# Geometric Exploration of T-Links using Lorenz Link Correspondence

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**Abstract**

Interested in studying T-links in hyperbolic space, we will explore other links and surfaces to obtain a different perspective that might guarantee hyperbolicity. First, we will utilize the one-to-one correspondence between T-links and Lorenz links and will classify subsets of hyperbolic and non-hyperbolic T-links by investigating several methods of nesting fully augmented links. We will also acknowledge link isotopies on the Lorenz Template to relate T-links to each other further. Next, we identify which properties of the braidword imply hyperbolicity for nested and unnested T-links. We will conclude identifying the smallest genus surface a link can embed on for all T-links, and discuss the next possible methods for studying T-links.

## 1 Introduction and Preliminaries

To fully understand T-links, we must first grasp the idea of crossings and T-links braids. Let there be some set of $r_k$ vertical parallel strands with a downward orientation. With these strands we are interested in positive crossings, where some left stand (one of two strands) crosses over the right strand. After the crossing, the two strands continue downward in their new positions in the T-link braid. In a T-link, all of these positive crossings start with the left most and first strand, and the crossing can continue along any number of the strands in the braid, up to $r_k$. What is called a crossing component consists of crossings along the same number of strands, where there are $s$ number of crossings. Any crossing involves at least two strands and at most $r_k$ strands, and occurs at least once. Given some collection of crossing components, where the number of strands in each component is always increasing, a proper T-link braid is formed, denoted

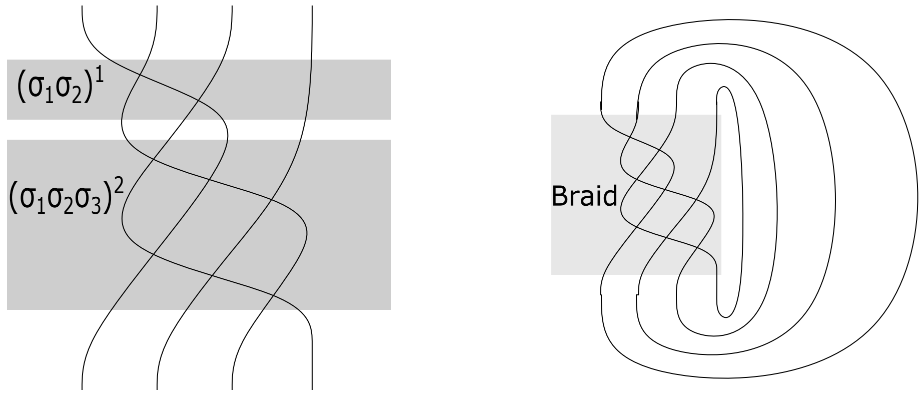
$$(\sigma_1 \cdots \sigma_{r_1-1})^{s_1} \cdots (\sigma_1 \cdots \sigma_{r_k-1})^{s_k}$$

The top and bottom strands of that T-link braid are then connected left to right respectively to form the T-link, denoted

$$\mathrm{T}((r_1, s_1) \ldots (r_k, s_k))$$

where $1 < r_1 < \cdots < r_k$ when the link is at its most simplified form, and $0 < s_i$ for all $i \in \mathbb{Z}^+$.

Let us consider the following example of the braidword in Figure 1a and its analogous T-link in Figure 1b of three strands crossing once followed by four strands crossing twice (two component T-link), with the corresponding descriptions $(\sigma_1\sigma_2\sigma_3)^1(\sigma_1\sigma_2\sigma_3\sigma_4)^2$ and $T((3,1)(4,2))$, both of which can be referred to as the braidword.




(a) The braid $(\sigma_1\sigma_2)^1(\sigma_1\sigma_2\sigma_3)^2$

(b) The T-link $T((3,1)(4,2))$

Figure 1

A general T-link braid, $(\sigma_1 \cdots \sigma_{r_1-1})^{s_1} \cdots (\sigma_1 \cdots \sigma_{r_k-1})^{s_k}$ is demonstrated by the template in Figure 2, where a block of $r_i$ width and $s_i$ height represents the crossing component $(r_i, s_i)$ in the braid and T-link (once the braid is closed and connected).

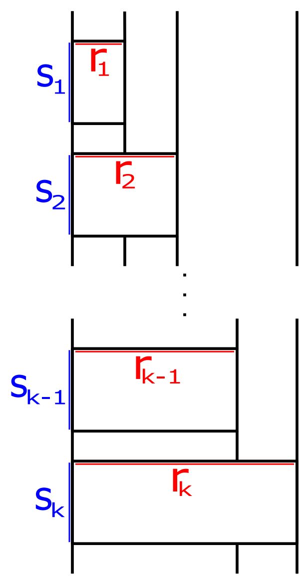



Figure 2: Generic T-Link Braid

We will also be interested in links known as $(p,q)$ Lorenz links. A $(p,q)$ Lorenz link is any link that can be successfully defined and projected using the Lorenz template, seen in Figure 3, and denoted

$$\langle r_1^{s_1}, \ldots, r_k^{s_k} \rangle$$

with the same corresponding entries at the T-link.[2].

