Embedded Totally Geodesic Surfaces in Fully Augmented Link Complements

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Abstract

This project studies the existence of embedded totally geodesic surfaces in a FAL complement. To do so, we use a geodesic analogue of normal surface theory. In particular, we study the structure of geodesic disks in a fundamental region. We then analyse how they glue up to create embedded totally geodesic surfaces in the FAL.

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1 Introduction

In this paper, we study the complement of fully augmented links in S^3 . Decomposing fully augmented link (FAL) complements into rectangular-cusped polyhedra in hyperbolic space allows us to study their geometry. In particular, we study the existence of embedded totally geodesic surfaces and the requirements for their existence in a FAL complement.



Figure 1: A fully augmented link (the Borromean Rings) decomposed into its corresponding ideal polyhedron

Futer and Purcell lay out a comprehensive description of a standard two cell decomposition and how to subdivide a FAL complement into two hyperbolic ideal polyhedra, P_+ and P_- [2]. We now describe how different components of a FAL correspond to the vertices and faces of P_{\pm} . We analyse how components of the FAL lift to P_+ (see Figure 1): The projection plane is divided into regions bounded by knot arcs and orthogonal crossing circle disks. These regions will lift to planes that we will call unshaded faces. The crossing circles will be sliced and flattened along the projection plane to create two halves of the crossing disk. These halves of the disk will lift to ideal triangles in \mathbb{H}^3 that we refer to as shaded triangles. The crossing circle of any crossing disk will become an ideal point, or puncture, between the two shaded triangular regions of the flattened crossing disk. Knot arcs that lie between regions of the projection plane will lift to the other ideal points in the polyhedron that we will denote as knot punctures. (See Futer and Purcell's introduction for more details).



Figure 2: Two truncated ideal shaded triangle and their crossing puncture p under a Möbius transformation

Ideal points in the polyhedra will be truncated by taking out horospheres centered at the ideal points (see Figure 2). Four planes meet at each ideal point, so by removing a horospheres a rectangular region called a truncated boundary face will remain. Truncated boundary faces are represented by rectangles connecting shaded triangles on opposite sides and unshaded faces on the other. This is discussed further in Section 2. We use the properties of the truncated ideal polyhedra in the fundamental region $\mathcal{F} = P_+ \cup P_-$, to study the geometry of any surface that intersects it.

In this paper, we show how embedded totally geodesic surfaces lift to the polyhedron P_{\pm} . An embedded totally geodesic surface S in a FAL complement lifts to a union of disjoint planes, call it \tilde{S} , in the universal cover [1]. Note that the lift \tilde{S} of S intersects $\mathcal{F} = P_+ \cup P_-$ in a union of disjoint geodesic disks. Let H be a plane in \tilde{S} that intersects P_+ , we say $D = H \cap P_+$ is a geodesic disk with boundary curve $\gamma = H \cap \partial P_+$. Conversely, one can begin with geodesic disks in \mathcal{F} and consider if they glue up to form a geodesic surface. Throughout this paper, we refer to $\gamma = \partial D$ to denote that γ is the boundary of a geodesic disk in P_{\pm} .



Figure 3: The projection of P_+ and its mirrored copy, P_- in the fundamental region

The boundaries of geodesic disks intersecting a polyhedral plane display convenient geometric properties that we can analyze through Möbius transformation. A Möbius transformation is a mapping of a truncated boundary face $p \in P_+$ to infinity and maps the four planes intersecting p to vertical planes (see Figure 2). Other faces $\in P_+$ remain as hemispheres. Conveniently enough, these vertical planes and hemispheres adhere to the geometry of Euclidean space. This geometry aids in uncovering geodesic disks that display properties of orthogonality in P_+ . This is discussed in Section 3.1.

There are various manners to verify that a curve γ is the boundary of a geodesic disk. The curve must be a simple closed curve, otherwise the disk is not the intersection of a plane with P_{\pm} . If S is a geodesic surface, we study curves $\gamma \in P_{\pm}$ to show finitely many geodesic disks in P_{+} . This dovetails nicely into showing that there are a finite amount of embedded totally geodesic surfaces in a FAL complement. To do so, we discuss how multiple geodesic disks under a gluing map φ may become an embedded totally geodesic surface.

