Canonical Expressions of Algebraic Curvature Tensors

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Algebraic curvature tensors can be expressed in a variety of ways, and it is helpful to develop invariants that can distinguish between them. One known invariant of an algebraic curvature tensor R is its structure group. Another potential invariant is the signature of R, which could be defined in a number of ways. A better understanding of how the structure groups of sums and differences of canonical algebraic curvature tensors differ in general could be helpful in further study on the signature conjecture because if $R = R_{\tau_1} + R_{\tau_2}$ and $R = R_{\psi_1} - R_{\psi_2}$, the structure groups $G_{R_{\tau_1}+R_{\tau_2}}$ and $G_{R_{\psi_1}-R_{\psi_2}}$ must be equal. The author conducted research in both the structure groups of sums and differences of canonical algebraic curvature tensors in dimension 3 and the signature conjecture. As a result, this report contains two sections. The first is dedicated to the signature conjecture, and the second concerns structure groups.

1 The Signature Conjecture

1.1 Abstract

This project shows that any algebraic curvature tensor defined on a vector space V with $dim(V) = n$ can be expressed using only canonical algebraic curvature tensors from forms with rank k or higher for any $k \in \{2, \ldots, n\}$, and that such an expression is not unique. We also provide bounds on the minimum number of algebraic curvature tensors of rank k needed to express any given R.

1.2 Introduction

Throughout, V is a real vector space with finite dimension n. A multilinear form $R : V \times V \times V \times V \to \mathbb{R}$ is an algebraic curvature tensor if $\forall x, y, z, w \in V$, R satisfies

$$R(x, y, z, w) = R(z, w, x, y) = -R(y, x, z, w), \text{ and}$$
$$R(x, y, z, w) + R(x, z, w, y) + R(x, w, y, z) = 0.$$

The space of all algebraic curvature tensors on V is denoted $\mathcal{A}(V)$. Given a symmetric bilinear form φ, we can define the canonical algebraic curvature tensor R_φ

$$R_\varphi(x, y, z, w) = \varphi(x, w)\varphi(y, z) - \varphi(x, z)\varphi(y, w).$$

For any positive real number c, $R_{\sqrt{c}\varphi} = cR_\varphi$.
Since algebraic curvature tensors are multilinear forms, we can R can be defined by where it maps some basis vectos e_i. $R(e_i, e_j, e_k, e_l)$ is denoted R_{ijkl}. For a canonical algebraic curvature tensor$R = R_\varphi$, $R_{ijkl} = \varphi(e_i, e_l)\varphi(e_j, e_k) - \varphi(e_i, e_k)\varphi(e_j, e_l)$. A diagonal matrix representation of φ exists since φ is symmetric, and $\varphi(e_i, e_j)$ is the entry in the i^{th} row and j^{th} column of this diagonal matrix. Thus $\varphi(e_i, e_j) \neq 0$ only if $i = j$, so $R_{ijkl} \neq 0$ only when two idices are used exactly twice, e.g. R_{ijji}. Note that R_{jiij}, R_{ijij}, etc. are defined by their relation to a given R_{ijji} using the properties of algebraic curvature tensors. Thus it suffices to define R by all the possible R_{ijji}, and R_{ijji} is the product of the i^{th} and j^{th} diagonal entries of φ.
In [3], Gilkey showed that any algebraic curvature tensor R can be expressed in the form

$$R = \sum_{i=1}^{m} \epsilon_i R_{\varphi_i}$$

for $\epsilon_i = 1$ or -1 and some symmetric bilinear forms φ_i. For a given R, define

$$\nu(R) = \min\{m | R = \sum_{i=1}^{m} \epsilon_i R_{\varphi_i}\}.$$

For any positive integer n, define

$$\nu(n) = \max_{R \in \mathcal{A}(V)} \nu(R)$$

where V has dimension n. Gilkey also showed in [3] that any R can be expressed as

$$R = \sum_{i=1}^{m} \epsilon_i R_{\psi_i}$$

for $\epsilon_i = 1$ or -1 and some antisymmetric bilinear forms ψ_i. Then we have the analogous definitions [5]

$$\eta(R) = \min\{m | R = \sum_{i=1}^{m} \epsilon_i R_{\psi_i}\} \text{ and}$$
$$\eta(n) = \max_{R \in \mathcal{A}(V)} \eta(R).$$

If we instead use bilinear forms τ_i which may be symmetric or antisymmetric [5],

$$\mu(R) = \min\{m | R = \sum_{i=1}^{m} \epsilon_i R_{\tau_i}\} \text{ and}$$
$$\mu(n) = \max_{R \in \mathcal{A}(V)} \eta(R).$$

This paper focuses on symmetric bilinear forms. For some positive integer $k \geq 2$, we define

$$\nu_k(R) = \min\{m | R = \sum_{i=1}^{m} \epsilon_i R_{\varphi_i}, \text{ where } \forall i,\ Rank(\varphi_i) \geq k\}.$$

Then, for any positive integer n, we define

$$\nu_k(n) = \max_{R \in \mathcal{A}(V)} \nu_k(R)$$

where V has dimension n. Note that if $Rank(\varphi) = 1$ or 0, R_φ is the zero tensor. Thus any minimal expression for $R \neq 0$ contains only forms of $Rank$ 2 or higher, so the absolute minimal number of canonical tensors needed, $\nu(R)$ is equal to $\nu_2(R)$ for all $R \neq 0$, and $\nu_2(n) = \nu(n)$. It was shown in [4] that $\nu(n) \leq \frac{n(n+1)}{2}$.