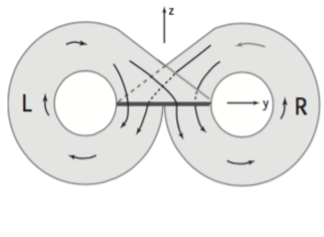
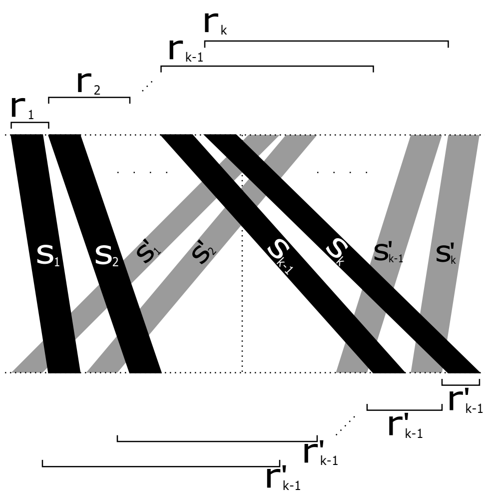
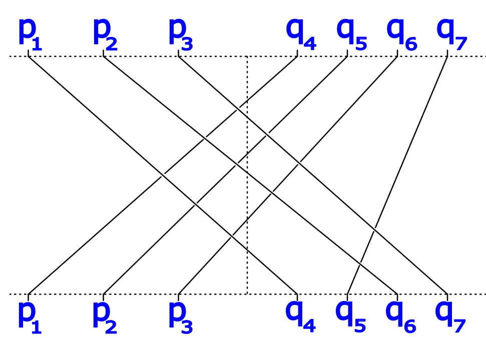



Figure 3: Birman and Kofman's Lorenz Template

Maneuvering through this Lorenz template, strands within the center are permitted to move from the left lobe to left (left left, LL), left lobe to right (left right, RR), right to left (right left, RL), and right right (RR). On both the left and right lobes, strands will reconnect to the top of each respective side. Another take on a general Lorenz template can be viewed in Figure 4a.




(a) $\langle r_1^{s_1}, \ldots, r_k^{s_k} \rangle$




(b) Lorenz Link $\langle 3^1, 4^2 \rangle$

Figure 4

On a Lorenz Template $p = \sum_{i=1}^{k} s_i$ strands exists on the left and $q = r_k$ strands exists on the right. Deriving the template in Figure 4a, notice that each crossing component (a band of $s_i$ strands) in a link $r_i^{s_i}$ implies the leftmost available $s_i$ strands in $p$ connect from the top of the template to the bottom $r_i$ strands to the right. The first component $r_1^{s_1}$ always uses the left most $s_1$ strands, while all consecutive components use the left most strands of the remaining in $p$ until the last crossing component $r_k^{s_k}$ uses the last remaining strands available in $p$.After all $p$ strands are connected from top to bottom crossing over all $q$ strands connecting from the top to the available strands across the whole template on the bottom (left to right). Also note that strands from the same side are always parallel. After the template is filled using the information from the Lorenz link description, the top and bottom strands from each side loop around the template and connect, forming a Lorenz link.

The T-link $T((3,1),(4,2))$ as a Lorenz link, $\langle 3^1, 4^2 \rangle$, can be seen in Figure 4b. Using Lorenz notation, our link has $p = 3 = \sum_{i=1}^{2} s_i$ strands on the left and $q = 4 = r_2$ strands on the right. The first strand $p_1$ on the top will move three strands to the right and connect to the fourth strand, $q_4$ on the bottom (satisfying $3^1$).The next two strands $p_2$ and $p_3$ on the top will move four strands to the right and connect to the sixth and seventh strands $q_6$ and $q_7$ on the bottom respectively (satisfying $4^2$). Notice that the first strands on the bottom were left open after $3^1$ and one strand was left open between the previous component and $4^2$. Now that the $p$ strands are connected, the four $q$ strands attach to the available bottom strands from left to right. Hence, $q_4$, $q_5$, and $q_6$ on the top will connect to $p_1$, $p_2$, and $p_3$ on the bottom, and $q_7$ on the top will connect to $q_5$ on the bottom. The top and bottom $p$ and $q$ strands will then connect and the Lorenz link will be complete.

Interested in these Link's, one of the best ways to study them is by looking at the space around the knot, the compliment, in hyperbolic space. Unfortunately, the geometric structure of T-links is sometimes too complicated and requires a simpler form. But, these same T-links can be studied in hyperbolic space as simpler links called fully augmented links (FAL). To construct a FAL, we surround a twist region in our T-link with a circle known as a crossing disk, which must surround either two strands of the link, or a single strand and another disk. With these disks, the link is augmented and any collection of full twists ($k$ strands crossing $k$ times) is removed from the link. After all full twists are removed, we have a FAL that can represent the original link, and has simpler geometric structure.[6] These crossing disks are applied to the link through what is called nesting, and links can be nested in several different ways to achieve specific conditions on that link. But after studying a FAL, all gathered information on that link is true for the original link, as well as links with any number of full twists added to the strands within the disks, using a process known as Dehn Filling.[6] These additionally constructed links are called offspring of the FAL, and the FAL with its offspring (including the original link) form a family of T-links, all sharing similar properties to the FAL.

