Linear Independence of Algebraic Curvature Tensors in the Higher-Signature Setting

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Abstract

We examine the conditions under which linear independence of three canonical algebraic curvatures tensors occurs in a vector space of dimension 3. Previous studies have completely characterized these conditions when we can assume that one of the three tensors is defined by a positive definite inner product. In this paper, we aim to extend those results by assuming that the form is only known to be non-degenerate. Within this setting, we consider the particular case where one of the other two tensors has an associated endomorphism with Jordan type $J(-,3)$. We show that, in these circumstances, linear independence occurs if and only if certain conditions are met, and we specify those conditions.

1 Introduction

A *manifold* is a topological space which is Hausdorff and locally Euclidean. Each point p on a given manifold M is associated with a *tangent space* consisting of all vectors tangent to M at the point p. Using the tools of differential geometry, we are able to describe the "curvature" of a given manifold by examining the tangent space of each point on the manifold. But in order to do so, we must first choose a means by which we can define length of, and angles between, tangent vectors at a given point on the manifold. This is accomplished by using a certain type of *symmetric bilinear form*.

Definition 1.1. A **symmetric bilinear form** on a vector space $V \subseteq \mathbb{R}^n$ is a function

$$\alpha : V \times V \to \mathbb{R}$$

which satisfies the following properties:

(i) Symmetry: $\alpha(v, w) = \alpha(w, v)$ for all $v, w \in V$, and

(ii) Bilinearity: $\alpha(bv + cw, z) = b\alpha(v, z) + c\alpha(w, z)$ for all $v, w, z \in V$ and $b, c \in \mathbb{R}$. Note that linearity in the second slot follows from symmetry.

We write $\alpha \in S^2(V)$ to denote that α is a symmetric bilinear form on the vector space V. We say that a symmetric bilinear form α is *non-degenerate* if for all non-zero $v \in V$ there exists some $w \in V$ such that $\alpha(v, w) \neq 0$. We say that a symmetric bilinear form is *positive-definite* if $\alpha(v, v) \geq 0$ for all $v \in V$, and $\alpha(v, v) = 0$ if and only if $v = 0$. A positive-definite symmetric bilinear form is known as an *inner product*. Note that non-degeneracy is a weaker property than positive-definiteness, and thus every inner product is also non-degenerate.

For a given symmetric bilinear form α and some basis $\{e_1, e_2, e_3\}$ for V, let D_α denote the matrix whose (i, j) entry is $\alpha(e_i, e_j)$.

For each point p on a given manifold M, we can choose an inner product for the tangent space of p, and these choices of inner products are collectively known as a *metric* for M. This metric allows us to describe the curvature of M at any given point. The curvature of the manifold M at a point $p \in M$ is given by an *algebraic curvature tensor*:

Definition 1.2. For a given vector space $V \subseteq \mathbb{R}^n$, an **algebraic curvature tensor** is a function of the form

$$R : V \times V \times V \times V \to \mathbb{R}$$

which satisfies the following properties for all $x, y, z, w \in V$:

(i) Multilinearity: $R(ax + bx', y, z, w) = aR(x, y, z, w) + bR(x', y, z, w)$ for all $x' \in V$ and $a, b \in \mathbb{R}$. Linearity is similar for the second, third, and fourth slots,

(ii) $R(x, y, z, w) = -R(y, x, z, w)$,

(iii) $R(x, y, z, w) = R(z, w, x, y)$, and

(iv) The Bianchi Identity: $R(x, y, z, w) + R(z, x, y, w) + R(y, z, x, w) = 0$.

Let $\mathcal{A}(V)$ denote the vector space of all algebraic curvature tensors on V.

Definition 1.3. A **model space** is a triple $\mathcal{M} = \{V, \varphi, R\}$ where V is a vector space, φ is symmetric bilinear form on V, and R is an algebraic curvature tensor.