2 Classification of Interior and Boundary Segments

In this section, we study the structure of geodesic disks and how they intersect the fundamental region. An embedded totally geodesic surface S in a FAL complement lifts to disjoint geodesic disks in $\mathcal{F} = P_+ \cup P_-$. By analyzing the boundary curves $\gamma = \partial D$ in F, we can see what restrictions exist for γ .

Two cells in the standard cell decomposition of a FAL are comprised of crossing disks and regions of the projection plane. Each component of a FAL has a solid torus neighborhood which we call a cusp. Removing the interiors of the solid tori yield a manifold with torus boundary components. The boundary components are decomposed into rectangles by the standard cell decomposition. This process corresponds to truncating the vertices using appropriate horospheres at every ideal point in the polyhedron. The boundary of the truncated polyhedron inherits a cell decomposition consisting of unshaded faces, shaded triangles, and the truncated rectangles we call punctures. Punctures lie at ideal points and will also be called boundary faces.

Definition 2.1. Let γ be the boundary of a geodesic disk D in P_+ . Then, any segment γ_i of γ can be one of the following:

- 1. Interior edge: a segment that lies on the intersection of a shaded triangle and an unshaded face
- 2. Boundary segment: a segment that transverses a boundary face.
- 3. Unshaded segment: a segment that transverses an unshaded face
- 4. Shaded segment: a segment that transverses a shaded triangle

We can see that if a geodesic disk D intersects P_+ , it can intersect each face in a finite number of ways. Then, $\gamma = \partial D$ can transverse faces and pass through ideal points. The first that we will look at is an interior edge. An *interior edge* is a segment of γ that lies on the intersection of an unshaded face and a shaded triangle. The endpoints of this segment lie on the corners of boundary faces, as seen in Figure 4(a).



Figure 4: Segments of γ

We can see that γ can also transverse an unshaded face such as in Figure 4(b) from the midpoint of an altitude to a truncated boundary face. We know that this is a possible segment of γ because its endpoints are on a truncated boundary face and a midpoint of an altitude. Additionally, γ is simple. Having an endpoint of a segment of γ the meridian of a truncated boundary face signifies that γ can enter and exit a boundary face in two ways: either as a boundary face transversal (Figure 5(a)) or boundary face to an interior edge segment (Figure 5(b)).

In Figure 5, we can see that γ enters the boundary face from an unshaded face and exits into another unshaded face such as in Figure 5(a), or onto an interior edge such as in Figure 5(b). If we perform a Möbius transformation on the boundary face transversal, we can see that $\gamma = \partial D$ where D is a vertical plane that intersects both unshaded faces which become vertical planes in H^3 . We study the properties of this transversal in Section 3.1.



Figure 5: More Segments of γ

Similarly, using a Möbius transformation on the boundary face in Figure 5(b), the boundary face to interior edge segment, we can see that $\gamma = \partial D$ where D is another vertical plane that intersects the point at infinity and continues to the interior edge. The interior edge becomes the intersection between the vertical unshaded face and the adjacent vertical shaded triangle.

Boundary segments can also exit into unshaded faces and enter other boundary faces such as the *boundary face to boundary face* segment in Figure 6(a). The only requirement for these segments is that the boundary faces that γ intersects do not share an interior edge, otherwise γ becomes an interior edge segment.



Figure 6: More Segments of γ

Additionally, γ can exit a shaded triangle from the midpoint of its altitude into another midpoint of an altitude such as in Figure 6(b). This is possible only if the interior edges that the midpoints lie on are not adjacent. That is, the interior edges containing the midpoints of the altitudes do not share a boundary face. Morgan et al. [3] shows that this would give you a surface with a self-intersection.

The *altitude* segment seen in Figure 7(a) displays some convenient properties. We know that embedded totally geodesic surfaces must intersect crossing disks in simple geodesics

[3]. It was proven that the altitude segment intersects the face of the shaded triangles orthogonally. This is the only possible intersection of a curve γ in a shaded triangle and it must enter and exit the crossing disks at the midpoint of the altitude [3].



Figure 7: Even More Segments of γ

Another convenient section of γ is the *alternating interior edge*. This section, seen in Figure 7(b), exits an interior edge by crossing through a puncture and continues along another interior edge.