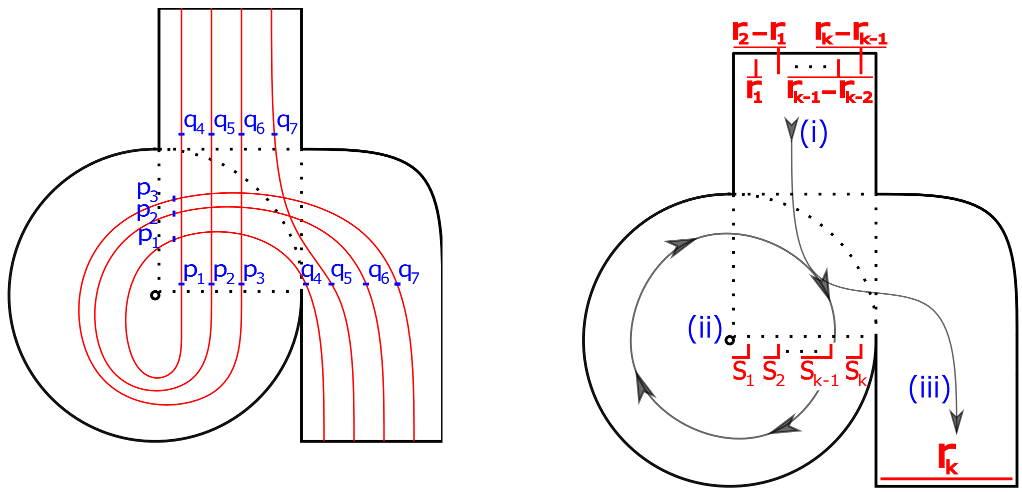
We will later use these FAL's and nesting to determining the hyperbolicity of a link. A link is hyperbolic if the links compliment can be viewed in hyperbolic space. Lastly, we will also test hyperbolicity searching for an annulus on a given T-link, and by observing T-links on genus surfaces. An annulus is a continuous band between two strands that is not punctured or split by any other strands in the link. If located within a T-link or Lorenz link, an annulus implies that the link is non-hyperbolic. A genus surface is a three dimensional surface that contains a hole known as a genus. The shell of a ring with one hole is considered to be a genus one surface, and a ring with $k$ holes would produce a genus $k$ surface. These will be used to study links capable of lying flat on the genus surface.

## 2 T-link and Lorenz Link Correspondence

To better understand T-links, Birman and Kofman introduced a one-to-one correspondence between T-links and Lorenz links.[2, Theorem 1] We will explore this relationship to solidify the transition between the two links, as well as discover isotopies between different T-links. With these connections between different types of links, we will then use information from one link to make implications on another.

Birman and Kofman's transition between the two links is best described as uncoiled transition link where defining characters from either link can be relayed and interpreted in terms of the corresponding link. This process is demonstrated with transition link template seen in Figure 5. This transition link consists of top strands in section (i) moving either through section (ii) and ending as bottom strands in section (iii), or starting as top strands from (i) and moving directly to (iii) and ending again as bottom strands. Provided either description of a link (T-link or Lorenz link), the transition link consists of $r_k$ strands in (i) and (iii), and $\sum_{i+1}^{k} s_i$ strands in (ii). Constructing the transition link from the Lorenz template is relatively easy, for a modified template exists within the transition link, as seen in Figure 5a. Using the same strands and rules as on the Lorenz template, the transition link is assembled and only requires a connection of strands in (ii) and around outside of the template between (i) and (iii)

Maneuvering between the T-link and transition, using the available entries seen in Figure 5b we can then follow the first crossing components of the T-link from (i) to (ii), add any addition strands moving from (i) to (iii), and then repeat with each addition crossing components until the link is complete and then requires a connection between all strands. Using the relationship between T-links and Lorenz links, we can easily transition between the two different braidwords to quickly switch between the new links.[2, Lemma 1]




(a) Transition Link of T((3,1)(4,2))

(b) General Transition Link Template

Figure 5

Along with this relationship, we bring attention to another isotopy between Lorenz links (and thus T-links). Provided some Lorenz template for some link, there exists an equivalent Lorenz link that is derived from a 180° rotation of the template. For example, $\langle 3^1, 4^2 \rangle$ rotates, interchanges its $p$ and $q$ sides and ends with the equivalent Lorenz link, $\langle 2^1, 3^3 \rangle$, displayed in Figure 6. So, for every Lorenz link, there exists this rotated isotopic link with the exact same structure. Using this isotopy, any implications made on one link will follow to its rotated link.

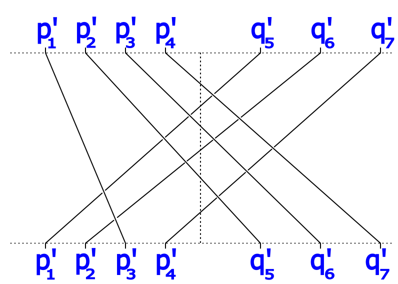



Figure 6: $\langle 2^1, 3^3 \rangle$, the rotated equivalent to $\langle 3^1, 4^2 \rangle$

Given only a T-link, we can exploit this new found relationship to construct a method to transfer from the T-link to its equivalent rotated link, without ever seeing its Lorenz projection.