Given a manifold M and a metric g on M, we can construct a model space for any point $p \in M$ from the tangent space at p, the metric at p, and curvature tensor at p. Studying the curvature of M at a point p then amounts to studying the curvature given by the corresponding model space. This paper takes the approach of studying these model spaces, rather than a manifold more generally. The following is of use in such an endeavor:

Definition 1.4. If φ is a symmetric bilinear form on a vector space $V \subseteq \mathbb{R}^n$, then a **canonical algebraic curvature tensor** with respect to φ is the function $R_\varphi : V \times V \times V \times V \to \mathbb{R}$ defined by

$$R_\varphi(v_1, v_2, v_3, v_4) = \varphi(v_1, v_4)\varphi(v_2, v_3) - \varphi(v_1, v_3)\varphi(v_2, v_4).$$

We note that for any symmetric bilinear form α and $c \in \mathbb{R}$, the following properties follow from the definition of canonical algebraic curvature tensor (see [5] for the computations):

$$R_\alpha = R_{-\alpha}, \qquad cR_\alpha = R_{\sqrt{c}\alpha}.$$

From these properties, we see that

$$cR_\alpha = \epsilon R_{\sqrt{|c|}\alpha}, \tag{1}$$

where $\epsilon = \text{sign}(c) = \pm 1$.

We also note that for any $\alpha \in S^2(V)$ we have $R_\alpha \in \mathcal{A}(V)$ [3]. Moreover, since $\{R_\alpha \mid \alpha \in S^2(V)\}$ is a spanning set for $\mathcal{A}(V)$, we can attempt to efficiently express any algebraic curvature tensor as a linear combination of the form $R = \sum c_i R_\alpha$. For this reason, studying linear combinations of canonical algebraic curvature tensors can help us better understand the structure of $\mathcal{A}(V)$, which in turn can help in describing the curvature given by a model space.

2 Linear Independence

One property of $\mathcal{A}(V)$ which has been studied extensively concerns linear independence of a set $\{R_{\alpha_i}\}$ of canonical algebraic curvature tensors. The aim of this paper is to extend those studies. In particular, we will consider sets of three canonical algebraic curvature tensors, in a vector space V where $\dim(V) = 3$.

We say that three canonical algebraic curvature tensors $R_{\tilde{\varphi}}$, $R_{\tilde{\psi}}$, and $R_{\tilde{\tau}}$, are **linearly dependent** if there exist $c_1, c_2, c_3 \in \mathbb{R}$ such that

$$c_1 R_{\tilde{\varphi}} + c_2 R_{\tilde{\psi}} + c_3 R_{\tilde{\tau}} = 0 \tag{2}$$

where at least one of c_1, c_2, or c_3 is non-zero. We can divide this problem into three cases:

1. Exactly one of c_1, c_2, or c_3 is non-zero.
2. Exactly two of c_1, c_2, or c_3 are non-zero.
3. Each of c_1, c_2, and c_3 is non-zero.

We note that the first two cases are equivalent to a proper subset of $\{R_{\tilde{\varphi}}, R_{\tilde{\psi}}, R_{\tilde{\tau}}\}$ being linearly dependent, which has been studied previously (see [1]). This paper will therefore focus on the third case, which we refer to as **proper linear dependence**.

Previous studies on this question have started from the assumption that $\tilde{\varphi}$ is positive-definite, and have completely determined the conditions under which linear dependence occurs, given that assumption. For example, Ye showed in [5] that if $\tilde{\varphi}$ is positive-definite, and some specific eigenvalue relationships hold, then the set $\{R_{\tilde{\varphi}}, R_{\tilde{\psi}}, R_{\tilde{\tau}}\}$ is properly linearly dependent if and only if $\tilde{\psi}$ and $\tilde{\tau}$ are simultaneously diagonalizable with respect to $\tilde{\varphi}$.

In this paper, we aim to generalize this line of inquiry, assuming only that $\tilde{\varphi}$ is non-degenerate. We refer to such symmetric bilinear forms, which are neither positive-definite nor negative-definite, as *higher signature* forms.

3 Some Simplifications and Matrices

Let us begin by simpifying Equation 2. Divide both sides of Equation 2 by c_1 and let $\varphi = \tilde{\varphi}$, $\psi = \sqrt{|\frac{c_2}{c_1}|}\tilde{\psi}$, and $\tau = \sqrt{|\frac{c_3}{c_1}|}\tilde{\tau}$. Then by Equation 1 we see that Equation 2 reduces to

$$R_\varphi + \epsilon R_\psi = \delta R_\tau \tag{3}$$

where $\epsilon = \operatorname{sign}(c_2/c_1)$ and $\delta = \operatorname{sign}(c_3/c_1)$. Our problem now reduces to analyzing Equation 3. Let us now introduce some concepts which will help in this.

