

Belted Sum Decompositions in Nested Links

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1 Introduction

Hyperbolic geometry has been useful in the study of knots since Thurston's groundbreaking research in 1979 [1]. For example, his work implies that there are three types of knots: torus knots, satellite knots, and hyperbolic knots. For the purposes of this paper, we are interested in the properties of two classes of hyperbolic links: Fully Augmented Links (FALs) and Nested Links.

Adams proved that FALs are always hyperbolic [2]. An FAL is a link in which each twist of two strands and has been augmented. The notion of augmenting a twist region was formalized by Adams in [2]. The process consists of embedding a trivial component called a crossing circle around the region and removing all full twists. An illustration of this process can be seen in Figure 1. One convenient feature of an FAL is that each crossing disk is a thrice punctured sphere where the punctures are the intersections of two knot circles with the crossing disk and the boundary of the crossing disk.

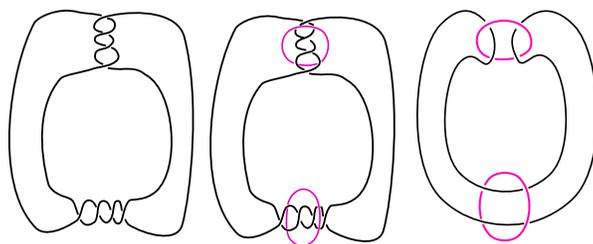


Figure 1: Fully augmenting yields the Borromean Rings

A Generalized Fully Augmented Link (GFAL) is a Fully Augmented Link that can have more than two strands per twist region. While GFALs are defined similarly to FALs, they have notable differences. For example, the crossing circles bound n -punctured spheres which may not be geodesic. This project focuses on a class of GFAL called nested links because they share convenient properties with FALs. In particular, in each augmented twist region of a nested link, each crossing circle is still a thrice punctured sphere. Figure 2 shows a

nesting with one crossing disk inside of the other, and both are thrice punctured spheres.

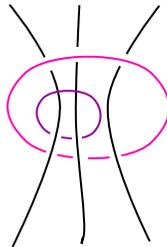


Figure 2: Nested Link

Both FALs and GFALs can be complicated in nature and thus difficult to study. Due to the work of Adams, FALs can be broken down into simpler components using the notion of Belted Sum Decomposition [2]. Morgan et al proved that every FAL can be broken down into prime FALs, or FALs that are not belted sums [3]. This paper extends results of [4] to the class of nested links.

2 Cell Decomposition

To see the geometry of an FAL, one can start with a cell decomposition. Cell decomposition is a process which takes an FAL to its circle packing. The process begins by defining 0-cells, 1-cells, 2-cells, and 3-cells. In an FAL, there are no 0-cells. 1-cells are the intersections of the plane of projection and crossing disks. The 2-cells in an FAL are the crossing disks and the plane of projection. Finally, the 3-cells are \mathbb{B}_+^3 and \mathbb{B}_-^3 , which are the regions above and below the plane of projection respectively.

The next step in cell decomposition is to cut along the planar 2-cells, which will divide the manifold into a top half and a bottom half. Then, slice along the crossing 2-cells and shrink each crossing circle arcs to a point. Finally, shrink the knot circle arcs. The unshaded regions will be circles in the circle packing and the shaded regions will remain triangles. An example of this process can be seen in Figure 3.