**Theorem 2.1.** *For every T-link, there exists an isotopic T-link (due to Lorenz rotation),* $T((r'_1, s'_1) \ldots (r'_k, s'_k))$, *constructed component-wise using,*

$$(r'_n, s'_n) = \left( \sum_{i=1}^{k} s_i - \sum_{i=1}^{n-1} s_i,\ r_n - r_{n-1} \right)$$

*up to reordering of each crossing component in ascending order of $r'_i$.*

*Proof.* This Theorem is best demonstrated by Figure 4a. Given some T-link with, its crossing component entries determine the nature of all $p$ bands in the figure, while all $q$ follow suit to fill the remaining template and construct the link. To form this rotated link, we are interested in the crossing components that determine the nature of the $q$ bands, while the $p$ bands follow suit.

So, $(r'_1, s'_1)$ is the first band in $q$ derived from the first gap produced by $(r_1, s_1)$. In regards to movement, the band crosses all $p = \sum_{i=1}^{k} s_i$ strands, thus $(r'_1 = \sum_{i=1}^{k})$. In regards to band size, the number of strands is determined by the gap created by $(r_1, s_1)$, namely the movement $r_1$, thus $s'_1 = r_1$, and $(r'_1, s'_1) = (\sum_{i=1}^{k} s_i, r_1)$. Similarly the second crossing component in $q$, $(r'_2, s'_2)$, crosses all $p$ bands but the first, thus the movement is $r'_2 = \sum_{i=1}^{k} s_i - \sum_{i=1}^{1} s_i$. The second component also lives in gap between $(r_1, s_1)$ and $(r_2, s_2)$, hence its size is determined by the difference in movement between the first two $p$ bands, thus $s_1 = r_2 - r_1$ and $(r_2, s_2) = (\sum_{i=1}^{k} s_i - \sum_{i=1}^{1} s_i, r_2 - r_1)$.

For any general crossing component $(r'_n, s'_n)$, demonstrated in the Figure, $r'_n$ always depends on the difference between the total strand size of all the bands and the total size of the bands $(r'_n, s'_n)$ doesn't cross, i.e $r'_n = \sum_{i=1}^{k} s_i - \sum_{i=1}^{n-1} s_i$ since $(r'_n, s'_n)$ crosses the $n^t h$ band and further right. $s'_n$ will always depend on the gap permitted between $(r_n, s_n)$ (the last band on the left it crosses) and $(r_n, s_{n-1})$ (the first band on the left it does not cross), thus $s'_n$ is decided by the difference in movement of the two $p$ bands, i.e. $s'_n = r_n - r_{n-1}$. We can then conclude that $(r'_n, s'_n) = (\sum_{i=1}^{k} s_i - \sum_{i=1}^{n-1} s_i, r_n - r_{n-1})$. □

In consequence to determining isotopic T-link using this crossing component wise method, the components of the T-link will appear in descending order of $r'_i$, and the the T-link must be reordered in terms of ascending $r'_i$ to maintain a consistent braidword.

Considering T((2,1)(3,3)), the rotated isotopic T-link to T((3,1)(4,2)), we can derive the link using the crossing component wise construction where

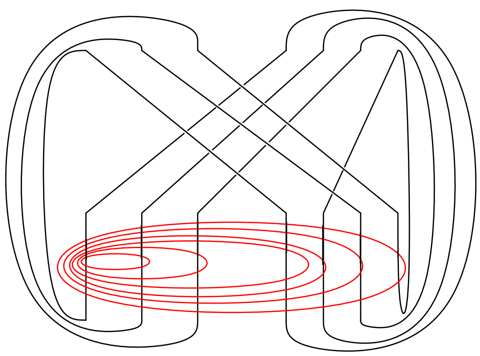
$$\begin{aligned}(r'_1, s'_1) &= (\sum_{i=1}^{2} s_i, r_1) \\ &= (1+2, 3) \\ &= (3, 3)\end{aligned}$$

$$\begin{aligned}(r'_2, s'_2) &= (\sum_{i=1}^{2} s_i - \sum_{i=1}^{1} s_i, r_2 - r_1) \\ &= ((1+2) - 1, 4 - 3) \\ &= (2, 1)\end{aligned}$$

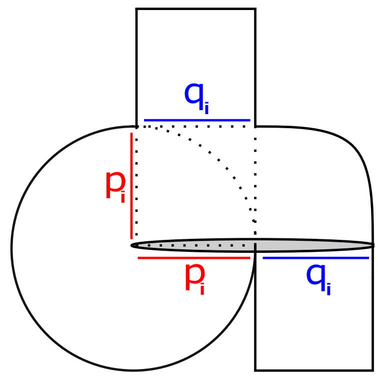
The two crossing components must then be reordered in terms of increasing $r_i$, and thus T((2,1)(3,3)) is confirmed as the isotopic rotated T-link of T((3,1)(4,2)) using this crossing component reconstruction.

