General Curvature Homogeneity Theories

Alexandro Luna

Abstract

This papers studies past notions of curvature homogeneity while also discussing G-modeled pseudo-Riemannian manifolds. It is shown that for any $m \in \mathbb{N} \cap [3, \infty)$, there exists a manifold M of dimension m, a model space $\mathcal{M}$, and a one dimensional Lie group such that M is G-modeled up to order 0 with respect to $\mathcal{M}$. In addition, the manifold fails to be curvature homogeneous as well as homothety curvature homogeneous.

1 Introduction

Past studies of curvature homogeneity, for example in [1], have involved the comparison of pseudo-Riemmanian manifolds and model spaces. This paper dives into a deeper theory of curvature homogeneity, which was first studied in [3]. We wish to examine a new branch of manifolds known as G-modeled manifolds. This theory involves the ingredients of a pseudo-Riemannian manifold, a model space, and a Lie group G. One of the first examples of a G-modeled manifold (see [3]) appears in the form of a three dimensional manifold being G-modeled with respect to a model, and a one dimensional Lie group G. In this paper, we extend this particular result by increasing the dimension of the manifold to any arbitrary finite dimension m, while leaving the dimension of the Lie group the same.

1.1 Manifolds and Model Spaces

For the remainder of the paper, when a vector space is mentioned, we assume that it is real and finite dimensional.

Definition 1. *Let V be a vector space. A function $\phi : V \times V \to \mathbb{R}$ is called an inner product on V if*

1. *ϕ is bilinear,*
2. *$\phi(v, w) = \phi(w, v)$ for any $v, w \in V$,*
3. *$\phi(v, v) > 0$ if $v \neq 0$ (positive definite).*

Furthermore, if M is a manifold, then a metric g on M is a choice of inner product on each tangent space. We denote the tangent space at a point $P \in M$ as $T_P M$. We say that the tuple (M, g) is a pseudo-Riemannian manifold if g admits an inner product on each $T_P M$ that satisfies the following property:

4. *For any $v \in V \setminus \{0\}$, there exists a $w \in V$ such that $\phi(v, w) = 0$.*

Given a pseudo-Riemannian manifold (M, g) as discussed above, if ∇ is Levi-Civita connection on M, the we define the Riemannian curvature tensor R on the vector fields X, Y, Z, W as

$$R(X, Y, Z, W) := g\left(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, W\right).$$

Furthermore, we define $\nabla^i R$ to be the i^{th} covariant derivative of R. If P is a point on M, then we denote $T_P M$ to be the tangent space at P, and g_P, $\nabla^i R_P$ to be the metric g and i^{th} covariant derivative at P, respectively.

Definition 2. *A map $R_0 : V^4 \to \mathbb{R}$ is called an algebraic curvature tensor on V if*

1. *R_0 is multilinear,*
2. *$R_0(x, y, z, w) = -R_0(y, x, z, w)$,*
3. *$R_0(x, y, z, w) = R_0(z, w, x, y)$, and*
4. *$R_0(x, y, z, w) + R_0(z, x, y, w) + R_0(y, z, x, w) = 0$,*

for all $x, y, z, w \in V$. The set of algebraic curvature tensors on V is denoted by $\mathcal{A}(V)$. The tuple (V, ϕ, R_0) is called a model space. For simplicity, we denote the set of all model spaces over V as $\mathcal{M}(V)$.

Remark 1. *We note that in the above definition, R_0 was an element of $\otimes^4 V^*$ that satisfies the same algebraic properties as the Riemannian curvature tensor. We may further extend the definition above by saying that a model space is a tuple $(V, \phi, R_0, R_1, \ldots, R_k)$ where V is a vector space, ϕ is an inner product on V, and each R_i, for $i = 0, 1, \ldots, k$, is an element of $\otimes^{4+i} V^*$ that satisfies the same algebraic properties as $\nabla^i R$.*

Given two model spaces $\mathcal{V} = (V, \phi_1, R_0, \ldots, R_k)$ and $\mathcal{W} = (W, \phi_2, S_0, \ldots, S_k)$, we say that $\mathcal{V}$ is isomorphic to $\mathcal{W}$ and write $\mathcal{V} \cong \mathcal{W}$, if there exists an invertible linear map $A : V \to W$ such that $\mathcal{V} = A^* \mathcal{W}$, where A^* represents precomoposition by A, and

$$A^* \mathcal{W} := (W, A^* \phi_2, A^* S_0, \ldots, A^* S_k).$$

Remark 2. *It is important to note that in some cases, for a model space $\mathcal{M}$, we sometimes define $A^* \mathcal{M}$ exactly as above, but do not precompose A with the inner product.*

1.2 Curvature Homogeneity Theories

The next definitions discuss the types of curvature homogeneity we wish to study. If a model is of the form $\mathcal{M} = (V, \phi, R_0, \ldots, R_k)$, then we say that $\mathcal{M}$ is a k-model and in some cases use the notation $\mathcal{M}^k$ in its place. Also, if (M, g) is a pseudo-Riemannian manifold and $P \in M$, then the tuple

$$\mathcal{M}_P^k := (T_P M, g_P, R_P, \ldots, \nabla^k R_P)$$

is a k-model space.