Definition 3.1. Suppose $A : V \to V$ is a linear transformation on a vector space $V \subseteq \mathbb{R}^n$. Then the **adjoint** of A with respect to a symmetric bilinear form α is the linear transformation $A^* : V \to V$ satisfying

$$\alpha(Av, w) = \alpha(v, A^* w)$$

for all $v, w \in V$. If $A = A^*$, then we say that A is **self-adjoint**.

Lemma 1. *If $\psi, \tau \in S^2(V)$, then there exist linear maps $\Psi : V \to V$ and $T : V \to V$, self-adjoint with respect to φ, such that*

$$\psi(v, w) = \varphi(\Psi v, w) \quad \textit{and} \quad \tau(v, w) = \varphi(Tv, w)$$

for all $v, w \in V$.

We refer to the linear transformations Ψ and T as the *associated endomorphisms* for ψ and τ, respectively. Much of what follows in this paper will concern the matrices corresponding to Ψ and T. In particular, we will need the following concepts:

Definition 3.2. A **sip matrix** is a square matrix whose entries are all zeroes except for ones on the main skew diagonal. We write sip_k to denote a $k \times k$ sip matrix.

Definition 3.3. The **Jordan normal form** of a matrix A is a block diagonal matrix where each block is one of the two forms

$$\begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & \lambda \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} B & I & & & \\ & V & I & & \\ & & \ddots & \ddots & \\ & & & \ddots & I \\ & & & & B \end{pmatrix}$$

where all blank entries are zero, $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and $B = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, for real eigenvalues λ and complex eigenvalues $a + ib$ of A.

We will also need the following theorem from linear algebra:

Theorem 3.1. *For every square matrix A, there exists an invertible matrix P such that $P^{-1}AP = J$ is in Jordan normal form. In other words, every square matrix is similar to a direct sum of Jordan blocks.*

Now, it is established in [4] that there exists a basis $\mathcal{F} = \{f_1, f_2, f_2\}$ with respect to which Ψ is in Jordan normal form, and with respect to which D_φ is a corresponding direct sum of sip matrices. We note that in a vector space of dimension 3 there are four possible types of Jordan normal form for Ψ:

$$\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} \qquad \begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix} \qquad \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \qquad \begin{pmatrix} a & b & 0 \\ -b & a & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

In this paper, we will assume that Ψ has the first of these Jordan types. So we have

$$[\Psi]_{\mathcal{F}} = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} \qquad \text{and} \qquad [D_\varphi]_{\mathcal{F}} = \begin{pmatrix} 0 & 0 & \varepsilon \\ 0 & \varepsilon & 0 \\ \varepsilon & 0 & 0 \end{pmatrix}$$

Symbolically, this is denoted $[\Psi]_{\mathcal{F}} = J(\lambda, 3)$ and $[D_\varphi]_{\mathcal{F}} = \pm\,\mathrm{sip}_3$. We note, though, that for any symmetric bilinear form α we have $R_\alpha = R_{-\alpha}$. Therefore, we can assume without loss of generality that $\varepsilon = 1$, giving us

$$[D_\varphi]_{\mathcal{F}} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Let us now also specify the matrix of the associated endomorphism T. Let us denote the entries of the matrix for $[T]_{\mathcal{F}}$ as follows:

$$[T]_{\mathcal{F}} = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix}$$

Now, since T is self-adjoint with respect to φ, we see that $\tau(f_1, f_2) = \varphi(Tf_1, f_2) = \varphi(f_1, Tf_2)$. So we have:

$$\begin{aligned}\varphi(Tf_1, f_2) &= \varphi(f_1, Tf_2)\\ \varphi(T_{11}f_1 + T_{21}f_2 + T_{31}f_3, f_2) &= \varphi(f_1, T_{12}f_1 + T_{22}f_2 + T_{32}f_3).\end{aligned}$$