# 3 Nesting Methods

Relying on FAL's, an exploration of possible nesting on T-links is necessary to observe and generalize consistent conditions of hyperbolicity under a certain nesting. The nestings of interest are referred to as a complete, right, and left nesting. The nesting with the most immediate result is the complete nesting, which spans across the $p+q$ crossing strands. This nesting originated from work on the projection of a one crossing component T-link onto the fundamental region for a Torus and is represented on a Lorenz link and transition link in Figure 7.[4]



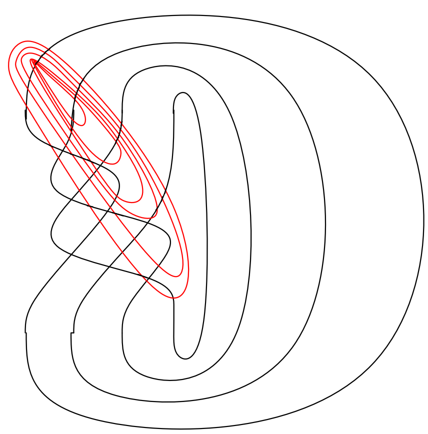
(a) Complete Nested $\langle 3^1, 4^2 \rangle$



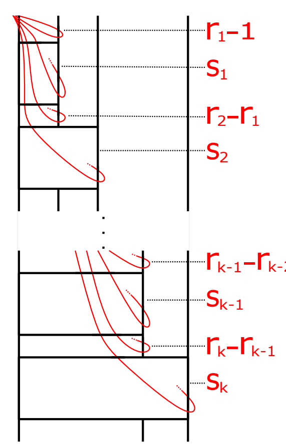

(b) Completely Nesting across Transition Link

Figure 7

It is further represented on T((3,1)(4,2)) in Figure 7a, where each crossing disk threads over vertical strands of the T-link, under horizontal crossing strands of the link, and then under the entire T-link on the back side of the disk. We then move to a general template for a complete nesting on a T-link braid in Figure 7b where each crossing disk strand actually represents a collection of strands determined by the entry information on the right side of the figure.



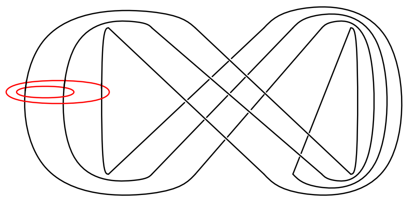
(a) Completely Nested T((3,1)(4,2))



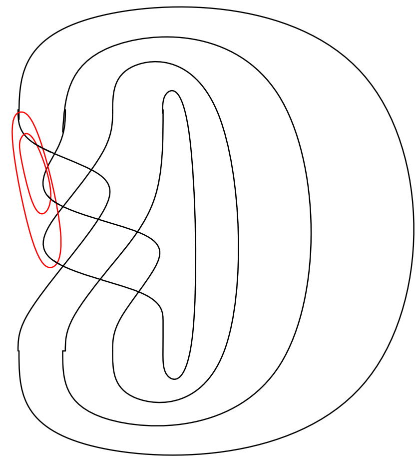

(b) Completely Nested $\langle 3^1, 4^2 \rangle$

Figure 8

The second pertinent nesting is a nesting along the left lobe of a Lorenz link, like in Figure 9a, which is used in determining hyperbolicity when $p > q$. This same nesting correlates to the nesting within the crossing components on a T-link, like in Figure 9b.



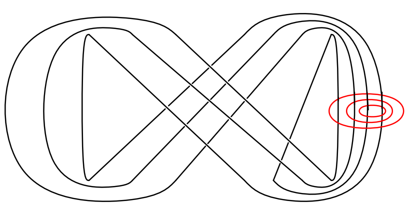
(a) Left Nested $\langle 3^1, 4^2 \rangle$



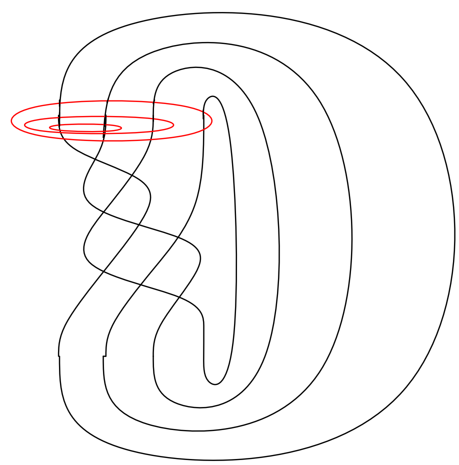
(b) Left Nest on T((3,1)(4,2))

Figure 9

The last form of nesting used is a along the right lobe of a Lorenz link, like in Figure 10a, which is used when $p < q$. This nesting on the T-link forms a more intuitive nesting just along the connecting band of the link, as seen in Figure 10b.



(a) Right Nested $\langle 3^1, 4^2 \rangle$



(b) Right Nested T((3,1)(4,2))

Figure 10

# 4 Hyperbolic Determination

Equipped with several nesting methods, we can determine the hyperbolicity of certain families of T-links using Coenen's work on nested Lorenz links, their cell decompositions, and their nerve.[5] Using Coenen's theorem on complete nesting and the Lorenz Hyperbolicity Theorem, the complete crossing and larger lobe will be utilized on Lorenz links. After determining the hyperbolicity of nested T-links, specific properties of the components of a link's braidword can be identified to imply specific conditions of hyperbolicity.