Definition 3. *Let (M, g) be a pseudo-Riemannian manifold and let $\mathcal{M}^k = (V, \phi, R_0, \ldots, R_k)$ be a model space of order k. Then, we say that (M, g) is:*

- *Curvature homogeneous up to order k (CH_k) with model $\mathcal{M}^k$, if $\mathcal{M}_P^k \cong \mathcal{M}^k$ for every $P \in M$*
- *Homothety curvature homogeneous up to order k (HCH_k) with model $\mathcal{M}^k$, if there exists a nonzero smooth function $\lambda : M \to \mathbb{R}$ such that for every $P \in M$,*

$$\mathcal{M}_P^k \cong (V, \phi, \lambda R_0, \lambda^{\frac{3}{2}} R_1, \ldots, \lambda^{\frac{k+2}{2}} R_k).$$

Remark 3. *It is clear to see that $CH_k \Rightarrow HCH_k$ by simply setting $\lambda = 1$.*

While the above types of curvature homogeneity have been studied in the past, we can generalize these notions by also considering a Lie group that acts on the set of model spaces over V. Adding this aspect of a group action has only recently been studied in [3]. This idea is clearly laid out in the following definition.

Definition 4. *Let (M, g) be a pseudo-Riemmanian manifold and $\mathcal{M} = (V, \phi, R_0, \ldots, R_k)$ be a model space. Suppose that $G \leq Gl(V)$ is a Lie group and $A \mapsto A \cdot \mathcal{M}$ is an action of G on the set of model spaces over V. Then, (M, g) is G-modeled up to order k provided the following hold:*

1. *For every $P \in M$ there exists an $A \in G$ such that $\mathcal{M}_P^k \cong A \cdot \mathcal{M}$*
2. *For every $A \in G$ there exists a $P \in M$ such that $\mathcal{M}_P^k \cong A \cdot \mathcal{M}$.*

1.3 Invariants of Curvature Homogeneous Manifolds

As briefly discussed before, we are interested in manifolds that live in the category of Definition 4 but not that of Definition 3. The following result is a useful curvature homogeneous invariant, which was first used in [2] and [3].

Proposition 1. *Suppose (M, g) is an HCH_0 manifold with model $\mathcal{M}$. Define*

$$\tau := \sum_{i,j,k,l} g^{il} g^{jk} R_{ijkl} \quad \textit{and} \quad ||R||^2 := \sum_{i_1,j_1,\ldots,i_4,j_4} g^{i_1 j_1} g^{i_2 j_2} g^{i_3 j_3} g^{i_4 j_4} R_{i_1 i_2 i_3 i_4} R_{j_1 j_2 j_3 j_4}, \tag{1}$$

then we have that

$$\frac{\tau(P_1)^2}{||R(P_1)||^2} = \frac{\tau(P_2)^2}{||R(P_2)||^2}$$

for any points distinct points $P_1, P_2 \in M$.

We use the contraposititive of this statement, in practice.

Corollary 1. *Let (M, g) be a manifold. As defined above, if $\frac{\tau^2}{||R||^2}$ is non-constant on M, then (M, g) is not HCH_0, and hence not CH_0 either.*

2 G-modeled Manifolds of 4 Dimensions

In [3], it was shown that there exists a 3-dimensional manifold that is G-modeled up to order 0, with respect a model space $\mathcal{M}$ and a Lie group $G \cong \mathbb{R}^+$ acting on $\mathcal{M}$. This manifold was also not HCH_0 nor CH_0. While it is our overall goal to construct an arbitrary finite dimensional manifold satisfying these conditions, we begin by showing that there exists such a manifold of 4 dimensions.