It then follows from the bilinearity of φ that

$$T_{11}\varphi(f_1, f_2)+T_{21}\varphi(f_2, f_2)+T_{31}\varphi(f_3, f_2) = T_{12}\varphi(f_1, f_1)+T_{22}\varphi(f_1, f_2)+T_{32}\varphi(f_1, f_3).$$

By inspection of the matrix $[D_\varphi]_{\mathcal{F}}$, this becomes

$$T_{11}(0) + T_{21}(1) + T_{31}(0) = T_{12}(0) + T_{22}(0) + T_{32}(1)$$

which gives us

$$T_{21} = T_{32}.$$

Computations for other choices of two basis vector inputs are similar, and the result is that our matrix now has the simpler form

$$[T]_{\mathcal{F}} = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{12} \\ T_{31} & T_{21} & T_{11} \end{pmatrix}.$$

4 The System of Equations

Having specified the matrices of interest, let us now begin relating Equation 3 to the contents of these matrices.

For an arbitrary canonical algebraic curvature tensor R_α on V, let $R_{\alpha(ijkl)}$ denote $R_\alpha(f_i, f_j, f_k, f_l)$. It follows from the properties of algebraic curvature tensors that many choices of i, j, k, l yield equivalent canonical ACTs. In fact, there are exactly six distinct, non-zero canonical ACTs, and we will represent them by the following choices of i, j, k, l:

$$R_{\alpha(1221)}, R_{\alpha(1331)}, R_{\alpha(2332)}, R_{\alpha(1231)}, R_{\alpha(2132)}, R_{\alpha(3123)}$$

We start by evaluating Equation 3 for the first of these inputs:

$$R_{\varphi(1221)} + \epsilon R_{\psi(1221)} = \delta R_{\tau(1221)} \tag{4}$$

Expanding Equation 4 gives

$$\begin{aligned}&\varphi(f_1, f_1)\varphi(f_2, f_2) - \varphi(f_1, f_2)\varphi(f_2, f_1)\\ +&\epsilon[\psi(f_1, f_1)\psi(f_2, f_2) - \psi(f_1, f_2)\psi(f_2, f_1)] = \delta[\tau(f_1, f_1)\tau(f_2, f_2) - \tau(f_1, f_2)\tau(f_2, f_1)]\end{aligned}$$

By inspection of the matrix $[D_\varphi]_\mathcal{F}$, we see that $\varphi(f_1, f_1) = \varphi(f_1, f_2) = 0$, so the above equation becomes

$$\epsilon[\psi(f_1, f_1)\psi(f_2, f_2) - \psi(f_1, f_2)\psi(f_2, f_1)] = \delta[\tau(f_1, f_1)\tau(f_2, f_2) - \tau(f_1, f_2)\tau(f_2, f_1)]$$

By Lemma 1 this gives us

$$\epsilon[\varphi(\Psi f_1, f_1)\varphi(\Psi f_2, f_2) - \varphi(\Psi f_1, f_2)\varphi(\Psi f_2, f_1)] = \delta[\varphi(T f_1, f_1)\varphi(T f_2, f_2) - \varphi(T f_1, f_2)\varphi(T f_2, f_1)]$$

Now, by inspection of the matrices $[\Psi]_\mathcal{F}$ and $[T]_\mathcal{F}$ we see that

$$\begin{aligned}\Psi f_1 &= \lambda f_1\\ \Psi f_2 &= f_1 + \lambda f_2\\ \Psi f_3 &= f_2 + \lambda f_3\end{aligned}$$

So the left-hand side of our equation becomes

$$\epsilon[\varphi(\lambda f_1, f_1)\varphi(f_1 + \lambda f_2, f_2) - \varphi(\lambda f_1, f_2)\varphi(f_1 + \lambda f_2, f_1)]$$

And since φ is bilinear, this becomes

$$\epsilon[\lambda\varphi(f_1, f_1)[\varphi(f_1, f_2) + \lambda\varphi(f_2, f_2)] - \lambda\varphi(f_1, f_2)[\varphi(f_1, f_1) + \lambda\varphi(f_2, f_1)]]$$

And, once again, since $\varphi(f_1, f_1) = \varphi(f_1, f_2) = 0$, this evaluates to zero.

