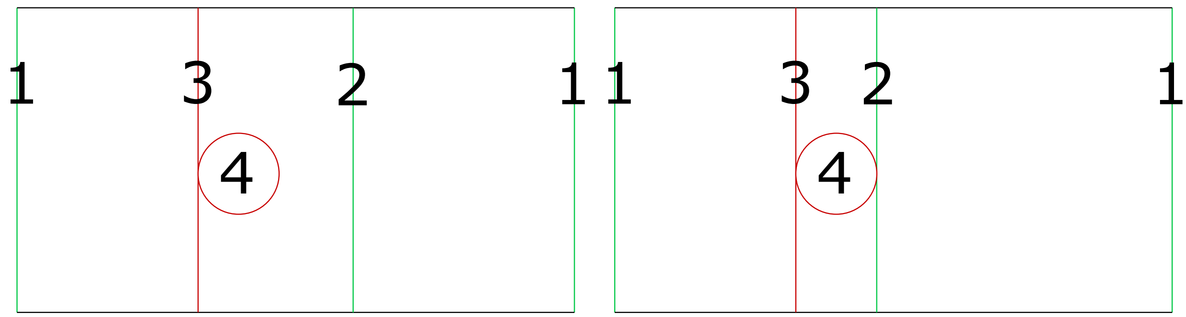

Figure 16: Two ways which case 4pdD could look, eithe with or without tangency.

In case 4pdD, since it is assumed that each pair forms a valid belted sum decomposition, they both separate the FAD. Thus, it can be seen that they will still separate the FAD resulting from a decomposition performed by the other pair. Thus they can be performed in any order to yield the same result.

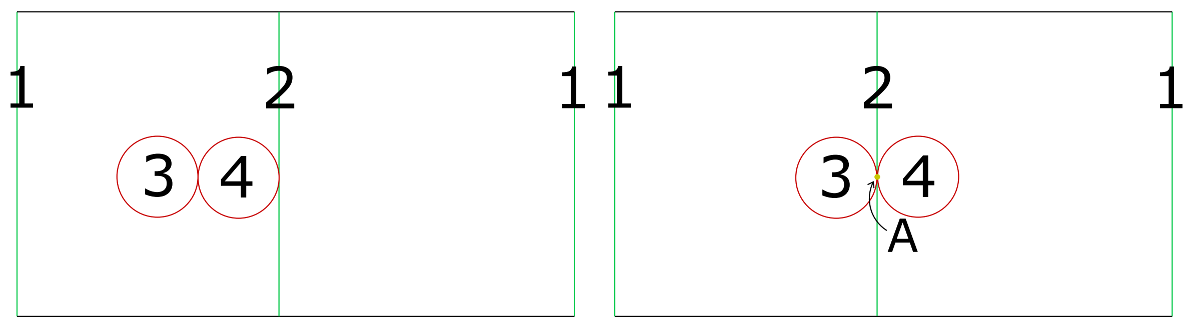

Figure 17: Two ways which case 4ddD could look, depending on tangencies.

In case 4ddD on the right of the figure, , the same argument as in 4pdD can be used. In the case on the left, an argument based on gluings can be used to show that both pairs of thrice punctured spheres cannot simultaneously be considered to be thrice punctured spheres. If 1 and 2 formed a valid belted sum decomposition, they must separate the FAD. It is know that the space underneath 3 must glue only to itself, to the space under 4, or to the rest of the space (or some combination of these). If it glues to 4, then 1 and 2 do not separate the space and are not a valid belted sum decomposition. If it glues to the rest of the space, then for 3 and 4 to be considered a valid belted sum decomposition, 4 must itself separate the space, which cannot be the case for a single thrice punctured sphere. If 3 only glues its bottom to itself, then it separates the space on its own, which again cannot be the case for a single thrice punctured sphere. Thus both pairs cannot simultaneously be valid belted sum decomposition, and the diagram on the right of Figure 17 must not be considered.