Let $M := \{(x_1, x_2, x_3, x_4) : x_1 > 0\}$, where (x_1, x_2, x_3, x_4) are the standard coordinates of $\mathbb{R}^4$ and define a metric on M via

$$g(\partial x_1, \partial x_1) = 1,\ g(\partial x_2, \partial x_3) = g(\partial x_2, \partial x_4) = e^{2f(x_1)},\ \text{and}\ g(\partial x_3, \partial x_3) = h(x_1)$$

where we define $f(x_1) := -x_1 + \ln(e^{x_1} - 1)$ and $h(x_1) := \frac{1}{4}e^{2x_1}$. Let $V = \text{span}\{X_1, X_2, X_3, X_4\}$. We construct a model space on V, and define it to be $\mathcal{M} := (V, \phi, R_0)$ where the following are the nonzero entries of R_0 and ϕ, up to the standard symmetries:

$$\phi(X_1, X_1) = \phi(X_2, X_3) = \phi(X_2, X_4) = 1,$$
$$R_0(X_1, X_3, X_3, X_1) = R_0(X_3, X_4, X_2, X_3) = -1,$$

and

$$R_0(X_1, X_2, X_3, X_1) = R_0(X_2, X_3, X_3, X_2) = R_0(X_1, X_2, X_4, X_1) = R_0(X_2, X_4, X_4, X_2) = 1.$$

Let $G \leq Gl(V)$ be the Lie group

$$G = \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & t & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} : t > 0 \right\},$$

which is isomorphic to $\mathbb{R}^+$. Let G act on the set of model spaces over V by

$$\begin{aligned} G \times \mathcal{M}(V) &\to \mathcal{M}(V) \\ (A, (V, \phi, R_0)) &\mapsto (V, \phi, A^* R_0). \end{aligned}$$

The following Lemma is due to calculations done in Maple.

Lemma 1. *Let (M, g) be constructed as above. The following hold:*

1. *The nonzero covariant derivatives of the coordinate frames are*

$$\nabla_{\partial x_2} \partial x_3 = \nabla_{\partial x_3} \partial x_2 = \nabla_{\partial x_2} \partial x_4 = \nabla_{\partial x_4} \partial x_2 = -e^{2f(x_1)} f'(x_1) \partial x_1,$$

$$\nabla_{\partial x_3} \partial x_3 = -\frac{h'(x_1)}{2} \partial x_1,\ \nabla_{\partial x_1} \partial x_2 = \nabla_{\partial x_2} \partial x_1 = f'(x_1) \partial x_2,$$

$$\nabla_{\partial x_1} \partial x_3 = \nabla_{\partial x_3} \partial x_1 = \frac{h'(x_1)}{2h(x_1)} \partial x_3 + \left(-\frac{h'(x_1)}{2h(x_1)} + f'(x_1) \right) \partial x_4,$$

and

$$\nabla_{\partial x_1} \partial x_4 = \nabla_{\partial x_4} \partial x_1 = f'(x_1) \partial x_4.$$

2. The nonzero curvature entries (up to the usual symmetries) are

$$R(\partial x_1,\partial x_3,\partial x_3,\partial x_1)=\frac{(h'(x_1))^2}{4h(x_1)}-\frac{h''(x_1)}{2},\ R(\partial x_3,\partial x_4,\partial x_2,\partial x_3)=-\frac{e^{2f(x_1)}(f'(x_1))(h'(x_1))}{2},$$

$$R(\partial x_2,\partial x_3,\partial x_3,\partial x_2)=R(\partial x_2,\partial x_4,\partial x_4,\partial x_2)=e^{4f(x_1)}(f'(x_1))^2,$$

and

$$R(\partial x_1,\partial x_2,\partial x_3,\partial x_1)=R(\partial x_1,\partial x_2,\partial x_4,\partial x_1)=-e^{2f(x_1)}((f'(x_1))^2+f''(x_1)).$$

Theorem 1. *As defined above, (M,g) is G-modeled up to order 0 with respect to the model space $\mathcal{M}$ (and group action as above). Moreover, (M,g) is neither CH_0 nor HCH_0.*

Proof. Using Lemma 1 part 2, we consider the frame $\{X_1,X_2,X_3,X_4\}$ (note that this is an abuse of notation when considering V above, but the significance of this abuse will become clear) where

$$X_1=\partial x_1,\quad X_2=\frac{\sqrt{|\Delta|}}{e^{2f(x_1)}}\partial x_2,\quad X_3=\frac{1}{\sqrt{|\Delta|}}\partial x_3,\quad\text{and}\quad X_4=\frac{1}{\sqrt{|\Delta|}}\partial x_4.$$