Some segments of γ are never the bondary of a geodesic disk. For example, Figure 8 will never exist as a tile because any segment of a curve γ entering a truncated boundary face through an interior face must intersect that interior face at an altitude. Using a Möbius transformation on this puncture, we see that the shaded triangles become parallel vertical faces in Euclidean space. Thus, any surface intersecting the unshaded triangle must continue and eventually intersect the parallel vertical face. Thus, γ cannot enter an interior face and exit a truncated puncture into an unshaded face.



Figure 8: This boundary segment will never occur in a curve γ

3 Results on Embedded Totally Geodesic Disks

The following structures theorems classify how a geodesic disk D intersects the faces of P_{\pm} so that we may eventually study how the disks glue up to embedded totally geodesic surfaces.

3.1 Results on Orthogonality

When we discuss the properties and restrictions on a curve $\gamma = \partial D$, we ignore the possibility of $\gamma = \partial A$, where A is an unshaded face or a shaded triangle. This is an example of a *standard embedded totally geodesic surface*. These are the projection planes and crossing disks in the FAL complement that lift to unshaded faces and shaded triangles in P_+ . Throughout this paper, when referring to geodesic disks and embedded totally geodesic surfaces, we refer to the non-standard geodesic disks and surfaces.

Lemma 3.1. Each segment of γ intersects any shaded or unshaded faces of ∂P^+ at most once.

Proof. Two hyperbolic planes, e.g. a geodesic disk and any face of P^+ , intersect in at most one geodesic line. This means that there is at most one interior edge or altitude in any interior face and at most one boundary segment in any boundary face.

Lemma 3.2. Consecutive segments of γ that pass through unshaded faces of P^+ are either both orthogonal, or neither orthogonal.

Proof. Denote the puncture between the unshaded faces that contain the segments of γ as p, see Figure 9. Using a Möbius transformation on p, we see that the unshaded faces on either side of p become vertical faces in the polyhedron that are parallel in Euclidean space. Because the unshaded faces are parallel, if D intersects one unshaded face orthogonally, it must pass through the other unshaded face orthogonally. Similarly, if D is not orthogonal to one of the unshaded faces, it cannot pass through the other unshaded face orthogonally.



Figure 9: Curve γ entering puncture p and exiting through an unshaded face.

Corollary 3.3. If $\gamma = \partial D$ contains an interior edge, then it is not orthogonal to any shaded triangle or any unshaded face.

Proof. Let's begin by assuming that D contain an interior edge and intersects an unshaded face orthogonally. Then, Lemma 3.2 implies that when γ exits the unshaded face, it must enter another unshaded face orthogonally as well.

Theorem 3.4. If $\gamma = \partial D$ contains an altitude, then D is orthogonal to every face it intersects.

Proof. Let D be a geodesic disk in P^+ with $\gamma = \partial D$. If D intersects the crossing disk B at an altitude, then $D \perp B, C$ where C is the adjacent unshaded face to B.



Figure 10: Possible paths of the curve γ

Knowing that $D \perp C$ is crucial to understanding how D exits C into a new face. In Figure 10 we can see that γ can exit C and enter into an altitude or a boundary face transversal into another unshaded face.

Case 1: Enter into an altitude

If γ enters in an altitude, e.g. E_1 , then we know that D intersects the shaded triangle orthogonally [3].

Case 2: Enter into an unshaded face

If γ exits the unshaded face C and enters the unshaded face E_2 through a boundary transversal, we know by Lemma 3.2 that D is orthogonal to both faces adjacent to that boundary segment.

We eliminate the possibility of γ containing an interior edge. The only way that γ can intersect an interior edge is from a boundary face to interior edge segment as seen in Figure 5. However, an interior edge is never orthogonal as proven in Corollary 3.3.

These are all possible segments that γ consists of and this, D is orthogonal to all faces it intersects.

Now, we know the properties of γ if it contains an altitude and we can define the following:

Definition 3.5. An altitude containing path is a path that contains an altitude and is orthogonal to every face it intersects.

Corollary 3.6. There is at most one embedded totally geodesic surface that intersects an altitude through a crossing disk.

Proof. Given $\gamma = \partial S$ containing an altitude segment *a* through puncture *p*, we know that γ must pass through only the faces that are orthogonal to the shaded triangle containing *a*. That is, γ has a unique path that it can travel to in order to be a simple, closed path in ∂P_+ and still be orthogonal to all faces that γ intersects. We can see this easily by using a Möbius transformation on the puncture in our altitude, *p*. When *p* is thrown to infinity, *S* becomes a vertical plane intersecting *a* and P_+ . At most one vertical plane can intersect *a* through *p*.