Next, since

$$\begin{aligned}T f_1 &= T_{11} f_1 + T_{21} f_2 + T_{31} f_3\\ T f_2 &= T_{12} f_1 + T_{22} f_2 + T_{21} f_3\\ T f_3 &= T_{13} f_1 + T_{12} f_2 + T_{11} f_3\end{aligned}$$

we see that the right-hand side of our equation becomes

$$\delta[\varphi(T_{11}f_1 + T_{21}f_2 + T_{31}f_3, f_1)\varphi(T_{12}f_1 + T_{22}f_2 + T_{21}f_3, f_2)$$
$$-\varphi(T_{11}f_1 + T_{21}f_2 + T_{31}f_3, f_2)\varphi(T_{12}f_1 + T_{22}f_2 + T_{21}f_3, f_1)]$$

And by the bilinearity of φ, this becomes

$$[T_{11}\varphi(f_1, f_1) + T_{21}\varphi(f_2, f_1) + T_{31}\varphi(f_3, f_1)][T_{12}\varphi(f_1, f_2) + T_{22}\varphi(f_2, f_2) + T_{21}\varphi(f_3, f_2)]$$
$$-[T_{11}\varphi(f_1, f_2) + T_{21}\varphi(f_2, f_2) + T_{31}\varphi(f_3, f_2)][T_{12}\varphi(f_1, f_1) + T_{22}\varphi(f_2, f_1) + T_{21}\varphi(f_3, f_1)]$$

By inspection of the matrix $[D_\varphi]_\mathcal{F}$, this becomes

$$[T_{11}(0) + T_{21}(0) + T_{31}(1)][T_{12}(0) + T_{22}(1) + T_{21}(0)]$$
$$-[T_{11}(0) + T_{21}(1) + T_{31}(0)][T_{12}(0) + T_{22}(0) + T_{21}(1)]$$

which reduces to $T_{31}T_{22} - T_{21}^2$. So, we have shown that Equation 4 is equivalent to

$$0 = T_{31}T_{22} - T_{21}^2$$

We can similarly compute the results for the other five possible inputs; the results of these computations are as follows:

$$R_{\varphi(1221)} + \epsilon R_{\psi(1221)} = \delta R_{\tau(1221)} \iff 0 = T_{31}T_{22} - T_{21}^2 \tag{5}$$

$$R_{\varphi(1331)} + \epsilon R_{\psi(1331)} = \delta R_{\tau(1331)} \iff -1 - \epsilon\lambda^2 = \delta(T_{31}T_{13} - T_{11}^2) \tag{6}$$

$$R_{\varphi(2332)} + \epsilon R_{\psi(2332)} = \delta R_{\tau(2332)} \iff -\epsilon = \delta(T_{22}T_{13} - T_{12}^2) \tag{7}$$

$$R_{\varphi(1231)} + \epsilon R_{\psi(1231)} = \delta R_{\tau(1231)} \iff 0 = T_{12}T_{31} - T_{21}T_{11} \tag{8}$$

$$R_{\varphi(2132)} + \epsilon R_{\psi(2132)} = \delta R_{\tau(2132)} \iff 1 + \epsilon\lambda^2 = \delta(T_{11}T_{22} - T_{21}T_{12}) \tag{9}$$

$$R_{\varphi(3123)} + \epsilon R_{\psi(3123)} = \delta R_{\tau(3123)} \iff -\epsilon\lambda = \delta(T_{21}T_{13} - T_{11}T_{12}) \tag{10}$$

From the above system of equations, the following is immediate:

Theorem 4.1. *If $[\Psi]_\mathcal{F} = J(\lambda, 3)$, $[D_\varphi]_\mathcal{F} = \pm sip_3$, and $[T]_\mathcal{F} = J(\eta, 3)$, then there does not exist a solution to $R_\varphi + \epsilon R_\psi = \delta R_\tau$ for any $\epsilon, \delta \in \{\pm 1\}$.*