Any symmetric bilinear form φ can be diagonalized, and Sylvester's law of inertia [7] states that the number of negative entries p, the number of positive entries q, and the number of 0 entries s along the diagonal is unique. (p,q,s) is called the <u>signature</u> of φ.

Throughout, we denote diagonal matrices

$$diag(\lambda_1, \lambda_2, \ldots, \lambda_n) = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix}.$$

We demonstrate the proof that $R_\varphi = R_A + R_B$ for some symmetric bilinear forms φ, A, and B the first time it arises in the proof of theorem 1.1. All similar claims are proved in the same way, so we do not demonstrate the calculations again. For any symmetric bilinear form φ with $Rank(\varphi) \geq 3$, there is no ψ for which $R_\varphi = -R_\psi$ [1]. Noting this, the following conjecture was made.

Conjecture 1.1 (The Signature Conjecture)**.** *For any algebraic curvature tensor R and expression*

$$R = \sum_{i=1}^{\nu_3(R)} \epsilon_i R_{\varphi_i}$$

where $Rank(\varphi_i) \geq 3$ $\forall i$, the number of i such that $\epsilon_i = -1$ is unique.

If one is presented with components of two algebraic curvature tensors on different bases that could perhaps be the same tensor, it is useful to develop quantities that can distinguish between these algebraic curvature tensors. These quantities are called <u>invariants</u>. If the signature conjecture were true, we could define the signature of an algebraic curvature tensor R to be the number of $+$ and $-$ signs used any expression of R in $\nu_3(R)$ terms, and the signature of R would be an invariant.

In section 1.3, we show that $\nu_3(R)$ is well defined for every R. In section 1.4, we show that the signature conjecture is not true as stated in conjecture 1.1, and we provide revised conjectures in section 1.5.

1.3 Bounds on $\nu_k(n)$

Gilkey's proof that $R = \sum_{i=1}^{m} \epsilon_i R_{\varphi_i}$ for every R requires that some φ_i can have rank 2. Thus to even consider the signature conjecture, we need to show that $\nu_3(R)$ is well defined. It is also useful to check that $\nu_k(R)$ is well defined, as a higher rank requirement is one way to strengthen the conjecture. In this section, we show that $\nu_k(R)$ is well defined for any R and any $k \in 3,\ldots,n$, and we provide an upper bound on $\nu_k(R)$.

Theorem 1.1. *$\nu_k(R) \leq 2\nu_{k-1}(R)$ for any $R \in \mathcal{A}(V)$ and any $k \in 3,\ldots,n$.*

Proof. Choose any $R \in \mathcal{A}(V)$. By definition, $\nu_{k-1}(R) \leq \nu_{k-1}(n)$. We can write

$$R = \sum_{i=1}^{\nu_{k-1}(R)} \epsilon_i R_{\varphi_i} \text{ with } Rank(\varphi_i) \geq k-1 \ \forall i,$$

For any φ_i with rank $k-1$, there is some basis where

$$\varphi_i = diag(0, \ldots, 0, \lambda_1, \ldots, \lambda_{k-1})$$

for $\lambda_i \in \mathbb{R}$. Define

$$A = diag\Big(0, \ldots, 0, 1, \frac{\lambda_1}{\sqrt{2}}, \ldots, \frac{\lambda_{k-1}}{\sqrt{2}}\Big) \text{ and}$$
$$B = diag\Big(0, \ldots, 0, -1, \frac{\lambda_1}{\sqrt{2}}, \ldots, \frac{\lambda_{k-1}}{\sqrt{2}}\Big).$$

One can check that $R_{\varphi_i} = R_A + R_B$. Let the number of diagonal entries equal to 0 in φ_i be s. The $ijji$ entry of R_{φ_i} is 0 if $i \leq s$ or $j \leq s$ and $\lambda_i\lambda_j$ if $i > s$ and $j > s$.
The $ijji$ entries of R_A and R_B are both 0 if $i < s-1$ or $j < s-1$ and $\frac{\lambda_i\lambda_j}{2}$ if $i > s$ and $j > s$. Without loss of generality, the $sjjs$ entry of R_A is $\frac{\lambda_j}{\sqrt{2}}$ and the $sjjs$ entry of R_B is $\frac{-\lambda_j}{\sqrt{2}}$. Then the $ijji$ entries of $R_A + R_B$ are 0 if $i \leq s$ or $j \leq s$ and $\lambda_i\lambda_j$ if $i > s$ and $j > s$, so $R_{\varphi_i} = R_A + R_B$. Replace R_{φ_i} in the expression $R = \sum_{i=1}^{\nu_{k-1}(R)} \epsilon_i R_{\varphi_i}$ with $R_A + R_B$. There are at most $\nu_{k-1}(R)$ R_{φ_i} to be replaced, so R can be expressed as a sum of at most $2\nu_{k-1}(R)$ forms of rank k. □

Corollary 1.1.1. *$\nu_k(n) \leq 2\nu_{k-1}(n)$ for any $k \in 3,\ldots,n$.*

Proof. By definition, $\nu_{k-1}(R) \leq \nu_{k-1}(n)\ \forall R$. The theorem shows that $\nu_k(R) \leq 2\nu_{k-1}(R) \leq 2\nu_{k-1}(n)$ for all R, so it is clear that $\nu_k(n) \leq 2\nu_{k-1}(n)$. □