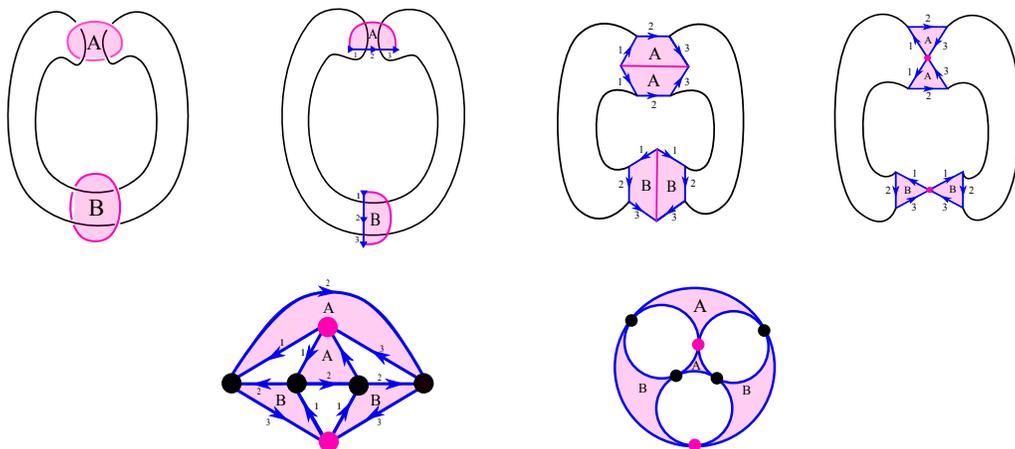


Figure 3: Standard Cell Decomposition of the Borromean Rings

3 The Nerve and the Dual

In various instances, it can be easier to study FALs by looking at their nerves. In order to do this, it is important to understand how to find the nerve corresponding to an FAL. Given an FAL, use the process of cell decomposition to find its corresponding circle packing. Then, place a vertex in the center of each circle. Finally, connect adjacent circles with edges through points of tangency. An edge is painted if it goes through a point of tangency that corresponds to a crossing circle (See Figure 4. The nerve of an FAL is a triangulation of S^3 [5], thus it follows that the dual to the nerve of an FAL will always be trivalent.

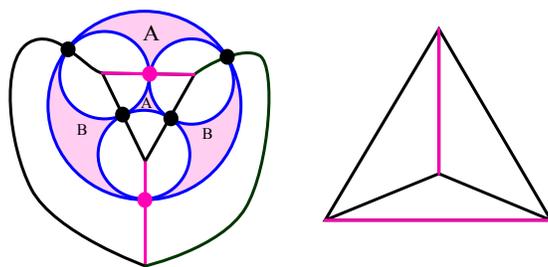


Figure 4: The circle packing of the Borromean Rings and Its Nerve

This project focused on studying links using the dual to the nerve. The dual to the nerve of an FAL will always be a trivalent, planar, connected graph. Conversely, given a perfect matching on a trivalent, planar graph, one can construct a corresponding FAL. A perfect matching is a graph in which each vertex is paired with exactly one adjacent vertex. The

pairing is represented by painting the edge between them. Note that each vertex in a perfect matching has exactly one painted edge.

A nested link is associated to each edge-symmetric spanning tree in a given dual. The edge of symmetry corresponds to the outermost crossing circle in a nesting. An example of a nested link and its dual can be seen in Figure 5. Both Nested Links and FALs can be described using painted duals.

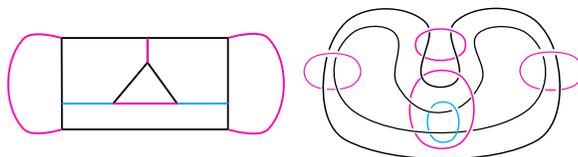


Figure 5: Left: A dual with an edge-symmetric spanning forest, Right: Corresponding Nested Link

4 Thrice-Punctured Spheres

Our main interest is in belted sum decompositions of Nested Links. It will be seen that thrice-punctured spheres are used in belted sum decompositions, so thrice-punctured spheres prove important in finding the results in this project. Thus, it was necessary to recognize a thrice-punctured sphere in a link based on its dual. Due to [3], we know that a 3-cut in a dual corresponds to a thrice-punctured sphere in an FAL. These facts led to the result in Lemma 1.

Lemma 1. In a perfect matching of a graph G , every 3 cut has an odd number of painted edges.

Proof. We know that every trivalent, connected graph has an even number of vertices. We can separate G into a left-hand side and a right-hand side using the 3-cut. Consider reducing the right-hand side to a single vertex. This gives a trivalent, planar, connected graph. Since the right-hand side is reduced to one vertex, this shows that the left-hand side has an odd number of vertices. A similar argument shows that the right-hand side must have an odd number of vertices.

Assume that no edges in the 3-cut are painted. Then, without loss of generality, consider the left-hand side. Since G is a perfect matching, the left-hand side must be a perfect matching. However, a perfect matching requires an even number of vertices. Thus, at least one edge in the 3-cut must be painted.

Now, assume 2 edges in the 3-cut are painted. Without loss of generality, consider the left-hand side. Since we know there is an odd number of vertices in the left-hand side and 2 are now painted, there is an odd number of unpainted vertices. Since every perfect matching of the remaining vertices will require an even number of vertices, this is only possible if the

third edge in the 3-cut is also painted. Therefore, in a perfect matching of a dual, every three-cut is either once-painted or thrice-painted. \square

Studying belted sum decompositions also led to interest in prime links and their duals. [3] showed that in the nerve of a prime FAL, every 3-cycle must be thrice-painted. Thus, it follows that in the dual of a prime FAL, every 3-cut must be thrice-painted.