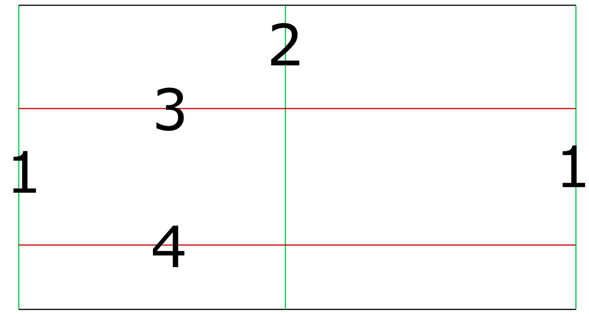

Figure 18: Case 4ppI

Because the pair of belted sum decompositions are assumed to intersect in this case, and because an intersecting pair of thrice punctured spheres cannot form a valid belted sum decomposition, we are forced to have two horizontal thrice punctured spheres which form a belted sum decompositon. By Theorem 2.5 this cannot be the case, and this case does not exist.

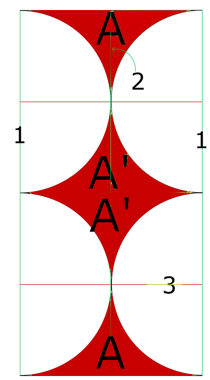

Figure 19: Case 4pdI.

The only way that case 4pdI could occur is in the above diagram of the Boromian Rings' FAD, with these exact two thrice punctured sphere pairs. If the shared point between A and A' in P^+ is sent to infinity, it can be seen that the two belted sum decomposition are symmetric an both split to two Whitehead links, which cannot be further decomposed. Thus, despite the fact that performing one decomposition destroys the option for the other, the effect is the same and so order does not matter.

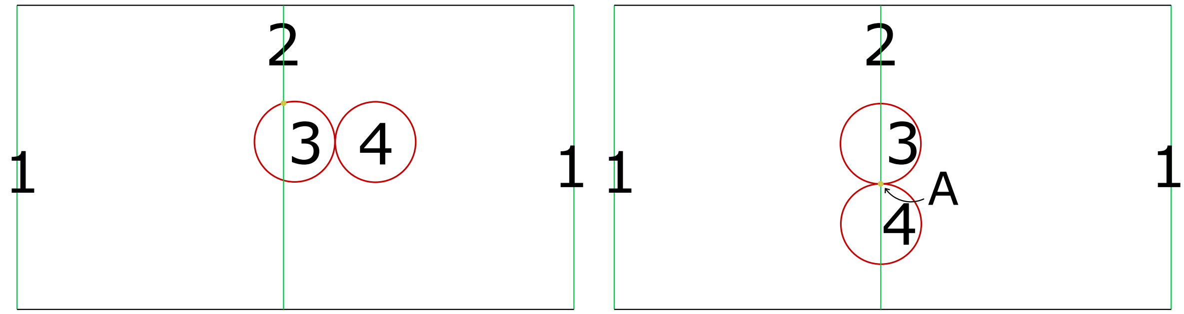

Figure 20: Two ways which case 4ddI could look, depending on types of intersection.

In each of these cases for 4ddI, send the shared point of one of the d-geodesics and the p-geodesic to infinity (this can be done since geodesic surfaces must intersect at geodesics, which end in ideal points). Then both of the surfaces become p-geodesics which still intersect by isometry. The only way this can happen is if the resulting diagram of that point being sent to infinity is the Boromian Rings, as in case 4pdI, which is already resolved. Otherwise this cannot exist.

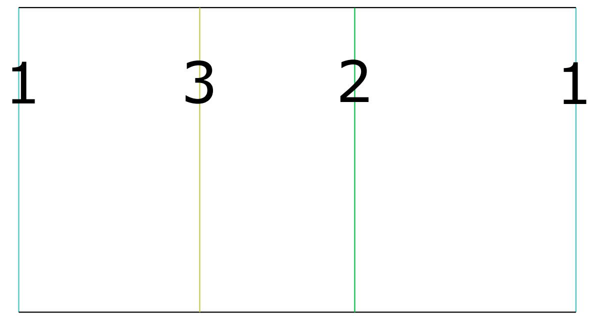

Figure 21: Case 3pD.