Corollary 1.1.2. *$\nu_k(n) \leq 2^{k-3}n(n+1)$ for any $k \in 3,\ldots,n$.*

Proof. In [4], it was shown that $\nu(n) \le \frac{n(n+1)}{2}$. When $k = 3$, $2^{k-3} = 1$. The previous theorem shows that $\nu_3(n) \le 2\nu(n)$, and Gilkey's result verifies that $\nu_3(n) \le n(n+1)$. If $\nu_k(n) \le 2^{k-3}n(n+1)$ for some k, then the theorem implies $\nu_{k+1}(n) \le 2\nu_k(n) \le 2^{k-2}n(n+1)$. Thus the corollary is true by induction. □

The following theorem demonstrates that in at least some cases, $\nu_k(n) < 2\nu_{k-1}(n)$.

Theorem 1.2. $\nu_3(3) = \nu(3) = 2$

Proof. In [2], it was shown that $\nu(3) = 2$ and any $R \in \mathcal{A}(V)$ when $dim(V) = 3$ is exactly one of the following: R_φ where $Rank(\varphi) = 3$, R_φ where $Rank(\varphi) = 3$, or $R = R_{\varphi_1} + R_{\varphi_2}$ and $R \neq R_\varphi$ for any φ where, on some basis,

$$\varphi_1 = diag(0, 1, \lambda_2) \text{ and } \varphi_2 = diag(1, 0, \lambda_1) \text{ for some nonzero } \lambda_i.$$

In the first case, $\nu_3(R) = 1$. In the second case, Gilkey showed that $R_\varphi \neq R_\psi$ for any φ with rank 2 and ψ with rank 3, so $\nu_3(R) \neq 1$. There is some basis where $R = diag(0, a, b)$. Then $R = R_A + R_B$ for $A = diag\big(1, \frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\big)$ and $B = diag\big(-1, \frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\big)$, so $\nu_3(R) = 2$.
In the third case, it is again clear that $\nu_3(R) > 1$, but $R_{\varphi_1} + R_{\varphi_2} = R_{\tau_1} + R_{\tau_2}$ where

$$\tau_1 = diag\Big(\frac{1}{\sqrt{3}}, -\sqrt{3}, \frac{\sqrt{3}\lambda}{2}\Big) \text{ and } \tau_2 = diag\Big(1, 1, \frac{\lambda}{2}\Big) \text{ if } \lambda = \lambda_1 = -\lambda_2,$$

$$\tau_1 = diag\Big(\frac{1}{\sqrt{3}}, \sqrt{3}, \frac{\sqrt{3}\lambda}{2}\Big) \text{ and } \tau_2 = diag\Big(1, -1, \frac{\lambda}{2}\Big) \text{ if } \lambda = \lambda_1 = \lambda_2,$$

and

$$\tau_1 = diag\Big(\sqrt{2}, \sqrt{2}, \frac{\lambda_1 + \lambda_2}{\sqrt{8}}\Big) \text{ and } \tau_2 = diag\Big(-\sqrt{2}, \sqrt{2}, \frac{\lambda_1 - \lambda_2}{\sqrt{8}}\Big)$$

otherwise. For any nonzero choice of λ_i, $Rank(\tau_i) = 3$, so $\nu_3(R_{\varphi_1} + R_{\varphi_2}) = 2$. Thus $\nu_3(3) = 2$. □

1.4 Counterexamples to the Signature Conjecture

In the original statement of the signature conjecture, we require that any expression of R uses forms of at least rank 3. To generate a counterexample, choose any real numbers a and b with $|b| > |a|$. R_τ where τ takes the form

$$\tau = diag\Big(0, \ldots, 0, \sqrt{b^2 - a^2}, \sqrt{b^2 - a^2}\Big)$$

is a counterexample, since $R_\tau = R_A + R_B = R_{\bar{A}} - R_{\bar{B}}$ where

$$A = diag\left(0, \ldots, 0, 1, \frac{\sqrt{b^2 - a^2}}{\sqrt{2}}, \frac{\sqrt{b^2 - a^2}}{\sqrt{2}}\right),\ B = diag\left(0, \ldots, 0, -1, \frac{\sqrt{b^2 - a^2}}{\sqrt{2}}, \frac{\sqrt{b^2 - a^2}}{\sqrt{2}}\right),$$

$\bar{A} = diag\big(0, \ldots, 0, a, b, b\big)$, and $\bar{B} = diag\big(0, \ldots, 0, b, a, a\big)$.

This problem cannot be resolved by choosing a higher minimal rank in what might be a revised signature conjecture, as the following theorem shows.