3.2 Results on Slopes

We discuss properties of the lifts of embedded totally geodesic surfaces to the universal cover of a FAL complement. It was shown in [3] that any embedded totally geodesic surface is non compact so, it is a punctured surface. Let S be a geodesic surface in a FAL cover and H a plane in \mathbb{H}^3 that projects to S and contains the puncture $p \in P_+$. Send p to infinity using a Möbius transformation so that H becomes a vertical plane in P_{\pm} as pictured in Figure 11.

In Figure 11, A and B are vertical shaded triangles and C and D are vertical unshaded faces. The vertical geodesic plane H will be represented by a straight line $\beta = H \cap P_+$. In some cases, the slope of our curve β is easier to analyse because β contains many restrictions as shown in Section 3.1 and by the possible segments of any $\gamma = \partial D$ in Section 2.



Figure 11: Fundamental Region $\mathcal{F} = P_+ \cup P_-$ where A, B are vertical shaded triangles and C, D are vertical unshaded faces.

We are interested in the curves β that does not intersect any crossing disk in $P_+ \cup P_-$, or at an altitude. Thus, β contains interior edges and intersects $P_+ \cup P_-$ at its unshaded faces.

Definition 3.7. A slope path is a curve $\beta = H \cap P_+$ that contains interior edges, intersects unshaded faces, and intersects the punctures between them.

We know that β must contain boundary segments in order to enter and exit the unshaded faces of P_+ . Additionally, slope paths must contain at least one interior edge. So we can use a Möbius transformation on the puncture p adjacent to the interior edge and see that the interior edge will be vertical. This way, we can ensure that β appear as a straight line in the universal cover that intersect the interior edge between a vertical shaded triangle and a vertical unshaded face, called lattice points. So, slope paths will always intersect *lattice points* of $P_+ \cup P_-$, or the punctures that represent the points of tangencies between the vertical shaded triangles and the vertical unshaded faces. These lattice points are labeled as p_1 and p_2 in Figure 12 where A, B are vertical shaded triangles and C, D are vertical unshaded faces. Thus, we analyse the slope of a curve $\beta = H \cap P_+$ by how it may intersect the lattice points in a universal cover. The slope can be understood by analyzing the longitudes and meridians of a fundamental region. A *longitude* is the unit between two adjacent vertical lattice points and a *meridian* is the unit between two adjacent horizontal lattice points (see Figure 12).

Lemma 3.8. The slope of any slope path β is

 $\frac{q \ longitudes}{p \ meridian}, \quad p,q \neq 0.$

Proof. Any curve β must intersect the fundamental region at its lattice points (labeled p1, p2, p3 and p4 in Figure 12. The slope path β may be the boundaries of vertical geodesic surfaces that cover the entire fundamental region and continue in the universal cover. \Box



Figure 12: Fundamental Region $\mathcal{F} = P_+ \cup P_-$ where p_1, p_2, p_3 and p_4 are lattice points

Slope paths are not orthogonal to any face in P_+ by Corollary 3.3 because they contain an interior edge. It is possible that these slope paths contain only interior edges and do not intersect any unshaded face. These slope paths are special and we refer to them as alternating interior edge cycles. We study them later in Section 3.3. However, we are also interested in the paths $\beta = H \cap P_+$ that do not contain altitudes or interior edges. They only intersected unshaded faces in $P_+ \cup P_-$.

Definition 3.9. An unshaded face path is a curve $\beta = H \cap P_{\pm}$ that contains only unshaded face segments.

These paths intersect the punctures along the unshaded faces C and D but never intersect the vertical shaded triangles A or B. We do not consider unshaded face paths as a variety of slope paths because slope paths must contain interior edges. In fact, unshaded face paths are distinguishable from slope paths because an unshaded face path β is always orthogonal to every face it intersects. **Lemma 3.10.** A plane $H \in \mathbb{H}^3$ that projects to S which does not intersect any crossing disk is orthogonal to every unshaded face it intersects.

Proof. If $\beta = \partial H$ and does not intersect any crossing disk, it does not contain an altitude or an interior edge segment. Thus, it can only intersect P_+ in unshaded faces and through their punctures. The curve β is not able to intersect any vertical shaded triangle, such as Aor B in Figure 11. Similarly, β is unable to intersect any lattice point because that would signify that the vertical plane H intersects the interior edge along a shaded triangle. This implies that β must run vertically in $P_+ \cup P_-$ to ensure that it avoids any copy of $A, B \in F$.