Proof. Assume that $[\Psi]_{\mathcal{F}} = J(\lambda, 3)$, $[D_\varphi]_{\mathcal{F}} = \pm\mathrm{sip}_3$, and $[T]_{\mathcal{F}} = J(\eta, 3)$. Assume towards a contradiction that there exists a solution to the equation $R_\varphi + \epsilon R_\psi = \delta R_\tau$. We note that there exists a solution to this equation if and only if Equations 5 - 10 all hold. But since $[T]_{\mathcal{F}} = J(\eta, 3)$, we see that Equation 7 reduces to $-\epsilon = -\delta$, and thus we have $\epsilon = \delta$. Similarly, Equation 10 reduces to $-\epsilon\lambda = -\delta\eta$, so we have $\lambda = \eta$. But we also note that Equation 9 reduces to $1 + \epsilon\lambda^2 = \delta\eta^2$. So by substitution this last equation becomes $1 = 0$, and we have a contradiction.

□

Let us now note that we can partition the possible forms of $\tilde{T}$ and $\tilde{\Psi}$ as follows:

- $\tilde{T}$ is not invertible
- $\tilde{T}$ and $\tilde{\Psi}$ are both invertible
- $\tilde{T}$ is invertible and $\tilde{\Psi}$ is not invertible

The next three sections of this paper look at these three respective cases.

5 If $\tilde{T}$ is not invertible

In this section we will make use of the following:

Definition 5.1. For some algebraic curvature tensor R, let the **kernel** of R, denoted $\ker(R)$, be defined as

$$\ker(R) = \{v \in V \mid R(v, x, y, z) = 0 \text{ for all } x, y, z \in V\}$$

We note that the kernel of an algebraic curvature tensor is not biased towards the first argument of (x, y, z, w) (see [2]).

The following Lemma is established in [3]:

Lemma 2. *Let α be a symmetric bilinear form and $R_\alpha \in \mathcal{A}(V)$. Then* $\operatorname{Rank} R_\alpha = \operatorname{Rank} \alpha$.

We may now proceed to our proof:

Theorem 5.1. *If $[\tilde{\Psi}]_{\mathcal{F}}$ is a constant multiple of $J(\lambda, 3)$, $[D_{\tilde{\varphi}}]_{\mathcal{F}} = \pm sip_3$, and $\tilde{T}$ is not invertible, then the set $\{R_{\tilde{\varphi}}, R_{\tilde{\psi}}, R_{\tilde{\tau}}\}$ is not properly linearly dependent.*

Proof. Assume that $[\tilde{\Psi}]_{\mathcal{F}}$ is a constant multiple of $J(\lambda, 3)$, $[D_{\tilde{\varphi}}]_{\mathcal{F}} = \pm \mathrm{sip}_3$, and $\tilde{T}$ is not invertible. Since $\tilde{T} = \sqrt{\left|\frac{c_1}{c_2}\right|}T$, we see that the assumption of $\tilde{T}$ not being invertible is equivalent to T not being invertible. And we note that this is equivalent to $\ker(T) \neq 0$. And since $\ker(T) \neq 0$, and $\dim(V) = 3$, we see that $\ker(T)$ must have dimension 1, 2, or 3.

We note that if $\dim(\ker(T)) = 2$ or $\dim(\ker(T)) = 3$, then it follows that $R_\tau = 0$, in which case the set $\{R_{\tilde{\varphi}}, R_{\tilde{\psi}}\}$ is linearly dependent, and thus the set $\{R_{\tilde{\varphi}}, R_{\tilde{\psi}}, R_{\tilde{\tau}}\}$ is not properly linearly dependent (see [2]).

So, we are then left with the case where $\dim(\ker(T)) = 1$. We will now show that this case cannot occur, since it leads to a contradiction.