Theorem 1.3. *For any symmetric bilinear form τ with rank $k-1$, $R_\tau = R_A + R_B = R_{\bar{A}} - R_{\bar{B}}$ for some symmetric bilinear forms A, B, $\bar{A}$, and $\bar{B}$ with rank k.*

Proof. Take any symmetric bilinear form τ of signature $(p, q, s+1)$ where $p + q = k - 1$. We can find a basis where

$$\tau = diag(\underbrace{0, \ldots, 0}_{s+1}, \underbrace{-1, \ldots, -1}_{p}, \underbrace{1, \ldots, 1}_{q}).$$

Then $R_\tau = R_A + R_B$ for

$$A = diag\Big(\underbrace{0, \ldots, 0}_{s}, 1, \underbrace{\frac{-1}{\sqrt{2}}, \ldots, \frac{-1}{\sqrt{2}}}_{p}, \underbrace{\frac{1}{\sqrt{2}}, \ldots, \frac{1}{\sqrt{2}}}_{q}\Big),$$

$$B = diag\Big(\underbrace{0, \ldots, 0}_{s}, -1, \underbrace{\frac{-1}{\sqrt{2}}, \ldots, \frac{-1}{\sqrt{2}}}_{p}, \underbrace{\frac{1}{\sqrt{2}}, \ldots, \frac{1}{\sqrt{2}}}_{q}\Big)$$

and $R = R_{\bar{A}} - R_{\bar{B}}$ for

$$\bar{A} = diag(\underbrace{0, \ldots, 0}_{s}, a, \underbrace{-b, \ldots, -b}_{p}, \underbrace{b, \ldots, b}_{q}), \text{ and}$$

$$\bar{B} = diag(\underbrace{0, \ldots, 0}_{s}, b, \underbrace{-a, \ldots, -a}_{p}, \underbrace{a, \ldots, a}_{q})$$

where $b = \frac{1}{a}$ and $\frac{1}{a^2} - a^2 = 1$, or $a = \pm\sqrt{\frac{\sqrt{5}-1}{2}} = \pm\frac{1}{\sqrt{\varphi}}$ where φ is the golden ratio. In other words, if

$$T_1 = diag(\underbrace{0,\dots,0}_{s}, \frac{1}{\varphi}, \underbrace{-1,\dots,-1}_{p}, \underbrace{1,\dots,1}_{q}), \text{ and}$$
$$T_2 = diag(\underbrace{0,\dots,0}_{s}, \varphi, \underbrace{-1,\dots,-1}_{p}, \underbrace{1,\dots,1}_{q}),$$

$R_{\bar{A}} = R_{\sqrt{\varphi}T_1}$ and $R_{\bar{B}} = R_{\frac{1}{\sqrt{\varphi}}T_2}$, so $R = \varphi R_{T_1} - \frac{1}{\varphi} R_{T_2}$.

□

Counterexamples of this type can be avoided by requiring that $\nu(R) = \nu_k(R)$ for a chosen minimal rank k.

Definition 1.1. *An algebraic curvature tensor R is absolutely minimal in rank k if $\nu_k(R) = \nu(R)$.*

The above counterexamples demonstrate that absolute minimality is necessary. The following result demonstrates that it is not sufficient when $k = 3$.

Theorem 1.4. *There exists an algebraic curvature tensor R such that $\nu(R) = 2$ and $R = R_{\tau_1} + R_{\tau_2} = R_{\psi_1} - R_{\psi_2}$ for some symmetric bilinear forms τ_1, τ_2, ψ_1, and ψ_2 with rank at least 3.*

Proof. Let $R = R_{\varphi_1} + R_{\varphi_2}$ where

$$\varphi_1 = diag(0, 1, \lambda_1) \text{ and } \varphi_2 = diag(1, 0, \lambda_2) \text{ with } \lambda_i \neq 0$$

for some nonzero λ_1 and λ_2. [2] showed that $\nu(R) = 2$. $R_{\psi_1} - R_{\psi_2}$ for

$$\psi_1 = diag(\lambda_1, \lambda_2, 2) \text{ and } \psi_2 = diag(\lambda_1, \lambda_2, 1)$$

and $R = R_{\tau_1} + R_{\tau_2}$ where τ_1 and τ_2 are defined as in the proof of Theorem 1.2. Since λ_1 and λ_2 were chosen to be nonzero, $Rank(\tau_1) = Rank(\tau_2) = Rank(\psi_1) = Rank(\psi_2) = 3$. □

Corollary 1.4.1. *For any integer n, there exists an algebraic curvature tensor $R \in \mathcal{A}(V)$ where $dim(V) = n$ such that $\nu(R) = 2$ and $R = R_{\tau_1} + R_{\tau_2} = R_{\psi_1} - R_{\psi_2}$ for some symmetric bilinear forms τ_1, τ_2, ψ_1, and ψ_2 with rank at least 3.*

Proof. Let $R = R_{\varphi_1} + R_{\varphi_2}$ where

$$\varphi_1 = diag(\underbrace{0, \ldots, 0}_{n-2}, 1, \lambda_1) \text{ and } \varphi_2 = diag(\underbrace{0, \ldots, 0}_{n-3}, 1, 0, \lambda_2) \text{ with } \lambda_i \neq 0$$

for some nonzero λ_1 and λ_2. The proof that $\nu(R) = 2$ given in [2] still holds when we extend R to dimension n by adding more 0 entries on the diagonal, so $\nu(R) = 2$. $R = R_{\psi_1} - R_{\psi_2}$ where

$$\psi_1 = diag(\underbrace{0, \ldots, 0}_{n-3}, \lambda_1, \lambda_2, 2) \text{ and } \psi_2 = diag(\underbrace{0, \ldots, 0}_{n-3}, \lambda_1, \lambda_2, 1),$$

and $R = R_{\tau_1} + R_{\tau_2}$ where

$$\tau_1 = diag\left(\underbrace{0, \ldots, 0}_{n-3}, \frac{1}{\sqrt{3}}, -\sqrt{3}, \frac{3\lambda}{2\sqrt{3}}\right) \text{ and } \tau_2 = diag\left(\underbrace{0, \ldots, 0}_{n-3}, 1, 1, \frac{\lambda}{2}\right)$$