The curve β must intersect P_{\pm} at punctures between unshaded faces. So, it can only intersect the unshaded faces C and D at its punctures. Now, we show that β is orthogonal to every face it intersects. It is known that if two circles are tangent, their centers and shared point of tangency are collinear. We know that β will exit C or D vertically and thus intersect the centers of all of the circles representing unshaded faces between them. Thus, any β passing through a puncture, into an unshaded face, and leaves through a puncture into another unshaded face will pass through their Euclidean centers and therefore be orthogonal to both unshaded faces. By Lemma 3.2, we know that β is orthogonal to all unshaded faces it passes through as long as it passes through at least one orthogonally.

3.3 Results on Cycles

We now study a curve $\gamma = \partial D$ that contains only interior edge segments. We are not interested in curves that contain interior edges that enclose faces in P_+ as they are the boundaries of standard geodesic disks. Rather, we analyze the properties of interior edge paths whose boundaries are non-standard embedded totally geodesic disks in the FAL complement. We call these alternating interior edge cycles.

If we begin with an interior edge γ_1 , we know that γ_1 connects two punctures, one of which we denote as p. In an alternating interior edge cycle, as γ_1 enters p, it can exit in one of three ways: through an interior edge on a shared unshaded face, through an interior edge on a shared shaded triangle, or on an interior edge γ_2 that lies on a boundary of some face that γ_1 does not touch.

If we define the length of the alternating interior edge cycle, we use the following definition:

Definition 3.11. An n-cycle is an alternating interior edge path containing n interior edges.

Properties:

- 1. If an *n*-cycle γ contains only alternating interior edges, then *n* is even. We can see this if we view γ as an oriented line, an unshaded face is either to the left or the right γ . As we pass through punctures onto another interior edge, an unshaded surface will be located on the opposite side. If we view each interior edge $\in \gamma$ as a sequence of left, right, left, right, ..., we can see that we need an even number of interior edges to close γ back to the original interior edge.
- 2. A 2-cycle is not possible. There are no faces in the polyhedron with only 2 sides, so γ cannot be a closed curve.

Now we analyse when interior edge cycles are the boundary of an embedded totally geodesic disk.



Figure 13: Caption

Lemma 3.12. The only 4-cycle interior edge cycle that is the boundary of an embedded totally geodesic disk is in the Borromean Rings (with or without half-twists).

Proof. We know that every path will consist of an even number of edges by Property 2. So, we begin with three alternating interior edges that form a path and observe how the fourth edge behaves so that $\gamma = \partial D$ and is closed. Begin with an arbitrary circle packing consisting of three tangential circles labeled them A, B, and C (see Figure 13). Choose any interior edge that lies in the shaded triangle between them and continue the path γ so that there are three alternating interior edges. If we choose the interior edge on circle A, the fourth edge must connect to the interior edge on B and the interior edge on C.

Thus, the fourth circle we draw must be tangent to circle A and circle C. If we draw a circle D so that it is tangent to B, C, and A, it will share exactly one puncture with each of the circles. This ensures that there an alternating interior edge between B and C. By making D tangent to A, it satisfies the condition that there is a puncture between B & D and a puncture between C & D to ensure an alternating interior edge will exist.

Note that if there were another circle in any of the four shaded triangles in the figure on the right in Figure 13, the interior edges would no longer be a 4-cycle.

Now we must show that this path $\gamma = \partial D$. Notice how the symmetry of A, B, and C makes the tangencies reflective. So, we can find a circle between the 2 tangent points on A that also intersect the punctures where C and B meet D. Thus, these punctures fall on the same circle and lie in the boundary of a geodesic disk.

So, circle D is tangent to A, B, and C and we see that this circle packing is actually the Borromean Rings. \Box

A general n-cycle $\in \partial P_+$ will not bound a geodesic disk in P_+ . In fact, it not clear that the boundary of a geodesic disk is ever an *n*-cycle. We now prove that geodesic disks with *n*-cycle boundaries indeed exist.

Theorem 3.13. For every $n \ge 6$, there exists a P_+ containing a geodesic disk whose boundary is an n-cycle.