Here, we define $\Delta:=\frac{(h'(x_1))^2}{4h(x_1)}-\frac{h''(x_1)}{2}$. It is an easy verification that

$$|\Delta|=-\Delta\quad\text{and}\quad\frac{h'(x_1)}{2}=|\Delta|$$

for any $x_1>0$. Now,

$$g(X_1,X_1)=g(X_2,X_3)=g(X_2,X_4)=1$$

and

$$R(X_1,X_3,X_3,X_1)=\frac{\Delta}{|\Delta|}=-1,\ \ R(X_1,X_2,X_3,X_1)=R(X_1,X_2,X_4,X_1)=-((f'(x_1))^2+f''(x_1)),$$

$$R(X_2,X_3,X_3,X_2)=R(X_2,X_4,X_4,X_2)=(f'(x_1))^2,$$

$$\text{and } R(X_3,X_2,X_4,X_3)=-\frac{f'(x_1)h'(x_1)}{2|\Delta|}=-f'(x_1).$$

It is trivial to see that $f'(x_1)=-((f'(x_1))^2+f''(x_1))$ and also that $f'(\{x_1:x_1>0\})=\mathbb{R}^+$. Hence, for any $t>0$, we can find a point $P_0=(y_1,y_2,y_3,y_4)\in M$ such that $f'(y_1)=t$, and when considering the frame (X_1,X_2,X_3,X_4), we have

$$R(X_1,X_3,X_3,X_1)=-1,\quad R(X_1,X_2,X_3,X_1)=R(X_1,X_2,X_4,X_1)=t,$$

$$R(X_2,X_3,X_3,X_2)=R(X_2,X_4,X_4,X_2)=t^2,\quad R(X_3,X_2,X_4,X_3)=-t.$$

In other words, given any $A\in G$, there exists a point $P\in M$ such that $R_P\equiv A^*R_0$ and hence $A^*\mathcal{M}=\mathcal{M}_P$. We conclude that (M,g) is G-modeled up to order 0 with model $\mathcal{M}$. To verify that this manifold is not HCH_0, we calculate the values in (1) (via Maple) as

$$\tau=\frac{2(-e^{2x_1}+2e^{x_1}-2)}{(e^{x_1}-1)^2}$$

and

$$||R||^2 = \frac{16(36e^{x_1} - 131e^{4x_1} - 32e^{6x_1} + 80e^{5x_1} - e^{8x_1} + 8e^{7x_1} - 94e^{2x_1})}{(e^{x_1}-1)^2}$$

for any $x_1 > 0$. Now, one could verify that $\frac{\tau^2}{||R||^2}$ is non-constant as x_1 varies. Thus, by Corollary 1, we find that our manifold is not HCH_0. By Remark 3, it is also not CH_0. $\square$

3 G-Modeled Manifolds of Finite Dimension $m \geq 3$

It is our goal to prove the following Theorem:

Theorem 2. *Let $m \in \mathbb{N} \cap [3, \infty)$. Then, there exists:*

- *a pseudo-Riemannian manifold (M, g),*
- *a 0-model $\mathcal{M}$, and*
- *a Lie group $G \cong \mathbb{R}^+$ and an action of G on $\mathcal{M}(V)$*

such that M is G-modeled up to order 0 with respect to the model space $\mathcal{M}$, and dim$(M) = m$. Furthermore, (M, g) is neither CH_0 nor HCH_0.

For the remainder of this section, suppose that n is an arbitrary element of $\mathbb{N}$.

3.1 Construction for $m = 2n + 1$

Lemma 2. *Let $M = \mathbb{R}^{2n+1}$ with coordinates $(x_1, x_2, \ldots, x_{2n+1})$. Suppose f is a function of only x_1. Let g be a metric on M with nonzero entries given by*

$$g(\partial x_1, \partial x_1) = 1$$

and

$$g(\partial x_2, \partial x_3) = g(\partial x_4, \partial x_5) = \cdots = g(\partial x_{2n}, \partial x_{2n+1}) = e^{2f(x_1)}.$$

The following hold:

1. *The nonzero covariant derivatives of the coordinate frames are*

$$\nabla_{\partial x_1} \partial x_k = \nabla_{\partial x_k} \partial x_1 = f'(x_1) \partial x_k \text{ and } \nabla_{\partial x_i} \partial x_j = \nabla_{\partial x_i} \partial x_j = -f'(x_1) e^{2f(x_1)} \partial x_1$$

   where $k = 2, 3, ..., 2n + 1$, and $\{i, j\} \in U := \{\{i, j\} : g_{x_i x_j} = g_{x_j x_i} = e^{2f(x_1)}\}$.

2. *The nonzero curvature entries up to symmetry are*

$$R(\partial x_1, \partial x_i, \partial x_j, \partial x_1) = -e^{2f(x_1)}((f'(x_1))^2 + f''(x_1)),\ R(\partial x_i, \partial x_j, \partial x_j, \partial x_i) = e^{4f(x_1)}(f'(x_1))^2$$

$$\text{and } R(\partial x_i, \partial x_a, \partial x_b, \partial x_j) = -e^{4f(x_1)}(f'(x_1))^2,$$

   where $\{i, j\}$ and $\{a, b\}$ are unique sets contained in U.