if $\lambda = \lambda_1 = -\lambda_2 \neq 0$,

$$\tau_1 = diag\left(\underbrace{0, \ldots, 0}_{n-3}, \frac{1}{\sqrt{3}}, \sqrt{3}, \frac{3\lambda}{2\sqrt{3}}\right) \text{ and } \tau_2 = diag\left(\underbrace{0, \ldots, 0}_{n-3}, 1, -1, \frac{\lambda}{2}\right)$$

if $\lambda = \lambda_1 = \lambda_2 \neq 0$, and

$$\tau_1 = diag\left(\underbrace{0, \ldots, 0}_{n-3}, \sqrt{2}, \sqrt{2}, \frac{\lambda_1 + \lambda_2}{\sqrt{8}}\right) \text{ and } \tau_2 = diag\left(\underbrace{0, \ldots, 0}_{n-3}, -\sqrt{2}, \sqrt{2}, \frac{\lambda_1 - \lambda_2}{\sqrt{8}}\right)$$

otherwise. □

1.5 Revisions to the Signature Conjecture

Since all the absolutely minimal counterexamples have $k = 3$, it may be sufficient to require $k \geq 4$. The revised signature conjecture would then be:

Conjecture 1.2. *Given an expression $R = \sum_{i=1}^{\nu(R)} \alpha_i R_{\varphi_i}$ where $\alpha_i = \pm 1$ and $Rank(\varphi_i) \geq 4$, the number of i for which $\alpha_i = -1$ is unique.*

The simplest form of a counterexample to this revised signature conjecture would be any R such that $\nu(R) = 2$ and $R = R_{\tau_1} + R_{\tau_2} = R_{\psi_1} - R_{\psi_2}$ for some τ_i and ψ_i with rank at least k for some $k \geq 4$. Since the majority of the counterexamples come from manipulating kernels, it would also be reasonable to amend this conjecture to $k = n$.

Conjecture 1.3. *Given an expression* $R = \sum_{i=1}^{\nu(R)} \alpha_i R_{\varphi_i}$ *where* $\alpha_i = \pm 1$, $Rank(\varphi_i) = n$, *and* $n \geq 4$, *the number of* i *for which* $\alpha_i = -1$ *is unique.*

In every counterexample we have demonstrated for $k > 3$, the signatures of the symmetric bilinear forms involved in an expression of R differ when the signs involved differ. We cannot simply require that the multiset of signatures of the φ_i is equal to the multiset of signatures of the ψ_j in any two absolutely minimal expressions $R = \sum_{i=1}^{\nu(R)} \alpha_i R_{\varphi_i} = \sum_{j=1}^{\nu(R)} \epsilon_j R_{\psi_j}$ where $Rank(\varphi_i) = Rank(\psi_j) = n$ in dimension 4 or higher. We must account for the fact that $R_\varphi = R_{-\varphi}$ and the signatures of φ and $-\varphi$ differ. This leads to the definition of an adjusted signature of φ and another possible revision of the signature conjecture.

Definition 1.2. *The adjusted signature of a bilinear form* φ *is the signature* (p, q, s) *of* φ *if* $q \geq p$ *and the signature* (q, p, s) *of* $-\varphi$ *if* $p > q$.

Conjecture 1.4. *In any two absolutely minimal expressions in dimension 4 or higher,* $R = \sum_{i=1}^{\nu(R)} \alpha_i R_{\varphi_i} = \sum_{j=1}^{\nu(R)} \epsilon_j R_{\psi_j}$ *where* $Rank(\varphi_i) = Rank(\psi_j) = n$ *and the multiset of adjusted signatures of the* φ_i *is equal to the multiset of adjusted signatures of the* ψ_j, *the number of* i *for which* $\alpha_i = -1$ *is equal to the number of* j *for which* $\epsilon_i = -1$.

We consider only $k \geq 4$ because the case $R = R_{\varphi_1} + R_{\varphi_2}$ where $\varphi_1 = diag(0, \ldots, 0, 1, \lambda)$, $\varphi_2 = diag(0, \ldots, 0, 1, 0, \lambda)$, and $\lambda < 0$ is a counterexample if $k = 3$. This can be seen by checking the signatures of the rank 3 τ_i and ψ_i defined in the previous section such that $R = R_{\tau_1} + R_{\tau_2} = R_{\psi_1} - R_{\psi_2}$.

1.6 Future Work

1. What is the nature of all counterexamples to the signature conjecture as originally stated? Does there exist an R in dimension 4 or higher for which $\nu(R) = 2$, $R = R_{\tau_1} + R_{\tau_2}$ for some τ_i with rank n, and $R = R_{\psi_1} - R_{\psi_1}$ for some ψ_i with rank n?

2. In the dimension 3 case, it was shown that $\nu_3(3) = \nu(3) = 2$, so $\nu_3(3) < 2\nu_2(3) = 4$. Can the bounds on $\nu_k(n)$ be improved upon in other cases?

3. When does $R_\varphi = R_{\tau_1} + R_{\tau_2} = R_{\psi_1} - R_{\psi_2}$ where $Rank(\varphi) = k$ and $Rank(\tau_i) = Rank(\psi_i) = k - 1$? Some cases to this are already known [8], but a more complete classification could be useful in proving one of the revised signature conjectures.