Lemma 2. If a trivalent graph has multiple 3-cuts with incident edges, it does not have a prime coloring.

Proof. By definition, a prime coloring is a perfect matching of the graph in which the edges in all 3-cuts are painted. If a graph has 3-cuts with incident edges, then at least two of the edges share a vertex. Therefore, they cannot both be painted in a perfect matching graph. Thus, the graph does not have a prime coloring. \square

5 Belted Sum Decompositions

In order to study prime FALs, it is necessary to know how to recognize when an FAL is a belted sum. A belted sum in an FAL is characterized by its *buckle*. A buckle is a once-painted 3-cycle in the nerve of an FAL, which corresponds to a once-painted 3-cut in the dual. Once the buckle is identified, you can think of enclosing the portion of the FAL bordered by the thrice punctured sphere with a second thrice-punctured sphere, which we call a *sock*. In order to decompose the link, we slice along the two thrice-punctured spheres making up the belted sum to get an inside link and outside link. Then, in each case the punctures in one thrice-punctured sphere will be glued to the punctures in the other thrice-punctured sphere without adding crossings. See Figure 6 for an example. This process can be repeated until the components are prime FALs.

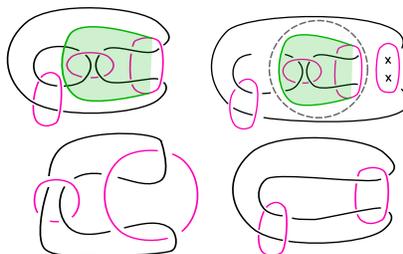


Figure 6: Belted Sum Decomposition of an FAL into two copies of the Borromean Rings

The sum of two perfectly matched graphs Γ and Γ' , denoted $\Gamma \oplus \Gamma'$, can be found in the following way. Consider two perfectly matched, n -regular graphs Γ, Γ' . Remove an arbitrary vertex from each. Since Γ is n -regular, there are n edges that were connected to the vertex removed. Call these edges $\{e_0, \dots, e_n\}$. Further, each edge in this set is connected to another

vertex in Γ . Call these vertices $\{v_0, \dots, v_n\}$. Similarly in Γ' , call these edges $\{e'_0, \dots, e'_n\}$ and the vertices $\{v'_0, \dots, v'_n\}$. Since both Γ, Γ' were well painted, we know that exactly one edge in the set $\{e_0, \dots, e_n\}$ is painted and exactly one edge in the set $\{e'_0, \dots, e'_n\}$ is painted. Without loss of generality, choose e_0 and e'_0 to be the painted edges. Then, glue Γ and Γ' such that $e_0 \sim e'_0, \dots$, and $e_n \sim e'_n$ (See Figure 7).

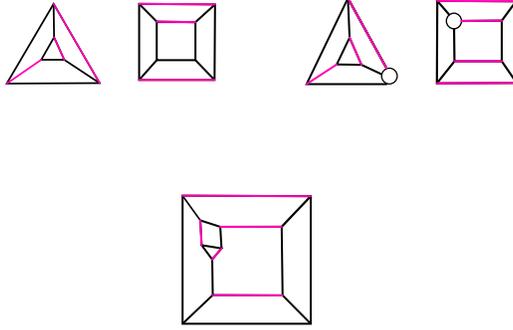


Figure 7: The sum of two perfectly matched graphs

Lemma 3. Let Γ and Γ' be two perfectly matched, n -regular graphs. Then, $\Gamma \oplus \Gamma'$ will be a perfectly matched, n -regular graph.

Proof. Trivially, we can see that all vertices in Γ that are not in $\{v_0, \dots, v_n\}$ remain well painted. Similarly, all vertices in Γ' that are not in $\{v'_0, \dots, v'_n\}$ remain well painted.

Then, since $e_0 \sim e'_0$, we know that no painted edges were added to v_0 or v'_0 , thus these vertices remain perfect matching. Finally, since $e_k \sim e'_k$ for all $0 < k \leq n$, we know that no painted edges were added to or removed from v_k or v'_k , so they remain perfect matching. Thus, the $\Gamma \oplus \Gamma'$ is perfect matching. \square

Theorem 4. Let Γ, Γ' be two trivalent, perfectly matched graphs. Then, $\Gamma \oplus \Gamma'$ is the belted sum of the FALs corresponding to Γ, Γ' .