Beginning with complete nesting, it turns out that all completely nested Lorenz links are hyperbolic.[5] This provides an immediate option to study T-links in hyperbolic space, but there exists nesting that assures a hyperbolic T-link with simpler geometry structure. Complete nesting is seen as a first option for studying hyperbolic links, but it is the not easiest nesting to work with in regards to geometric structure.

The remaining nesting that can be used to determine hyperbolicity is larger lobe nesting of a Lorenz link. According to the Lorenz Hyperbolicity Theorem, given some left nested $(p, q)$ Lorenz link where $p > q$ and there exists no RR's, there are four possible defining cases for the Lorenz link. One case in which the nested link is non-hyperbolic, and the three remaining cases where the nested link is hyperbolic. These cases are determined by the nature of the last bottom strands in $p$.[5]

The goal from a T-link perspective is to deduce from the braidword of a link and validate on which case follows. Unfortunately the nature of these two bottom $p$ strands is not directly available from the braidword since the origin of these strands is typically from the top of $q$. But, to properly read the braidword and fix some T-link to one of these four cases, the isometry of Lorenz links through 180° rotation is crucial. Under the same conditions required by the Lorenz Hyperbolicty Theorem, the rotation of these Lorenz links are accessible under the Theorem. These rotations happen to be, right nested $(p, q)$ Lorenz links where $p < q$ and there exists no LL's, where the new defining strands now originate from the beginning of the top of $p$ and end at the first strands in $q$, which correlate to the first crossing component of the link. Using this rotated link, we can now use the first component of the braidword, determine hyperbolicity, and make the same implications on the rotated Lorenz link.

Before grasping this perspective, we must know what Lorenz links may follow one of the four conditions. Those Lorenz links contain LL's and no RR's, or RR's and no LL's (the rotated link more suitable for T-links). The only other possible links are one that contain both LL's and RR's. The hyperbolicity of these LL's and RR's links cannot be determined by any current work on these links. Knowing the appropriate links for the theorem, it is ideal to notice when a link contains LL's, RR's, or both.

**Lemma 4.1.** *A Lorenz link contains LL's if and only if* $r_1 < \sum_{i=1}^{k} s_i$.

*Proof.* Given that the entry $r_1$ in the crossing component $r_1^{s_1}$ of some Lorenz link (and its corresponding T-link) is less than the value of $p = \sum_{i=1}^{k} s_i$, at least one strand of the component will reside on the left since the movement is not great enough to send all strands of the band into $q$. Thus, a LL exists if and only if $r_1 < p = \sum_{i=1}^{k} s_i$. □

**Lemma 4.2.** *A Lorenz link contains RR's if and only if* $s_k < r_k$.

*Proof.* Given that the entry $s_k$ in the crossing component $r_k^{s_k}$ of some Lorenz link is less than the value of $q = r_k$, not all strands in $q$ will map to strands from $r_k^{s_k}$, leaving gaps between the that and the previous crossing component. That gap will be filled by at least one strand (from $r_k'^{s_k'}$) and there will exist a RR mapping from the top to bottom of $q$. Thus, a RR exists if and only $s_k < q = r_k$. □

If only one of the these lemmas is true at a time for some Lorenz link, either the original link or its rotation can follow the Lorenz Hyperbolicty Theorem. But if both lemmas hold, a link contains LL's and RR's hence nothing about its hyperbolicty can be determined using the theorem.

Addressing the argument of the theorem, we will now validate the interpretation of each case using the braid word of the a $(p, q)$ larger nested link with no LL's ($p < q$ and right nested) while still representing no RR's as the isotopic rotated link ($p > q$ and left nested).

**Theorem 4.3.** *Let there exist some larger nested Lorenz link. The right nested Lorenz link (and its isotopic rotated link) is hyperbolic if its braidword satisfies one of three cases,*

$$Case\ 1 : r_1 = \sum_{i=1}^{k} s_i, s_1 = 1$$

$$Case\ 2 : r_1 \geq \sum_{i=1}^{k} s_i + 2$$

$$Case\ 3 : r_1 = \sum_{i=1}^{k} s_i + 1$$

*The nested Lorenz link is non-hyperbolic of its word satisfies the following case,*

$$Case\ 4 : r_1 = \sum_{i=1}^{k} s_i, s_1 \geq 2$$

*[5]*

*Proof.* Each case is proved using a $(p, q)$ right nested Lorenz link with $p < q$, then all isotopies are considered. The reason for this is that the Lorenz Hyperbolicity Theorem is interested in the first two bottom strands in $q$ and where they originate. The strands can exist in these four different ways and the braidword will be interpreted for each case.

Case 1:

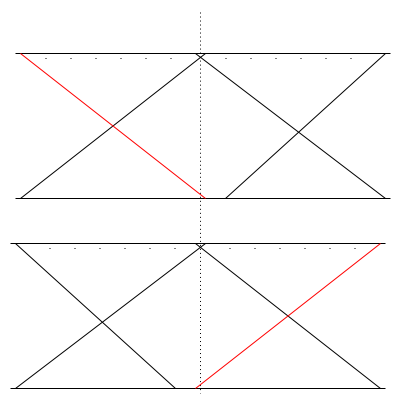


Figure 11: $r_1 = \sum_{i=1}^{k} s_i,\ s_1 = 1$

Case 1 in the Lorenz Hyperbolicity Theorem (LHT) requires the only one strand from $r_1^{s_1}$ to map across all of $p = \sum_{i=1}^{k} s_i$ to connect to the first strand in $q$. That second strand, as a gap strand between $r_1^{s_1}$ and $r_2^{s_2}$ will connect with $q$ and create the desired scenario in LHT. Thus, it must follow that $r_1 = \sum_{i=1}^{k} s_i$ and $s_1 = 1$, and the nested link is hyperbolic. This scenario is demonstrated for both rotations of the Lorenz link in Figure 11.