4. Given R, what is

$$\bar{\nu}_k(R) = \min_N \left\{ R = \sum_{i=1}^{N} \alpha_i R_{\varphi_i} | Rank(\varphi_i) = k \right\}?$$

5. Which revision from section 1.5, if any, of the signature conjecture holds?

6. Are there bounds on $\eta_k(R)$ and $\eta_k(n)$? On $\mu_k(R)$ and $\mu_k(n)$?

7. Is $\{R|\nu(R) = 1\}$ a dense subset of $\mathcal{A}(V)$?

2 Structure Groups

2.1 Abstract

How does the subgroup of $Gl(n)$ which preserves R where $R = R_{\varphi_1} \pm R_{\varphi_2}$ relate to the subgroups that preserve each R_{φ_i} individually? Clearly any element of $Gl(n)$ that preserves both R_{φ_i} preserves R, but it is not clear whether these are the only elements that preserve R. We provide examples of $A \in Gl(n)$ which preserve R but not R_{φ_1} or R_{φ_2} in a special case and state a conjecture that would explain when such A exist.

2.2 Introduction

The structure group of an algebraic curvature tensor $R \in \mathcal{A}(V)$, denoted G_R, is the group of elements $A \in Gl(n)$ such that

$$R(x, y, z, w) = R(Ax, Ay, Az, Aw)\ \forall x,\ y,\ z, \text{ and } w \in V.$$

G_R is always a Lie group. The dimension of G_R for any R in dimension 3 is known [6], but this was determined by studying the Lie algebras rather than the groups themselves, so the general forms of the elements were not previously known.

Here we examine the special case G_R where $R = R_{\varphi_1} \pm R_{\varphi_2}$ for $\varphi_1 = diag(0, 1, \lambda_2)$ and $\varphi_2 = diag(1, 0, \lambda_1)$ where λ_1 and λ_2 are nonzero. Note that $-R_{\varphi_2} = R_{\bar{\varphi}_2}$ where $R_{\bar{\varphi}_2} = diag(1, 0, -\lambda_1)$, so it suffices to study $R = R_{\varphi_1} + R_{\varphi_2}$.

We chose this example because it was shown in [2] that such an R has $\nu(R) = 2$, eliminating the possibility that $R_{\varphi_1} + R_{\varphi_2} = R_\varphi$ for some φ. Additionally, work in the previous section showed that $R_{\varphi_1} + R_{\varphi_2} = R_{\tau_1} + R_{\tau_2} = R_{\psi_1} - R_{\psi_2}$ where $Rank(\tau_i) = Rank(\psi_i) = 3$. Therefore, this example also provides insight into the structure groups of sums and differences of rank 3 forms. While one might expect that $G_{R_A+R_B}$ and $G_{R_C-R_D}$ are different for arbitrary rank 3 forms A, B, C, and D, they must be equal in the special case of $G_{R_{\tau_1}+R_{\tau_2}}$ and $G_{R_{\psi_1}-R_{\psi_2}}$. Understanding when this equality does and does not occur could be helpful in proving or disproving the revisions to the signature conjecture in section 1.5.

Throughout, $\{e_1, \ldots, e_n\}$ denotes an orthonormal basis for V. For any $x \in V$, x_i denotes the real number such that $x = \sum_{i=1}^n x_i e_i$. Finally, $\varphi_1 = diag(0, 1, \lambda_2)$ and $\varphi_2 = diag(1, 0, \lambda_1)$.

2.3 Elements of $G_{R_{\varphi_1} \pm R_{\varphi_2}}$

Theorem 2.1. $G_{R_{\varphi_1}} \cap G_{R_{\varphi_2}} \subsetneq G_{R_{\varphi_1} \pm R_{\varphi_2}}$

Proof. We can calculate, for some arbitrary $x, y, z, w \in V$,

$$R(x, y, z, w) = \lambda_1(x_1y_3 - x_3y_1)(z_3w_1 - z_1w_3) + \lambda_2(x_2y_3 - x_3y_2)(z_3w_2 - z_2w_3)$$

A matrix $A \in Gl(3)$ is in the structure group G_R if

$$R(Ax, Ay, Az, Aw) = R(x, y, z, w)$$

For an arbitrary 3×3 matrix A,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

and

$$Ax = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{bmatrix}.$$

We can directly compute $R(Ax, Ay, Az, Aw)$ to show that A is in G_R iff the elements of A satisfy the following system of equations:

$$
\begin{aligned}
\lambda_1(a_{11}a_{32}-a_{31}a_{12})^2+\lambda_2(a_{21}a_{32}-a_{31}a_{22})^2 &= 0\\
\lambda_1(a_{11}a_{32}-a_{31}a_{12})(a_{11}a_{33}-a_{31}a_{13})+\lambda_2(a_{21}a_{32}-a_{31}a_{22})(a_{21}a_{33}-a_{31}a_{23}) &= 0\\
\lambda_1(a_{11}a_{32}-a_{31}a_{12})(a_{12}a_{33}-a_{32}a_{13})+\lambda_2(a_{21}a_{32}-a_{31}a_{22})(a_{22}a_{33}-a_{32}a_{23}) &= 0\\
\lambda_1(a_{11}a_{33}-a_{31}a_{13})(a_{11}a_{32}-a_{31}a_{12})+\lambda_2(a_{21}a_{33}-a_{31}a_{23})(a_{21}a_{32}-a_{31}a_{22}) &= 0\\
\lambda_1(a_{11}a_{33}-a_{31}a_{13})^2+\lambda_2(a_{21}a_{33}-a_{31}a_{23})^2 &= \lambda_1\\
\lambda_1(a_{11}a_{33}-a_{31}a_{13})(a_{12}a_{33}-a_{32}a_{13})+\lambda_2(a_{21}a_{33}-a_{31}a_{23})(a_{22}a_{33}-a_{32}a_{23}) &= 0\\
\lambda_1(a_{12}a_{33}-a_{32}a_{13})(a_{11}a_{32}-a_{31}a_{12})+\lambda_2(a_{22}a_{33}-a_{32}a_{23})(a_{21}a_{32}-a_{31}a_{22}) &= 0\\
\lambda_1(a_{12}a_{33}-a_{32}a_{13})(a_{11}a_{33}-a_{31}a_{13})+\lambda_2(a_{22}a_{33}-a_{32}a_{23})(a_{21}a_{33}-a_{31}a_{23}) &= 0\\
\lambda_1(a_{12}a_{33}-a_{32}a_{13})^2+\lambda_2(a_{22}a_{33}-a_{32}a_{23})^2 &= \lambda_2
\end{aligned}
$$

When λ_1 and λ_2 have the same sign, some solutions are

$$
A=\begin{bmatrix} a & \sqrt{\frac{\lambda_2}{\lambda_1}(1-a^2)} & 0\\ -\sqrt{\frac{\lambda_1}{\lambda_2}(1-a^2)} & a & 0\\ 0 & 0 & 1\end{bmatrix} \text{ for } a\in[0,1] \text{ and}
$$

$$
A=\begin{bmatrix} a & \sqrt{\frac{\lambda_2}{\lambda_1}(1-a^2)} & 0\\ \sqrt{\frac{\lambda_1}{\lambda_2}(1-a^2)} & -a & 0\\ 0 & 0 & 1\end{bmatrix} \text{ for } a\in[0,1].
$$

When the signs of λ_1 and λ_2 are different, assume without loss of generality that $\lambda_1>0$ and $\lambda_2<0$. Then some solutions are

$$
A=\begin{bmatrix} a & \sqrt{\frac{|\lambda_2|}{\lambda_1}(a^2-1)} & 0\\ \sqrt{\frac{\lambda_1}{|\lambda_2|}(a^2-1)} & a & 0\\ 0 & 0 & 1\end{bmatrix} \text{ for } a\geq 1 \text{ and}
$$

$$
A=\begin{bmatrix} a & -\sqrt{\frac{|\lambda_2|}{\lambda_1}(a^2-1)} & 0\\ \sqrt{\frac{\lambda_1}{|\lambda_2|}(a^2-1)} & -a & 0\\ 0 & 0 & 1\end{bmatrix} \text{ for } a\geq 1.
$$

For any of these solutions, whenever $a\neq 1$, the solution is not in $G_{R_{\varphi_1}}$ or $G_{R_{\varphi_2}}$. $\square$

Note that in either case, there is a continuous path of solutions in the component of the identity and a continuous path not in the component of the identity. Note also that any of the elements above where $a \neq 1$ combine a space-like or time-like basis vector with a light-like one.

2.4 Future Work

1. In general, do all of the elements of $G_{R_{\varphi_1}+R_{\varphi_2}}$ which do not preserve R_{φ_1} and R_{φ_2} map basis vectors of one type to a linear combination of themselves with those of another type, e.g. space-like with light-like, in φ_1 and φ_2 respectively?

2. Does this process work using any two basis vectors of different types, or does it require the use of light-like basis vectors?

3. What is the structure group of $R = R_{\varphi_1} + R_{\varphi_2} + R_{\varphi_3}$, and how does it compare with $G_{R_{\varphi_i}}$?

4. How do the structure groups of $R_A + R_B$ and $R_C - R_D$ differ when the ranks of A, B, C, and D are at least 3, so that $-R_D \neq R_{\bar{D}}$ for any $\bar{D}$?

3 Linear Dependence Relationships

Throughout the course of this project, a number of linear dependence relationships arose. They are documented in this section so they can be conveniently located for use in future projects.

1. Given symmetric bilinear forms φ and ψ with signatures $(p, q+1, k)$ and $(p+1, q, k)$ respectively, $R_\varphi - R_\psi = R_\tau$ for some τ with signature $(p, q, k+1)$ if φ and ψ are simultaneously diagonalizable,

$$\varphi = diag(\underbrace{0, \ldots, 0}_{k}, a, \underbrace{-b, \ldots, -b}_{p}, \underbrace{b, \ldots, b}_{q}), \text{ and}$$
$$\psi = diag(\underbrace{0, \ldots, 0}_{k}, b, \underbrace{-a, \ldots, -a}_{p}, \underbrace{a, \ldots, a}_{q}).$$

2. $R_{\varphi_1} + R_{\varphi_2} = R_{\tau_1} + R_{\tau_2}$ when, for some nonzero λ,

$$\varphi_1 = diag(\underbrace{0, \ldots, 0}_{n-2}, 1, \lambda),$$

$$\varphi_2 = diag(\underbrace{0, \ldots, 0}_{n-3}, 1, 0, \lambda),$$

$$\tau_1 = diag\Big(\underbrace{0, \ldots, 0}_{n-3}, \frac{1}{\sqrt{3}}, \sqrt{3}, \frac{3\lambda}{2\sqrt{3}}\Big), \text{ and}$$

$$\tau_2 = diag\Big(\underbrace{0, \ldots, 0}_{n-3}, 1, -1, \frac{\lambda}{2}\Big).$$