Proof. Similar to Lemma 3.12, we can analyze what is necessary for an interior edge 6-cycle to exist as a boundary curve of an embedded totally geodesic disk. Then, we generalize for all n > 6. Begin with an arbitrary circle configuration of five circles so that we can construct

an alternating interior edge path of length five. See the figure on the left in Figure 14 for a possible construction and labeling of the tangent circles. We label our circles A, B, C, D,



Figure 14: A possible construction of a 6-cycle

and E. We must add a sixth circle, F, that is tangent to circles A & B and D & E. This ensures that there is a shaded triangle between A, B, & F and D, E, & F which allows the alternating interior edge path γ to be closed and simple. We can draw circle F around all five circles so that it shares a puncture with A, B, D, and E exactly once such as in the figure on the right in Figure 14.

We can choose a circle packing such that these six punctures $\in \gamma$ lie on a circle and thus on ∂D . If we rearrange our circles so that three congruent circles lie tangent within three larger congruent circles, such as in Figure 15, we can create some convenient symmetric properties. There are three lines of symmetry in Figure 15. The punctures that γ intersect are all equidistant from the center of the three congruent smaller unshaded faces and thus lie on a circle. This implies that $\gamma = \partial D$.

Generalizing for all n > 6, we can see that if we arrange any *n*-cycle into a circle packing with $\frac{n}{2}$ congruent smaller circles lying inside $\frac{n}{2}$ congruent larger circles, we create $\frac{n}{2}$ lines of symmetry. If γ contains the punctures where the smaller circles meet the larger circles, we can see that $\gamma = \partial D$ because all punctures are equidistant from the center of this figure. See Figure 16. Note that anywhere there exists an unshaded triangle not bounded by an interior edge, we can place more unshaded faces where P_+ continues to contain an *n*-cycle. Thus, multiple *n*-cycles exist as the boundary of a geodesic disk.



Figure 15: An example of a symmetric circle packing with a 6-cycle



Figure 16: An example of a symmetric circle packing with an 8-cycle

3.4 Finiteness Theorem

We study all possible curves of $\gamma \in \partial P_+$ which can be the boundary of a geodesic disk. The five possible simple, closed paths that γ can be are a standard geodesic disk, an alternating interior edge cycle, an unshaded face path, a slope path, or an altitude containing path. We count these paths to prove the following theorem:

Theorem 3.14. There are finitely many embedded totally geodesic disks in a polyhedra P_{\pm} .

Proof. We count the number of paths that we can find in any polyhedron given n, the number of crossing circles in the FAL complement and m, the number of unshaded faces in P_+ . With the structure theorems in the previous sections, we show the upper bound for the number of paths $\gamma = \partial D$ where γ can consist of a restricted set of segments from those listed in Section 2. We break these paths into five cases. If γ contains an altitude, we know that there is exactly one embedded totally geodesic disk that passes through that altitude.

Case 1: Standard Geodesic Disks

Our standard geodesic disks are the regions of the projection plane and the the crossing disk in a FAL complement that lift to unshaded faces and shaded triangles. If there are n crossing circles, then after the cell decomposition, we reveal two halves of a crossing disk that lift to shaded triangles. Thus, there are two geodesic disks per crossing circle. Similarly, each region of the the projection plane that is bounded by crossing circles and knot arcs will lift to an unshaded face in P_+ .

Case 2: Alternating Interior Edge Cycles

We found in Lemma 3.12 that the shortest alternating interior edge cycle is a 4-cycle in the Borromean Rings. Thus, we exclude the 4-cycle from our count and the next shortest possible path for any alternating path becomes a 6-cycle. Additionally, in an alternating path, there is only one edge to which γ can travel signifying that each interior edge belongs to at most one alternating interior edge cycle. By taking the total number of interior edges in any fundamental region and dividing it by the smallest number of interior edges needed in an alternating interior edge cycle, we find an upper bound for the total possible number of alternating paths. The total number of interior edges is 6n, where n is the number of crossing circles in the FAL. Each crossing circle, after decomposed, becomes a puncture between two halves of the crossing disks which are represented by shaded triangles. Each shaded triangle contains three interior edges. Thus, six interior edges per crossing circle.

$$\frac{2*3*n}{6} = n \text{ paths}$$

Note that in general, not every alternating interior edge cycle will contain exactly six interior edges. It may contain more, but no less. Additionally, not all of the interior edge cycles will lie on the boundary of intersection of a geodesic disk D and P_+ .