Case 2:

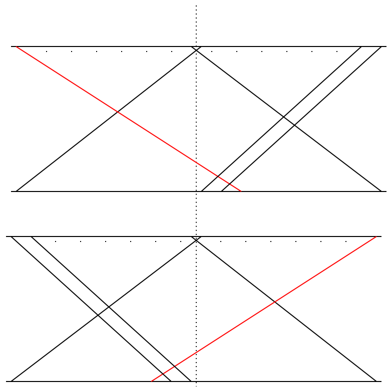


Figure 12: $r_1 \geq \sum_{i=1}^{k} s_i + 2$

Case 2 in LHT requires the first strand from $r_1^{s_1}$ to map at least across all of $p + 2 = \sum_{i=1}^{k} s_i + 2$ to connect to the third strand in $q$. This allows for two strands from $p$ to connect to the first strands left open, creating the desired scenario in LHT. Thus it must follow that $r_1 \geq \sum_{i=1}^{k} s_i + 2$ and the nested link is hyperbolic. This scenario is demonstrated for both rotations of the Lorenz link in Figure 12.

Case 3:

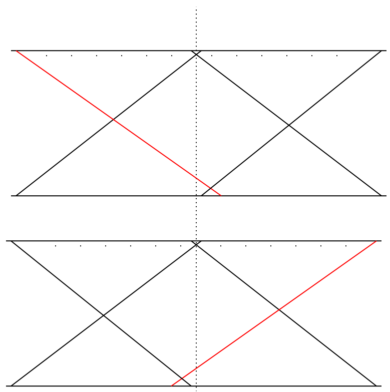


Figure 13: $r_1 = \sum_{i=1}^{k} s_i + 1$

Case 3 in LHT requires the first strand from $r_1^{s_1}$ to map across all of $p + 1 = \sum_{i=1}^{k} s_i + 1$ to connect to the second strand in $q$. This allows for one strand from $p$ to connect to the first strand left open, creating the desired scenario in LHT. Thus, it must follow that $r_1 = \sum_{i=1}^{k} s_i + 1$ and the nested link is hyperbolic. This scenario is demonstrated for both rotations of the Lorenz link in Figure 13.

Case 4:

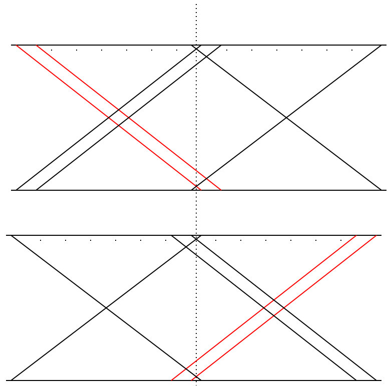


Figure 14: $r_1 = \sum_{i=1}^{k} s_i,\ s_1 \geq 2$

Case 4 in LHT requires the first two strands from $r_1^{s_1}$ to map across all of $p = \sum_{i=1}^{k} s_i$ to connect to the first two strands in $q$. This results in a bounded annulus living in the nested link. Thus, it must follow that $r_1 = \sum_{i=1}^{k} s_i$ and $s_1 = 2$, and the nested link is non-hyperbolic. This scenario is demonstrated for both rotations of the Lorenz link in Figure 14. $\square$

Despite not all cases guaranteeing a hyperbolic nested T-link, the geometric structure of the ones that are is simpler than a completely nested T-link. Hence, larger nesting is preferred if a link is hyperbolic in either case.

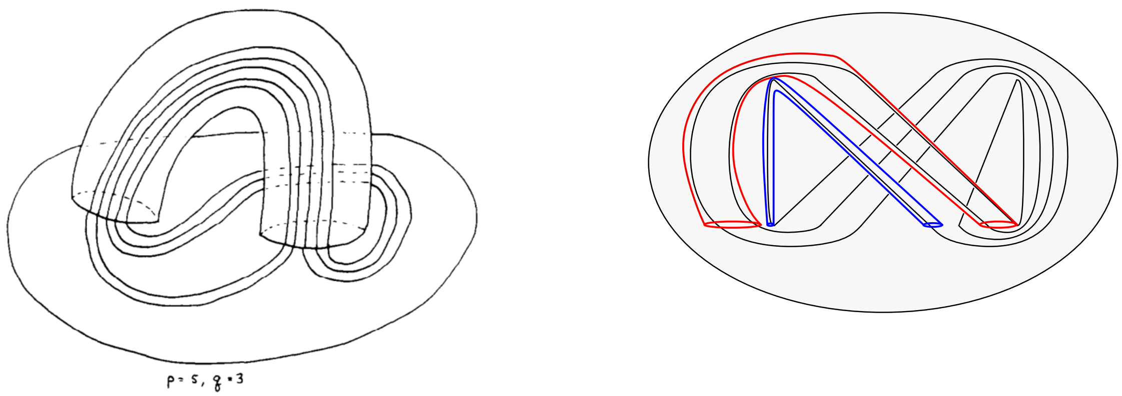
It is also important to note that Case 4 of LHT holds for unnested Lorenz links that still follow the required braidword conditions. This is because the annulus that exists in Case 4 is still bounded when nesting is removed. Also, LHT does not consider when $p = q$. But when that scenario is present, the T-link in question either contains both LL's and RR's, or is non-hyperbolic by the same principle as Case 4. Whenever an annulus is preserved within the unnested Lorenz, the additional case of the following corollary can occur.