Proof. 1. By construction, the only nonzero Christoffel symbols of the second kind (up to symmetries) are

$$\Gamma_{1kl} = f'(x_1)e^{2f(x_1)} \text{ and } \Gamma_{kl1} = -f'(x_1)e^{2f(x_1)},$$

where $\{k,l\} \in U$. Now, since

$$\Gamma_{1kl} = g\left(\sum_{m=1}^{2n+1} \Gamma_{1k}^m \partial x_m, \partial x_l\right) = \Gamma_{1k}^k g(\partial x_k, \partial x_l),$$

we have that

$$\Gamma_{1k}^m = \begin{cases} f'(x_1) & \text{if } m = k \\ 0 & \text{otherwise} \end{cases}.$$

This implies that $\nabla_{\partial x_1}\partial x_k = f'(x_1)\partial x_k$ and similar derivations show that $\nabla_{\partial x_k}\partial x_1$ obtains the same value. Also, we have that

$$\Gamma_{kl1} = g\left(\sum_{m=1}^{2n+1} \Gamma_{kl}^m \partial x_m, \partial x_1\right) = \Gamma_{kl}^1 g(\partial x_1, \partial x_1),$$

and hence

$$\Gamma_{kl}^1 = \begin{cases} -f'(x_1)e^{2f(x_1)} & \text{if } m = 1 \\ 0 & \text{otherwise} \end{cases}.$$

Thus, $\nabla_{\partial x_k}\partial x_l = -f'(x_1)e^{2f(x_1)}\partial x_1$ and by symmetry, we conclude that $\nabla_{\partial x_l}\partial x_k$ obtains the same value, proving the second assertion of (1). It is also clear that any other covariant derivative entries vanish.

2. Let $\{i,j\}, \{a,b\} \in U$. By part 1, we have

$$\begin{aligned} R(\partial x_1, \partial x_i, \partial x_j, \partial x_1) &= g\left(\nabla_{\partial x_1}\nabla_{\partial x_i}\partial x_j - \nabla_{\partial x_i}\nabla_{\partial x_i}\partial x_1, \partial x_j\right) \\ &= g\left(-2(f'(x_1))^2 e^{2f(x_1)} - f''(x_1)e^{2f(x_1)} - (f'(x_1))^2 e^{2f(x_1)}, \partial x_1\right) \\ &= -((f(x_1))^2 + f''(x_1)), \end{aligned}$$

$$\begin{aligned} R(\partial x_i, \partial x_j, \partial x_j, \partial x_i) &= g\left(\nabla_{\partial x_i}\nabla_{\partial x_j}\partial x_j - \nabla_{\partial x_j}\nabla_{\partial x_i}\partial x_j, \partial x_1\right) \\ &= g((f'(x_1))^2 e^{2f(x_1)}\partial x_j, \partial x_i) \\ &= -e^{4f(x_1)}(f'(x_1))^2, \end{aligned}$$

and

$$\begin{aligned} R(\partial x_i, \partial x_a, \partial x_b, \partial x_j) &= g\left(\nabla_{\partial x_i}\nabla_{\partial x_a}\partial x_b - \nabla_{\partial x_a}\nabla_{\partial x_i}\partial x_b, \partial x_j\right) \\ &= g\left(-(f'(x_1))^2 e^{2f(x_1)}\partial x_i, \partial x_j\right) \\ &= -e^{4f(x_1)}(f'(x_1))^2, \end{aligned}$$

as needed. It is left to show that the remaining curvature entries vanish. Let $p, q, r, s \in \{2, 3, \ldots, 2n+1\}$. First assume that $\{q, r\} \notin U$. Then,

$$\begin{aligned} R(\partial x_p, \partial x_q, \partial x_r, \partial x_s) &= g\left(-\nabla_{\partial x_q} \nabla_{\partial x_p} \partial x_r, \partial x_s\right) \\ &= \begin{cases} g\left((f'(x_1))^2 e^{2f(x_1)} \partial x_p, \partial x_s\right) & \text{if } \{p, r\} \in U \\ 0 & \text{otherwise} \end{cases}. \end{aligned}$$