Case 3: Unshaded Face Paths

Lemma 3.10 tells us that the only unshaded face paths that exist are orthogonal to every unshaded face that γ intersects. Thus, for each puncture that we send to infinity under a Möbius transformation, we count the maximum number of vertically sloped lines that we could have whose punctures lie on the vertical unshaded faces, such as C and D in Figure 17. The maximum number of vertically sloped lines we can count is the number of unshaded faces that lie in P_+ but are not the vertical unshaded faces C, D, or the unshaded faces that are adjacent to the vertical shaded triangles, such as E and F in Figure 17. I there are m unshaded faces in P_+ , then there exist at most m-4 vertically sloped lines.

Then, we multiply that by the total number of punctures in P_+ , which is 6n. However, we count these paths more than once, but we know that we have o



Figure 17: An arbitrary P_+ with vertical shaded triangles A, B and vertical unshaded faces C, D.

Case 4: Slope Paths

So that γ does not intersect any shaded face of P_+ at an altitude, it must intersect the vertical shaded faces A and B at its punctures, see the red dotted line in the Figure above. If γ begins from one puncture of P_+ , the maximum paths that γ can continue through are the punctures on the adjacent unshaded face labeled E in the Figure on the right. If k-2 is the amount of punctures on E not touching C or D, and p_1, p_2 are puncture on A, then we count k-2 positive slope paths from p_1 and k-2 negative slope paths from p_2 . Total, that is 2(k-2) ways that γ can exit a puncture on A and intersect E. Finally, we multiply by the total number of punctures in P_+ , 3n, to get an upper bound.

Case 5: Altitude containing paths

If a curve γ contains an altitude through a crossing circle puncture, then Corollary 3.6 shows that there is exactly one path γ containing this altitude and that $\gamma = \partial D$ is a vertical plane in P_+ . Although S may not intersect more than one altitude, we count one surface S for every altitude because we cannot generalize how many altitudes will appear in each altitude containing path. However, we do know that there is exactly one altitude passing through any crossing circle puncture and thus, we count a surface S for each of these. There are exactly n altitudes passing through crossing circle punctures if there are n crossing circles in the FAL. So, there are at most n paths.

Otherwise, the curve γ contains an altitude through a knot puncture. Similarly, there is exactly one geodesic disk D that contains any altitude through a knot puncture.

Type of Geodesic Disk Boundary	Upper Bound
Standard Geodesic Disks	2n+m
Alternating Interior Edge Cycles	n
Unshaded Face Paths	3n(m-4)
Slope Paths	6n(k-2)
Altitude Containing Paths	3n

Key:	
n, number of crossing circles	
m, number of unshaded faces	
k, see Case 4	

Figure 18: Upper Bound for Disks in a FAL complement

4 Surfaces Comprised of Geodesic Disks

4.1 Unshaded Face Path Projections

Now we analyze how geodesic disks whose boundaries are unshaded face path curves glue up. A geodesic disk D where $D \in P_+$ may glue up to other geodesic disks that together project to one embedded totally geodesic surface in a FAL complement. Let \mathcal{R} be the reflection interchanging P_{\pm} . Then, each unshaded face A is glued to its reflection $\mathcal{R}(A)$. The gluing map for an unshaded face path is $\varphi = \mathcal{R} \circ r_A$, where r_A is a hyperbolic reflection over a hemisphere containing A. The isometry φ determines how disks can glue to form totally geodesic surfaces. Let $\gamma_i \in A$ be an edge of ∂D , and let G be the plane containing D. The extension D_e of D across γ_i is defined by $D_e = \varphi(G) \cap \mathcal{F}$.

For these disks to glue up into geodesic surfaces, they must be disjoint disks in the fundamental region whose extensions under a gluing map φ are non-intersecting. Clearly, if D glues to part of an embedded totally geodesic surface in a FAL complement, then D_e must be a part of the same surface. Any other geodesic disk containing $\varphi(\gamma_i)$ forms a pleated surface when glued to D.

A geodesic disk D glues to $\mathcal{R}(D)$. We show this in the following lemma:

Lemma 4.1. Let $\gamma_i = A \cap G$ where A is an unshaded face and G contains a geodesic disk D, then $D_e = \mathcal{R}(G)$ if and only if $G \perp A$.