Proof. Consider two trivalent, perfect matching graphs Γ, Γ' . We know that $\Gamma \oplus \Gamma'$ will be a trivalent, perfectly matched graph.

To see that the corresponding fully augmented link is a belted sum, recall that a once-painted 3-cut in the dual of a fully augmented link corresponds to a buckle in the fully augmented link. Then, note that when the two trivalent graphs are glued along the loose edges, this creates a nontrivial “inner” graph and a nontrivial “outer” graph that are connected by those three edges. This, by definition, introduces a 3-cut in the resulting graph and thus, the corresponding fully augmented link will be have a buckle. This shows that the

FAL will be a belted sum. □

6 Belted Sums in Nested Links

This project's main focus was to fully classify belted sum decompositions in nested links. We used the duals of these links to study and understand them. Similar to FALs, when we are looking for a belted sum in the dual of a nested link, we are looking for 3-cuts. Since the duals of nested links can have more complicated paintings than those of FALs, there are more ways to paint a 3-cut. From this, we were able to organize these duals into cases in a natural way. We divided them into belted sum decompositions corresponding to once-painted 3-cuts, twice-painted 3-cuts, and thrice-painted 3-cuts. The following lemma was useful in proving results in all cases.

Lemma 5. Let S_1, S_2 be socks corresponding to a belted sum decomposition. Then each crossing circle puncture in S_1 must also be a puncture in S_2 .

Proof. Consider a Nested Link L . Suppose there exists a belted sum decomposition of L that includes a sock with the boundary of a crossing disk as a puncture. For sake of contradiction, assume that the sock is neither paired with the crossing disk or another sock sharing the boundary of the crossing disk. Then, slice along the thrice punctured spheres in the belted sum decomposition. Since we know the crossing disk is a thrice punctured sphere itself, there are at least two knot arcs that do not get severed in this operation. Thus, the manifold does not completely separate and this is not a belted sum decomposition, leading to a contradiction.

6.1 Once-painted 3-cuts

Theorem 6. Let S_1, S_2 be socks corresponding to two once-painted 3-cuts that share a painted edge in a planar, trivalent graph. Then, there is a valid belted sum decomposition in the associated Nested Link.

Proof. Consider a trivalent planar graph G where two 3-cuts share exactly one painted edge.

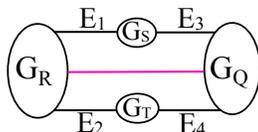


Figure 8: The Graph G

Then, consider the link corresponding to this graph. We know that the painted edge corresponds to a crossing disk that is a thrice punctured sphere. Note that the punctures of this crossing disk could be knot circles or nested crossing disks. Then, consider a sock found by starting at the crossing disk and enveloping tangle R . Note that this is a thrice punctured sphere, its punctures being the boundary of the crossing disk and the knot arcs K_1 and K_2 . Similarly, consider a sock found by starting at the crossing disk and enclosing tangle Q . This is also a thrice punctured sphere, with its punctures being the boundary of the crossing disk and the knot arcs K_3 and K_4 .

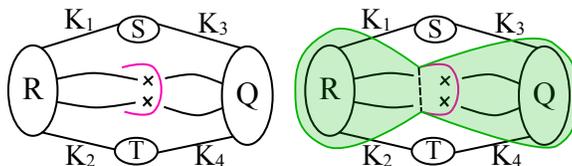


Figure 9: Left: The Nested Link, Right: The Nested Link With S_1 and S_2

Finally, we slice along S_1 and S_2 to get an inside piece and an outside piece. In both cases, we glue the punctures from S_1 to the punctures in S_2 . We can see that knot arc punctures will glue to knot arc punctures, and crossing circle punctures will glue to crossing circle puncture. Now we can see our manifold is separated into two pieces. Thus, this is a valid belted sum decomposition (see Figure 10 and Figure 11).