However, we note that if both $\{p, r\}, \{q, s\} \in U$, then this is a symmetry of a nonzero curvature entry. Thus, if either $\{p, r\}$ or $\{q, s\}$ are not in U, then this curvature entry vanishes, as required. Now suppose that $\{q, r\} \in U$. To distinguish from an existing nonzero curvature entry, it must be the case that $\{p, s\} \notin U$. Now,

$$R(\partial x_p, \partial x_q, \partial x_r, \partial x_s) = g\left(-(f'(x_1))e^{2f(x_1)} \partial x_p - \nabla_{\partial x_q} \nabla_{\partial x_p} \partial x_r, \partial x_s\right) = 0,$$

where the last inequality holds since $g(\partial x_p, \partial x_s) = 0$ and $\nabla_{\partial x_q} \nabla_{\partial x_p} \partial x_r = 0$. It remains to verify that any curvature entry with only one input of ∂x_1 is zero. Let $p, q, r \in \{2, 3, \ldots, 2n+1\}$. We have that

$$R(\partial x_1, \partial x_p, \partial x_q, \partial x_r) = \begin{cases} g\left(-e^{2f(x_1)}((f'(x_1))^2 + f''(x_1))\partial_x, \partial x_s\right) & \text{if } \{p, r\} \in U \\ g\left(-e^{2f(x_1)}(2f'(x_1))^2 + f''(x_1))\partial_x, \partial x_s\right) & \text{otherwise} \end{cases},$$

but since $g(\partial x_1, \partial x_s) = 0$, this value is zero in either case. Due to symmetry, we have that any curvature entry with only one input of ∂x_1 is zero.

$\square$

The goal now is to utilize the metric construction of Lemma 2 on a particular manifold, and show that it is $G-$modeled up to order 0 for some Lie group isomorphic to $\mathbb{R}^+$. In addition, we want our construction to omit a manifold which is neither CH_0 nor HCH_0. We now show given any $2n+1$, there exists a manifold M satisfying the above properties with $\dim(M) = 2n+1$.

Let $(x_1, x_2, \ldots, x_{2n+1})$ be the standard coordinates of $\mathbb{R}^{2n+1}$ and define

$$M := \{(x_1, x_2, \ldots, x_{2n+1}) | x_1 > 0\}$$

to be our manifold. We also let our metric g on M to be defined as in Lemma 2, while using the same notation for U, and also setting $f(x_1) := -x_1 + \ln(e^{x_1} - 1)$. With this mind, we let our model space be given by $\mathcal{M} := (V, \phi, R_0)$ where $V = \operatorname{span}\{X_1, X_2, \ldots, X_{2n+1}\}$, and the nonzero inner product entries (up to the standard symmetries) are given by

$$\phi(X_1, X_1) = \phi(X_2, X_3) = \cdots = \phi(X_{2n}, X_{2n+1}) = 1.$$

If we denote $\tilde{U} := \{\{i, j\} : \phi(X_i, X_j) = \phi(X_j, X_i) = 1 \text{ and } i \neq j\}$, then we define the nonzero algebraic curvature entries as

$$R_0(X_1, X_i, X_j, X_1) = R_0(X_i, X_j, X_j, X_i) = 1, \; R_0(X_i, X_a, X_b, X_j) = -1.$$

where $\{i,j\}$ and $\{a,b\}$ are distinct sets in $\tilde{U}$. We also let $G \leq Gl(V)$ be the Lie group defined to be the set of $(2n+1)\times(2n+1)$ matrices $A=[a_{ij}]$ for which

$$a_{ij} = \begin{cases} 1 & \text{if } i=j\equiv 1 \pmod 2, \\ t & \text{if } i=j\equiv 0 \pmod 2, \\ 0 & \text{otherwise} \end{cases}$$

where $t\in\mathbb{R}^+$, so that $G\cong\mathbb{R}^+$. Furthermore, we let G act on the set of model spaces over V (call this set $\mathcal{M}(V)$) via

$$\begin{gathered} G\times\mathcal{M}(V)\to\mathcal{M}(V) \\ (A,(V,\phi,R_0))\mapsto(V,\phi,A^*R_0), \end{gathered}$$

where A^* represents precomposition by A.

Theorem 3. *As defined above, the manifold (M,g) is G-modeled up to order 0 with respect to the model space $\mathcal{M}$ and the given group action on $\mathcal{M}(V)$. In addition, (M,g) is not HCH_0.*

Proof. By Lemma 2, we deduce that the nonzero curvature entries on the vector field (up to the usual symmetries) are given by

$$R(\partial x_1,\partial x_i,\partial x_j,\partial x_1)=-e^{2f(x_1)}((f'(x_1))^2+f''(x_1)),\ R(\partial x_i,\partial x_j,\partial x_j,\partial x_i)=e^{4f(x_1)}(f'(x_1))^2$$
$$\text{and } R(\partial x_i,\partial x_a,\partial x_b,\partial x_j)=-e^{4f(x_1)}(f'(x_1))^2,$$

where $\{i,j\}$ and $\{a,b\}$ are unique sets contained in U. Now consider the frame $\{X_1,X_2,\ldots,X_{2n+1}\}$ (using the same abuse of notation as before) where

$$X_i = \begin{cases} \partial x_i & \text{if } i\equiv 1 \pmod 2 \\ e^{-2f(x_1)}\partial x_i & \text{if } i\equiv 0 \pmod 2 \end{cases}.$$