3. $R_{\varphi_1} + R_{\varphi_2} = R_{\tau_1} + R_{\tau_2}$ when, for some nonzero λ,

$$\varphi_1 = diag(\underbrace{0, \ldots, 0}_{n-2}, 1, -\lambda),$$

$$\varphi_2 = diag(\underbrace{0, \ldots, 0}_{n-3}, 1, 0, \lambda),$$

$$\tau_1 = diag\Big(\underbrace{0, \ldots, 0}_{n-3}, \frac{1}{\sqrt{3}}, -\sqrt{3}, \frac{3\lambda}{2\sqrt{3}}\Big), \text{ and}$$

$$\tau_2 = diag\Big(\underbrace{0, \ldots, 0}_{n-3}, 1, 1, \frac{\lambda}{2}\Big).$$

4. $R_{\varphi_1} + R_{\varphi_2} = R_{\tau_1} + R_{\tau_2}$ when, for some nonzero λ_1 and λ_2 such that $|\lambda_1| \neq |\lambda_2|$,

$$\varphi_1 = diag(\underbrace{0, \ldots, 0}_{n-2}, 1, \lambda_2),$$

$$\varphi_2 = diag(\underbrace{0, \ldots, 0}_{n-3}, 1, 0, \lambda_1),$$

$$\tau_1 = diag\Big(\underbrace{0, \ldots, 0}_{n-3}, \sqrt{2}, \sqrt{2}, \frac{\lambda_1 + \lambda_2}{\sqrt{8}}\Big), \text{ and}$$

$$\tau_2 = diag\Big(\underbrace{0, \ldots, 0}_{n-3}, -\sqrt{2}, \sqrt{2}, \frac{\lambda_1 - \lambda_2}{\sqrt{8}}\Big).$$

5. $R_{\varphi_1} + R_{\varphi_2} = R_{\psi_1} - R_{\psi_2}$ when, for any real numbers λ_1 and λ_2,

$$\varphi_1 = diag(\underbrace{0, \ldots, 0}_{n-2}, 1, \lambda_2),$$
$$\varphi_2 = diag(\underbrace{0, \ldots, 0}_{n-3}, 1, 0, \lambda_1),$$
$$\psi_1 = diag(\underbrace{0, \ldots, 0}_{n-3}, \lambda_1, \lambda_2, 2), \text{ and}$$
$$\psi_2 = diag(\underbrace{0, \ldots, 0}_{n-3}, \lambda_1, \lambda_2, 1).$$

6. $R_{\varphi_1} + R_{\varphi_2} + R_{\varphi_3} = R_\tau$ when, for any real number λ,

$$\varphi_1 = diag(\underbrace{0, \ldots, 0}_{n-3}, -1, 1, \frac{\lambda}{2}),$$
$$\varphi_2 = diag(\underbrace{0, \ldots, 0}_{n-3}, 1, 1, -\frac{\lambda}{2}),$$
$$\varphi_3 = diag(\underbrace{0, \ldots, 0}_{n-3}, 1, -1, -\frac{\lambda}{2}), \text{ and}$$
$$\tau = diag(\underbrace{0, \ldots, 0}_{n-3}, \frac{1}{\sqrt{3}}, -\sqrt{3}, \frac{\sqrt{3}\lambda}{2}).$$

7. $R_{\varphi_1} + R_{\varphi_2} = R_\tau$ when, for any real number λ,

$$\varphi_1 = diag(\underbrace{0, \ldots, 0}_{n-3}, \sqrt{2}, \sqrt{2}, \frac{\lambda}{\sqrt{2}}),$$
$$\varphi_2 = diag(\underbrace{0, \ldots, 0}_{n-3}, -1, 1, \frac{\lambda}{2}),$$
$$\tau = diag(\underbrace{0, \ldots, 0}_{n-3}, \frac{1}{\sqrt{3}}, \sqrt{3}, \frac{\sqrt{3}\lambda}{2}).$$

8. $R_{\varphi_1} + R_{\varphi_2} = R_\tau$ when, for any real numbers λ_1 and λ_2,

$$\varphi_1 = diag(\underbrace{0, \ldots, 0}_{n-3}, 1, 1, \frac{\lambda_1}{2}),$$

$$\varphi_2 = diag(\underbrace{0, \ldots, 0}_{n-3}, 1, 1, \frac{\lambda_2}{2}),$$

$$\tau = diag(\underbrace{0, \ldots, 0}_{n-3}, \sqrt{2}, \sqrt{2}, \frac{\lambda_1 + \lambda_2}{2}).$$

9. $R_{\varphi_1} + R_{\varphi_2} = R_\tau$ when, for any real numbers λ_1 and λ_2,

$$\varphi_1 = diag(\underbrace{0, \ldots, 0}_{n-3}, -1, 1, \frac{\lambda_1}{2}),$$

$$\varphi_2 = diag(\underbrace{0, \ldots, 0}_{n-3}, 1, -1, \frac{\lambda_2}{2}),$$

$$\tau = diag(\underbrace{0, \ldots, 0}_{n-3}, -\sqrt{2}, \sqrt{2}, \frac{\lambda_1 - \lambda_2}{2}).$$

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