Suppose $\dim(\ker(T)) = 1$. We then have $\dim(\ker(R_\tau)) = 1$, and thus $\ker(R_\varphi + \epsilon R_\psi)$ also has dimension 1. So let $v = af_1 + bf_2 + cf_3 \in \ker(R_\varphi + \epsilon R_\psi)$ where $v \neq 0$. By the definition of kernel, we have:

$$\begin{aligned}
0 &= R_\tau(v, f_3, f_3, f_2) \\
0 &= R_\tau(af_1 + bf_2 + cf_3, f_3, f_3, f_2) \\
0 &= aR_{\tau(1332)} + bR_{\tau(2332)} + cR_{\tau(3332)} \\
0 &= -a\epsilon\delta\lambda - b\epsilon\delta
\end{aligned}$$

which gives us

$$b = -a\lambda \tag{11}$$

Similarly, we also have

$$\begin{aligned}
0 &= R_\tau(v, f_2, f_2, f_3) \\
0 &= R_\tau(af_1 + bf_2 + cf_3, f_2, f_2, f_3) \\
0 &= aR_{\tau(1223)} + bR_{\tau(2223)} + cR_{\tau(3223)} \\
0 &= a\delta(1 + \epsilon\lambda^2) - c\epsilon\delta
\end{aligned}$$

which gives us

$$c = \frac{a(1+\epsilon\lambda^2)}{\epsilon} \tag{12}$$

Finally, we also have

$$\begin{aligned}
0 &= R_\tau(v, f_3, f_3, f_1) \\
0 &= R_\tau(af_1 + bf_2 + cf_3, f_3, f_3, f_1) \\
0 &= aR_{\tau(1331)} + bR_{\tau(2331)} + cR_{\tau(3331)} \\
0 &= a\delta(-1-\epsilon\lambda^2) - b\epsilon\delta\lambda
\end{aligned}$$

Now, suppose $(-1-\epsilon\lambda^2) = 0$. The last line in the above calculation then becomes $0 = -b\epsilon\delta\lambda$, which means that $0 = b\lambda$. It cannot be the case that $\lambda = 0$, because then $(-1-\epsilon\lambda^2) = 0$ would give $-1 = 0$, which is a contradiction. So since it must be the case that $\lambda \neq 0$, the fact that $0 = b\lambda$ means that $b = 0$. And by Equation 11 we thus also have $a = 0$. And by Equation 12 we then also have $c = 0$.

Now suppose instead that $(-1-\epsilon\lambda^2) \neq 0$. The equation $0 = a\delta(-1-\epsilon\lambda^2) - b\epsilon\delta\lambda$ then becomes

$$a = \frac{b\epsilon\lambda}{(-1-\epsilon\lambda^2)} \tag{13}$$

Combining Equation 13 and Equation 11 then yields

$$\begin{aligned}
a &= \frac{-a\lambda^2\epsilon}{(-1-\epsilon\lambda^2)} \\
a(1+\epsilon\lambda^2) &= a\epsilon\lambda^2 \\
a + a\epsilon\lambda^2 &= a\epsilon\lambda^2 \\
a &= 0
\end{aligned}$$

And by Equations 11 and 12, this means that $a = b = c = 0$.

So we see that in either case we obtain $a = b = c = 0$, which contradicts our assumption that $v = af_1 + bf_2 + cf_3 \neq 0$. $\square$

Remark. Suppose that $\tilde{T}$ and $\tilde{\Psi}$ are both invertible. We note that this is equivalent to T and Ψ both being invertible. Since Ψ is invertible, it follows that $\text{Rank}(\Psi) \geq 3$. And since we also have that T is invertible, it follows from work by Diaz and Dunn in [1] that

$$\Psi T = \pm T\Psi.$$

So suppose that $\Psi T = -T\Psi$, and thus $\Psi T + T\Psi = 0$. We note that

$$\Psi T + T\Psi = \begin{pmatrix} 2T_{11}\lambda + T_{21} & T_{11} + 2T_{12}\lambda + T_{22} & T_{12} + 2T_{13}\lambda + T_{12} \\ 2T_{21}\lambda + T_{31} & T_{21} + 2T_{22}\lambda + T_{21} & T_{22} + 2T_{12}\lambda + T_{11} \\ 2T_{31}\lambda & T_{31} + 2T_{21}\lambda & T_{21} + 2T_{11}\lambda \end{pmatrix}$$

Since $\Psi T + T\Psi = 0$, we see that each entry in the above matrix must be zero. Now let us recall that $\lambda \neq 0$ by assumption. So, starting from the $(3,1)$ entry in the above matrix, we see it must be the case that $T_{31} = 0$. But then looking at the $(3,2)$ entry, we see it must then follow that $T_{21} = 0$. Looking next at the $(3,3)$ entry, we see it must then follow that $T_{11} = 0$. And if we continue to follow this line of reasoning, we find that

$$T_{11} = T_{12} = T_{13} = T_{21} = T_{22} = T_{31} = 0$$

and therefore $T = 0$. But this contradicts our assumption that T has full rank. Therefore $\Psi T = -T\Psi$ is impossible, and thus we see that Ψ and T commute.