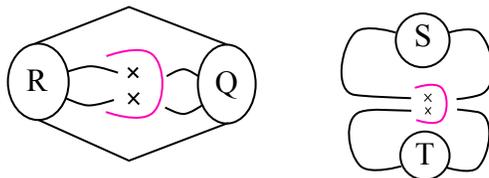


Figure 10: Left: Inside Link After Gluing, Right: Outside Link After Gluing

□

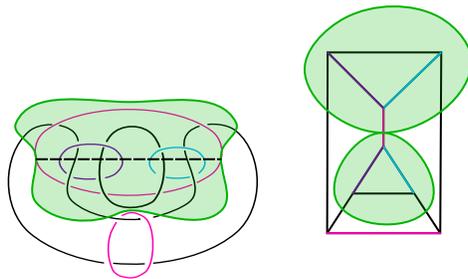


Figure 11: Example of Theorem 5

Theorem 7. Let S_1, S_2 be socks corresponding to two once-painted 3-cuts in a planar trivalent graph whose painted edges get glued. Then, there is a valid belted sum decomposition in the associated Nested Link.

Proof. Consider a trivalent, planar graph with two once-painted 3-cuts whose painted edges get glued (See Figure 12). We know the painted edges correspond to one nested crossing disk which is a thrice punctured sphere. Note that the punctures in this crossing disk could be knot arcs or nested crossing disks. Also note that we know the painted edges are part of an edge symmetric spanning forest where the edge of symmetry lies in G_R .

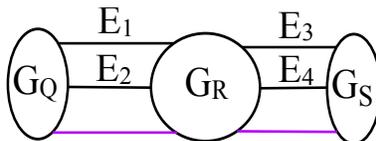


Figure 12: The graph G

Then, consider the link corresponding to this graph (see Figure 13). Note that knot arc K_1 in the link corresponds to edge E_1 in the dual and so on. From here, we can find socks S_1, S_2 whose punctures correspond to the edges in the 3-cuts in the dual.

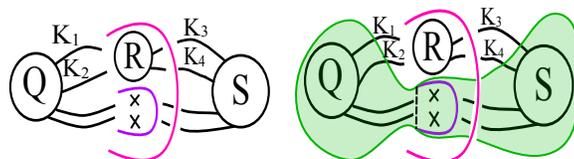


Figure 13: Left: Nested Link, Right: Nested Link with S_1 and S_2

Finally, we slice along S_1 and S_2 to get an inside piece and an outside piece. In both cases, we glue the punctures from S_1 to the punctures in S_2 . We can see that knot arc punctures will glue to knot arc punctures, and crossing circle punctures will glue to crossing circle puncture. Now we can see our manifold is separated into two pieces. Thus, this is a valid belted sum decomposition (see Figure 14 and Figure 15).

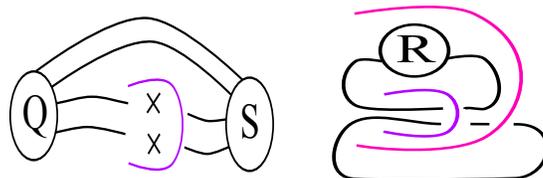


Figure 14: Left: Inside Link After Gluing, Right: Outside Link After Gluing

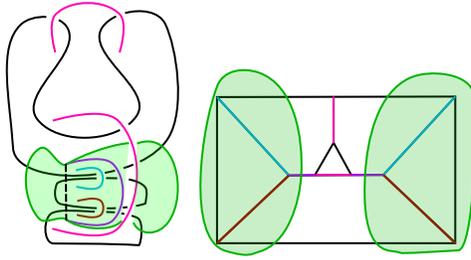


Figure 15: Example of Theorem 6

6.2 Twice-Painted 3-Cuts

When considering twice-painted 3-cuts, there were a few cases to consider. We first considered a twice-painted 3-cut where the painted edges correspond to the same crossing disk in the corresponding augmented nested link. We realized that this case could not occur, which is shown below.

Theorem 8. Let G be a trivalent, planar graph with a twice-painted 3-cut. Then, the painted edges cannot correspond to the same crossing circle in the nested link.

Proof. Let G have a twice-painted 3-cut. We can arrange G so that the 3-cut divides the graph into a left portion and a right portion.

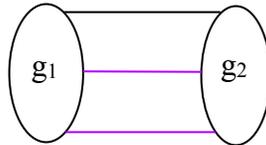


Figure 16: The graph G

Since G is trivalent, we know that it contains an even number of vertices. As in the proof of Lemma 1, reduce the right-hand side to a single vertex. This is still trivalent. This shows that the left-hand side has an odd number of vertices. A similar argument shows that the right-hand side has an odd number of vertices.