Proof. Note that if $r_A(G) = G$ if and only if $G \perp A$. In this case, we know that

$$D_e = \varphi(G) \cap \mathcal{F}$$
$$D_e = \mathcal{R}(r_A(G)) \cap \mathcal{F} \quad (*)$$
$$D_e = \mathcal{R}(G) \cap \mathcal{F} \quad (**)$$

Where (*) is by the definition of φ . Step (**) implies that $r_A(G) = G$ which is true if $G \perp A$ as the inversion of G under hemisphere A will return itself.

Corollary 4.2. If $D_e = \mathcal{R}(D)$, then both D and $\mathcal{R}(D)$ are orthogonal to ∂P_+ .

Proof. We know by Lemma 4.1 that if $D_e = \mathcal{R}(D)$, then D is orthogonal to every face it intersects. Finally, Euclidean reflections are a conformal mapping and thus, D_e is orthogonal to every face it intersects as well.

Knowing that D and D_e are orthogonal to every face they intersect in P_{\pm} gives insight on the surface they project to in a FAL complement.

Lemma 4.3. If $D_e = \mathcal{R}(D)$, then D and D_e will glue to an embedded totally geodesic surface in a FAL complement.

Proof. An unshaded face path $\beta = H \cap P_+$ will intersect P_+ orthogonally as a plane in \mathbb{H}^3 . All of the punctures that β intersects will lie on a straight line. However, if a puncture is mapped under a Möbius transformation such that β is not straight, the punctures of β will lie on a circle. We know that either way, the gluing map $\varphi = \mathcal{R} \circ r_A$ will be rewritten as $\varphi = \mathcal{R}$.

As we know, if D glues to part of an embedded totally geodesic in a FAL complement, then D_e must be a part of the same surface. However, D and D_e intersect P_{\pm} orthogonally. This implies that after they glue up to each other, they project to an embedded totally geodesic surface in a FAL complement. This is confirmed because D does not intersect D_e and the Euclidean reflection \mathcal{R} ensures that D_e will meet D at the same points.

4.2 Altitude Containing Path Projections

Altitude containing paths will glue up differently than unshaded face paths. Altitudes that pass through knot punctures have a restrictive gluing process. An altitude passing through knot punctures signify that the shaded triangles containing the altitude will glue up to different shaded triangles in P_+ . In fact, if $\gamma_a \in A$ is the segment passing through one shaded triangle and $\gamma_b \in B$ is the altitude passing through the other, γ_a will glue to $\gamma_{a'} \in A'$ and γ_b will glue to $\gamma_{b'} \in B'$. Unless A' and B' share a knot puncture so that A, B,A' and B' surround an unshaded face with four punctures, $\gamma_{a'}$ and $\gamma_{b'}$ will continue to glue up to the shaded triangles shared by their crossing circle punctures. This gluing process may be extensive and large after γ_a and γ_b continue creating associated gluing paths in P_+ . In fact, we conjecture that these paths become too restrictive and will eventually intersect.

However, if A' and B' share a knot puncture such as in Figure 19, then γ has only one associated gluing path. This duplicate altitude may lie on a disjoint geodesic disk D'that may glue up to a geodesic surface. To avoid counting the duplicates, we divide the total number of altitudes through knot punctures by the smallest number of duplicates any altitude may have. If there are n crossing circles in the FAL, there are 2n shaded triangles, each of which have two shared knot punctures with another shaded triangle. So, there are

$$\frac{2*2*n}{2} = 2n \text{ knot punctures.}$$

This signifies that we could have at most 2n altitudes through those knot punctures, but to account for the duplicates, we divide by two, the smallest number of duplicates that any altitude through a knot puncture can have.

$$\frac{2*n}{2} = n$$
 paths.



Figure 19: Altitude through a knot puncture (solid line), where A glues to A' and B glues to B', and its gluing duplicate (dashed line).

4.3 Finiteness Theorem for Embedded Totally Geodesic Surface

Theorem 4.4. There are finitely many embedded totally geodesic surfaces in a FAL complement.

Proof. We know that if there are a finite number of geodesic disks in P_+ and P_- , the disks will glue up and project to even fewer embedded totally geodesic surfaces in a FAL complement. If we continue a similar investigation of all of the geodesic disks we have counted in Theorem 3.14, we can see how these disks glue up and if they create non-pleated surfaces in a FAL complement.

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