The arument for case 3pD is the same as that for 4ppD.

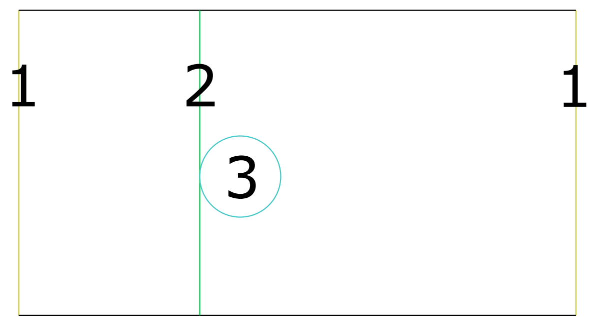

Figure 22: Case 3dD.

The argument for case 3dD is the same as that for case 4pdD.

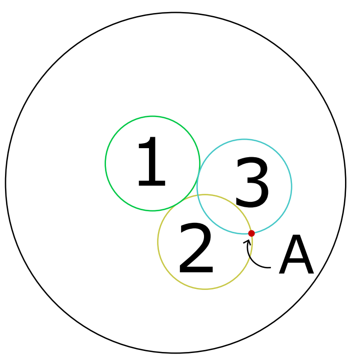

Figure 23: Case 3pI and 3dI, viewed with no points at infinity in the FAD.

Sending point A to infinity, we get two p-geodesics which intersect. Since they only case where this exists with two valid belted sum decompositions is 4pdI, we can see that this case does not exist, as it only utilizes three thrice punctured spheres.

□

Theorem 4.2. *When excluding twisted regluings, new belted sum decompositions can only be created by cutting horizontal 3 regions to the correct length, this creation is guaranteed to occur and can occur at any point in the decomposition without affecting anything else in the decomposition.*

Proof. Given a belted sum decomposition, the only way a new thrice punctured sphere can be created to form a new belted sum decomposition is if the surface that becomes the thrice punctured sphere intersects the belted sum decomposition. If this is the case, send their shared point to infinity and observe that the thrice punctured sphere being created now is a p-geodesic which intersects another p-geodesic to be created when cut. The only way this can happen is when a p-geodesic in region 3 is being cut to the correct length. However, by virtue of Theorem 4.1 the belted sum decomposition which created this new belted sum decomposition must always occur, and the potential for said belted sum decomposition to be created can only be destroyed by a symmetric belted sum decomposition (as only in case 4pdI) - that is, the new belted sum decomposition will always be created at some point in the decomposition, unless a symmetric decomposition has already been performed. Thus the only way for belted sum decompositions to be created does not affect the order of decomposition. □

It must also be noted that because this process of creation depends on cutting the FAD to a precise length, the process of creation must terminate.

Unfortunaly, when twist regluings are allowed, many belted sum options are able to be destroyed, and the number and location of these destructions depends randomly on the number and locations of twist regluings made. It is clear that this makes the order of decomposition along with the regluing type affect the order of decomposition. However, the following operation rectifies this issue.

Definition 4.1. *Pseudo-trivial belted sum decomposition* is the process of cutting along a single thrice punctured sphere in a manifold and then regluing the resulting pair of thrice punctured spheres with a twist.

Note that if the initial manifold had a twist gluing along this thrice punctured sphere, it can be untwisted after a pseudo-trivial belted sum decomposition, and vice versa. Pseudo-trivial belted sum decomposition is also not a true belted sum decomposition, rather a construct introduced to restore a version of prime uniqueness to the factorization. Being such, a key difference is that there is no fixed point which inherently restricts our regluing; however, since this will be a tool used only to "undo" regions which are twisted, we can simply apply these pseudo-trivial twists judisciously and fix the initial fixed points of the given twist in place to untwist.