Suppose that the two painted edges in the three cut correspond to the same crossing circle in the link and let the edge of symmetry of their edge-symmetric spanning tree be in the left-hand side. Then, consider separating the left-hand side into an independent graph. The tree containing the painted edges is still an edge symmetric spanning tree, and thus contains an even number of vertices. Since there is an odd number of total vertices, this shows that there is an odd number of vertices not spanned by this tree. Therefore, there is no edge-symmetric spanning forest in the left-hand side, which implies that the third edge in the 3-cut must also be painted. \square

After showing that this case was not possible, we looked at cases where the painted edges in a twice-punctured 3-cut do not represent the same crossing circle, which lead to the following results.

Theorem 9. Let S_1, S_2 be socks corresponding to two twice-painted 3-cuts in a trivalent, planar graph where the painted edges in distinct 3-cuts represent the same crossing circles. Then, there is a valid belted sum decomposition in the associated Nested Link.

Proof. Consider a trivalent, planar graph G with two twice-painted 3-cuts whose painted edges represent the same crossing circles (see Figure 17). $C_{1,0}$ and $C_{1,1}$ may be the same edge or glue to be the same crossing circle in the nested link. This is also the case for $C_{2,0}$ and $C_{2,1}$.

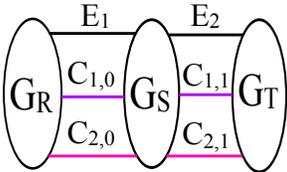


Figure 17: The Graph G

□

Then, consider the link corresponding to G (See Figure 18). Note that the edge E_1 in G corresponds to the knot arc K_1 in the link and so forth. Now we can find socks S_1, S_2 whose punctures correspond to the edges in the 3-cuts in the dual.

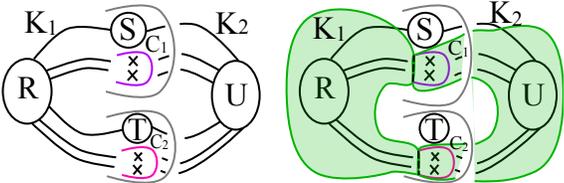


Figure 18: Left: Nested Link, Right: Nested Link with S_1 and S_2

Finally, we slice along S_1 and S_2 to get an inside link and an outside link. Now, in each case glue the punctures from S_1 to the punctures in S_2 . Note that in the outside link, the knot circles that go through C_1 and C_2 are no longer present. It is easy to see that the severed knot arcs can glue through either C_1 or C_2 . Then, in order to preserve the property that each crossing circle bounds a thrice punctured sphere, either C_1 or C_2 must become a knot circle. Without loss of generality, choose C_2 and stretch its boundary so that $C_{2,0}$ and $C_{2,1}$ glue together through C_1 . The link obtained by gluing $C_{1,0}$ to $C_{1,1}$ through C_2 would be equivalent up to homeomorphism.

Now, we can see that our manifold separated into two pieces. Thus, this is a valid belted sum decomposition (See Figure 19 and Figure 20).

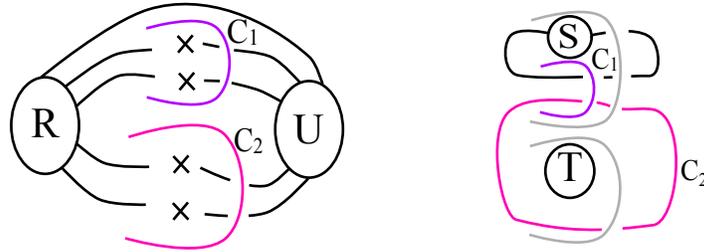


Figure 19: Left: Inside Link After Gluing, Right: Outside Link After Gluing

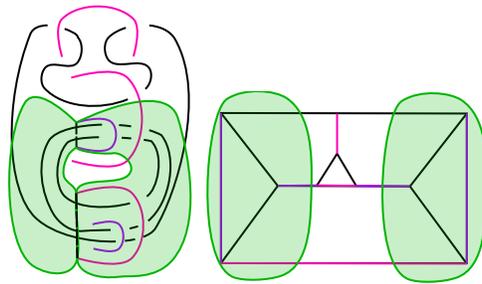


Figure 20: Example of Theorem 8

7 Open Questions

- Can these ideas be extended to classify belted sum decompositions in thrice-painted 3-cuts?

8 References

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