Noting that if $\{i,j\}\in U$, then i is incongruent to j modulo 2, it easily follows that

$$g(X_1,X_1)=g(X_2,X_3)=g(X_4,X_5)=\cdots=g(X_{2n},X_{2n+1})=1$$

and

$$\begin{gathered} R(X_1,X_i,X_j,X_1)=-((f'(x_1))^2+f''(x_1)),\ R(X_i,X_j,X_j,X_i)=(f'(x_1))^2, \\ R(X_i,X_a,X_b,X_j)=-(f'(x_1))^2, \end{gathered}$$

for any distinct pairs $\{i,j\}$ and $\{a,b\}$ in $\tilde{U}$. We already know that $f'(x_1)=-((f'(x_1))^2+f''(x_1))$ and also that f' is surjective onto $\mathbb{R}^+$. Thus, for any $t>0$, there exists a point $P=(y_1,y_2,...,y_{2n+1})\in M$ such that $f'(y_1)=t$, and on the frame $(X_1,X_2,\ldots,X_{2n+1})$, we find that

$$R(X_1,X_i,X_j,X_1)=t,\ R(X_i,X_j,X_j,X_i)=t^2,$$

and

$$R(X_i,X_a,X_b,X_j)=-t^2.$$

In other words, for any $A \in G$, there exists a $P \in M$ such that $A^*\mathcal{M} = \mathcal{M}_P$. We conclude that (M, g) is $G-$modeled up to order 0 with respect to the model space $\mathcal{M}$.

We now check that the manifold is not HCH_0. Calculating the values in (1), through basic counting, one could deduce that

$$\tau = (4n)t - \left(2n + 8\binom{n}{2}\right) t^2$$

and that

$$||R||^2 = 8nt^2 + \left(4n + 16\binom{n}{2}\right) t^4$$

for any $t > 0$. Hence, as a function of t,

$$F(t) := \frac{\tau^2}{||R||^2} = \frac{(4n^3 - 4n^2 + n)t^2 - (8n^2 - 4n)t + 4n}{(2n-1)t^2 + 2},$$

in which case,

$$F'(t) = \frac{(16n^3 - 16n^2 + 4n)t^2 + (16n^3 - 32n^2 + 12n)t - (16n^2 - 8n)}{((2n-1)t^2 + 2)}.$$

Since $F'(t) \not\equiv 0$, by Corollary 1, we conclude that our manifold is not HCH_0, and hence not CH_0 either. $\square$

3.2 Construction for $m = 2n + 2$

Lemma 3. *Let $M = \mathbb{R}^{2n+2}$ with coordinates $(x_1, x_2, \ldots, x_{2n+2})$. Suppose f is a function of only x_1. Let g be a metric on M with nonzero entries given by*

$$g(\partial x_1, \partial x_1) = 1$$

and

$$g(\partial x_2, \partial x_3) = g(\partial x_4, \partial x_5) = \cdots = g(\partial x_{2n}, \partial x_{2n+1}) = g(\partial x_{2n+2}, \partial x_{2n+2}) = e^{2f(x_1)}.$$

The following hold:

1. *The nonzero covariant derivatives of the coordinate frames are*

$$\nabla_{\partial x_1} \partial x_k = \nabla_{\partial x_k} \partial x_1 = f'(x_1)\partial x_k \text{ and } \nabla_{\partial x_i} \partial x_j = \nabla_{\partial x_i} \partial x_j = -f'(x_1)e^{2f(x_1)}\partial x_1$$

where $k = 2, 3, ..., 2n+1$, and $\{i, j\} \in \dot{U} := \{\{i, j\} : g_{x_i x_j} = g_{x_j x_i} = e^{2f(x_1)}, i \neq j\}$.

2. *The nonzero curvature entries up to symmetry are*

$$R(\partial x_1, \partial x_i, \partial x_j, \partial x_1) = R(\partial x_1, \partial x_{2n+2}, \partial x_{2n+2}, \partial x_1) = -e^{2f(x_1)}((f'(x_1))^2 + f''(x_1)),$$

$$R(\partial x_i, \partial x_j, \partial x_j, \partial x_i) = e^{4f(x_1)}(f'(x_1))^2,$$

$$\text{and } R(\partial x_i, \partial x_a, \partial x_b, \partial x_j) = R(\partial x_{2n+2}, \partial x_i, \partial x_j, \partial x_{2n+2}) = -e^{4f(x_1)}(f'(x_1))^2,$$

where $\{i, j\}$ and $\{a, b\}$ are unique sets contained in $\dot{U}$.