Theorem 4.3. *Allowing for the pseudo-trivial belted sum decomposition operation restores the fact that the order of decomposition is irrelevant.*

Proof. Because the only issue which makes order of decomposition matter is destruction of thrice punctured spheres by twist regluing, we can show that such spheres can be recreated by pseudo-trivial belted sum decomposition to make the arguments used in Theorem 4.1 applicable. Consider a thrice puncture sphere region that has been non-orientable by a twist regluing. Because it is preserved set0wise by the regluing, a pseudo-trivial belted sum decompotion can be performed on the twist region which destroyed it at any point in the decomposition, making it orientable once again. While unactive, it was not destroyed unless by a symmetric cut by the arguments of Theorem 4.1. Once orientable again, the arguments of Theorem 4.1 apply again. Thus when pseudo-trivial belted sum decomposition is allows, the order of decomposition and regluing truly is irrelevant. □

5 Primeness

Finally, we display an argument that the process of belted sum decomposition terminates. Without further hesitation:

Lemma 5.1. *A thrice punctured sphere must have at least two separate ideal points.*

Proof: Send an ideal point of the thrice punctured sphere to infinity. We can now study the way in which the p-geodesic that it lifts to intersects the horosphere plane. The p-geodesic is either vertical or horizontal. If it is vertical, it only punctures the horosphere once. If horizontal, it punctures the horosphere at most twice. Thus to attain three punctures, the thrice punctured sphere must have at least one more ideal point.

Theorem 5.2. *The process of belted sum decomposition terminates.*

Proof: A given FAD has a finite number of ideal points, n. Since each thrice punctured sphere must use at least 2 ideal points, there are $\binom{n}{2}$ pairs of ideal points available to house thrice punctured spheres. By othogonality considerations, at most two thrice punctured spheres can exist using the same pair of points. Thus at most $2\binom{n}{2}$ thrice punctured spheres can exist in a given FAD. By the arguments from the last section, it is possible that new thrice punctured spheres could be created in the FADs resulting from belted sum decomposition; however, this process of creation terminates. Thus the total number of thrice punctures spheres along which belted sum decomposition is possible in an FAD is finite. So each FAD resulting from a belted sum decomposition of the initial FAD must have fewer thrice punctured spheres along which betled sum decomposition is possible (expect perhaps in the finite number of cases where thrice punctured spheres are creates). It must then be the case that the belted sum decomposition process terminates. With this, we see that belted sum decomposition yields prime factorization on the set of FADs. Allowing for pseudo-trivial belted sum decomposition, the terminal pieces of this prime factorization can be segregated into equivalence classes based on whether or not they can be transformed into one another via pseudo-trivial belted sum decomposition. Thus, with pseudo-trivial belted sum decomposition, we get that FADs factor primely and uniquely into equivalence classes of FADs, and hence FALs follow suit. It must be noted also that Whitehead Links will be in this factorization due to the "quarter twist" of Type 1-2 FADs, but that Whitehead Links are the only non-FALs in this decomposition, due to the nature of the belted sum decompositions avaiable to be made in a Type 1-2 FAD.

6 Conclusions, Open Questions, and Acknowledgements

To conclude, it has been shown that FALs factor primely but not uniquely under belted sum decomposition. Uniqueness can be restored when pseudo-trivial belted sum decomposition is allowed, but only up to equivalence classes based on pseudo-trivial belted sum decomposition.

Open questions include the ability to generalize these results about FADs to more general tesellations of $\mathbb{H}^3$ and generating classes of prime FADs to studying their algebraic structure using the restrictions on the locations of thrice punctured sphere outline in this paper.

I'd like to extend my great appreciation to Dr. Rollie Trapp for his good-humored and warm guidance; both he and Dr. Corey Dunn provided such a comfortable and welcoming atmosphere during my eight weeks at CSUSB. I'd like to thank the NSF for providing the funding for this valuable and formative research program. Lastly I'd like to thank Amanda, Gabe, Gabby, Maxine, Maya, Ryan, and Wil for the group meals, and the laughter, and the support, and the Jug Band, and the friendship.

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