**Corollary 4.4.** *An unnested Lorenz link is non-hyperbolic if* $\forall i = 1, \ldots, k, \exists t \neq 1 \mid r_i = h_i t$ *and* $s_i = k_i t$ *for some* $h_i, k_i \in \mathbb{Z}^+$.

*Proof.* When every $r_i^{s_i}$ entry in some Lorenz link is a multiple of a $t$ entry where $t = r_a$ or $t = s_a$ for some $a \in 1, \ldots, k$, some band(s) of $k_i t$ strands in the link is preserved by all other bands since they are a width of strands that provide the required $k_i t$ width. Since the movement along the link consists of values $h_i t$, the proper width is still preserved with each movement guaranteeing that $k_i t$ width. This ensures that the band(s) is never punctured or split by other strands, and thus at least one annulus is bounded and the Lorenz link must be non-hyperbolic. $\square$

# 5 Genus Expansion

Studying T-links is not limited to nested Lorenz links, for we can also view these links standardly embedded on various genus surfaces. To standardly embed a T-link on a genus surface, its Lorenz link must lie flat, unknotted, and without crossings on the space.[1] In Figure 15a, we have a one crossing component link embedded on a genus one surface, or torus, where the genus is represented by the arch and the rest of the surface is flattened. [3, Fig. 6.1.] Also, in Figure 15b, the previous example link $\langle 3^1, 4^2 \rangle$ is embedded on a genus two surface, or two torus. With a split between the two crossing components, two arches are required for the link to live unknotted on the surface. Notice that the crossing components of both links live on arches of each genus surface, while the rest of both links live on the flattened surface.




(a) One Crossing Component Link on a Genus One Surface

(b) $\langle 3^1, 4^2 \rangle$ on a Genus Two Surface

Figure 15

Now, despite the surfaces our example links live on, the links can actually live embedded on higher genus surfaces. The one crossing component example however cannot live on any surface of lower genus than a torus.[1] But, we will show that any Lorenz link of $k$ crossing components can always live on a genus $k$ surface. Although, this genus $k$ surface does not determine the smallest possible embeddable surface for a $k$ crossing component.

**Theorem 5.1.** *Every Lorenz link $\langle r_1^{s_1}, ..., r_k^{s_k} \rangle$, and the respective T-links, can be embedded on a genus $k$ surface.*

*Proof.* First, notice again from Figure 15, we know that a one and two crossing component link can be embeddable on a one and two genus surface respectively. The defining left strands of a Lorenz template live within the arches, and the additional crossing component in $\langle 3^1, 4^2 \rangle$ causes a split between components and establishes a need for a second arch and thus a second genus.

Next, we will assume that an $n$ crossing component Lorenz link lives on an $n$ torus and show that an $n+1$ crossing component T-link lives on an $n+1$ torus.

So, let some $n$ component Lorenz link with $n$ distinct crossing components and $n$ distinct gap bands, live on a genus $n$ surface. To produce an $n+1$ link, we will add a new crossing band to $p$ and consequently an $r_{n+1} - r_n$ thick band to $q$ to produce the $(r'_{n+1}, s'_{n+1})$ pair that lives on the flattened segment of the surface. With the addition of a distinct and separated $p$ band, another arch is required to embed the newest defining band. This new defining arch and the past $n$ arches can be observed in Figure 16.

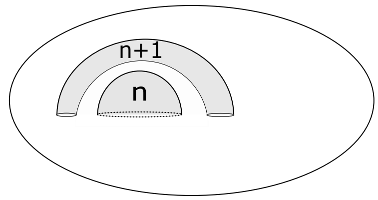



Figure 16: General $n + 1$ Torus

Our $n + 1$ crossing component link then lives on a $n + 1$ torus, and we conclude that an $n$ crossing component Lorenz link (and T-link) can be embedded on a genus $n$ surface (or $n$ torus). □

With this confirmation, along with studying the compliments of our T-links in hyperbolic space, we can now observe these links in another way, on some genus surface.

# 6 Further Research

The next course of action in terms of the hyperbolicity is to investigate T-links with LL's and RR's in their Lorenz projection. This would begin with searching for any consistent properties that guarantee an annulus bound within the Link. Another area of interest is viewing the projection of a two crossing component T-link on the genus two fundamental region. The projection for a one crossing component T-link is already known on the fundamental region for a torus, but there has been no work on the two torus, or a $k$ torus. Once light is shed on the projection for a two crossing component T-link on a genus two surface, that fundamental region can be investigated for all preexisting and possible nesting methods.

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