Proof. The proof of this Lemma is similar to the proof of Lemma 2 and thus is omitted. $\square$

Now, for our construction, let our manifold be given by

$$M := \{(x_1, x_2, \ldots, x_{2n+2}) \in \mathbb{R}^{2n+2} : x_1 > 0\}$$

and let the metric g on M be defined as in the previous Lemma. We again suppose that $f(x_1) = -x_1 + \ln(e^{x_1} - 1)$. Suppose $\mathcal{M} := (V, \phi, R_0)$ is the model space where $V = \text{span}\{X_1, X_2, \ldots, X_{2n+2}\}$, the nonzero inner product entries are

$$\phi(X_1, X_1) = \phi(X_2, X_3) = \cdots = \phi(X_{2n}, X_{2n+1}) = \phi(X_{2n+2}, X_{2n+2}) = 1.$$

Defining $\hat{U} := \{\{i, j\} : \phi(X_i, X_j) = \phi(X_j, X_i) = 1, i \neq j\}$, we let the nonzero algebraic curvature entries be given by

$$R_0(X_1, X_i, X_j, X_1) = R_0(X_1 X_{2n+2}, X_{2n+2}, X_1) = 1,\ R(X_i, X_j, X_j, X_i) = 1$$

$$\text{and } R(X_i, X_a, X_b, X_j) = R(X_{2n+2}, X_i, X_j, X_{2n+2}) = -1,$$

where $\{i, j\}$ and $\{a, b\}$ are unique sets in $\hat{U}$. We also let $G \leq Gl(V)$ be the set of $(2n+2) \times (2n+2)$ matrices $A = [a_{ij}]$ such that

$$a_{ij} = \begin{cases} 1 & \text{if } i = j \equiv 1 \pmod 2, \\ t & \text{if } i = j \equiv 0 \pmod 2 \text{ and } i \neq 2n+2, \\ \sqrt{t} & \text{if } i = j = 2n+2 \\ 0 & \text{otherwise} \end{cases}$$

where $t \in \mathbb{R}^+$, so that $G \cong \mathbb{R}^+$. We again let G act on $\mathcal{M}(V)$ via

$$\begin{gathered} G \times \mathcal{M}(V) \to \mathcal{M}(V) \\ (A, (V, \phi, R_0)) \mapsto (V, \phi, A^* R_0), \end{gathered}$$

where A^* represents precomposition by A.

Theorem 4. *As defined above, the manifold (M, g) is G-modeled up to order 0 with respect to the model space $\mathcal{M}$ and the given group action on $\mathcal{M}(V)$. In addition, (M, g) is not HCH_0.*

Proof. The details of this proof are similar to that of Theorem 3, and hence we only mention the change of frames. We consider the frame $\{X_1, X_2, \ldots, X_{2n+2}\}$ (using the same abuse of notation as before) where

$$X_i = \begin{cases} \partial x_1 & \text{if } i \equiv 1 \pmod 2, \\ e^{-2f(x_1)} & \text{if } i \equiv 0 \pmod 2 \text{ and } i \neq 2n+2, \\ e^{-f(x_1)} & \text{if } i = 2n+2 \end{cases}.$$

As before, if we set $f' = t$, then

$$R(X_1, X_i, X_j, X_1) = R(X_1, X_{2n+2}, X_{2n+2}, X_1) = t,\ R(X_i, X_j, X_j, X_i) = t^2,$$

$$\text{and } R(X_i, X_a, X_b, X_j) = R(X_{2n+2}, X_i, X_j, X_{2n+2}) = -t^2,$$

where $\{i, j\}$ and $\{a, b\}$ are unique sets contained in $\dot{U}$.

Again, as in Theorem 3, it can be verified that the value $\frac{\tau^2}{||R||^2}$ is non-constant, and using Corollary 1, we see that (M, g) is neither CH_0 nor HCH_0.

□

The combination of Theorems 3 and 4 proves Theorem 2.

4 Conclusion and Open Problems

While the work above tampers with the dimension of a G-modeled manifold, with the dimension of G being 1, it was shown in [3] that there is a 3 dimensional manifold that is G-modeled with respect to a model $\mathcal{M}$, where G had dimension 2. In fact, in this example, the manifold was $G-$modeled up to order 1. The next steps would be to investigate manifolds that are G-modeled up to order $k \geq 1$, where the Lie group has dimension $l \geq 1$. In addition, one might investigate more topologically interesting groups, such as a compact Lie group.

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