Lemma 3. *If Ψ and T commute, then*

$$[T]_{\mathcal{F}} = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ 0 & T_{11} & T_{12} \\ 0 & 0 & T_{11} \end{pmatrix}$$

Proof. Assume that Ψ and T commute. We note that

$$\Psi T = \begin{pmatrix} T_{21} + \lambda T_{11} & T_{22} + \lambda T_{12} & T_{12} + \lambda T_{13} \\ T_{31} + \lambda T_{21} & T_{21} + \lambda T_{22} & T_{11} + \lambda T_{12} \\ \lambda T_{31} & \lambda T_{21} & \lambda T_{11} \end{pmatrix} \quad \text{and} \quad T\Psi = \begin{pmatrix} \lambda T_{11} & T_{11} + \lambda T_{12} & T_{12} + \lambda T_{13} \\ \lambda T_{21} & T_{21} + \lambda T_{22} & T_{22} + \lambda T_{31} \\ \lambda T_{31} & T_{31} + \lambda T_{21} & T_{21} + \lambda T_{11} \end{pmatrix}$$

But since Ψ and T commute, these two matrices are equal, and we thus have

- $T_{21} + \lambda T_{11} = \lambda T_{11}$, which gives $T_{21} = 0$
- $T_{31} + \lambda T_{21} = \lambda T_{21}$, which gives $T_{31} = 0$
- $T_{22} + \lambda T_{12} = T_{11} + \lambda T_{12}$, which gives $T_{11} = T_{22}$

So we now have

$$[T]_{\mathcal{F}} = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ 0 & T_{11} & T_{12} \\ 0 & 0 & T_{11} \end{pmatrix}$$

□

We now proceed to the main theorem of this section:

Theorem 6.1. *If $[\Psi]_{\mathcal{F}} = J(\lambda, 3)$, $[D_\varphi]_{\mathcal{F}} = \pm sip_3$, and $\tilde{T}$ and $\tilde{\Psi}$ are both invertible, then a solution to $R_\varphi + \epsilon R_\psi = \delta R_\tau$ exists if and only if $\delta(1+\epsilon\lambda^2) > 0$ and*

$$[T]_{\mathcal{F}} = \frac{1}{\sqrt{\delta(1+\epsilon\lambda^2)}} \begin{pmatrix} \delta(1+\epsilon\lambda^2) & \delta\epsilon\lambda & \frac{-\epsilon}{\delta(1+\epsilon\lambda^2)} \\ 0 & \delta(1+\epsilon\lambda^2) & \delta\epsilon\lambda \\ 0 & 0 & \delta(1+\epsilon\lambda^2) \end{pmatrix}.$$

Proof. Assume that $\tilde{T}$ and $\tilde{\Psi}$ are both invertible, or equivalently that T and Ψ are both invertible. By the above remark, we see that T and Ψ commute. Therefore, by Lemma 3 we have

$$[T]_{\mathcal{F}} = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ 0 & T_{11} & T_{12} \\ 0 & 0 & T_{11} \end{pmatrix}$$

Let us now recall Equations 5 - 10, which hold if and only if Equation 3 holds. Given our new matrix for $[T]_{\mathcal{F}}$, Equations 5 - 10 become:

$$R_{\varphi(1221)} + \epsilon R_{\psi(1221)} = \delta R_{\tau(1221)} \iff 0 = 0 \tag{14}$$

$$R_{\varphi(1331)} + \epsilon R_{\psi(1331)} = \delta R_{\tau(1331)} \iff \delta(1+\epsilon\lambda^2) = T_{11}^2 \tag{15}$$

$$R_{\varphi(2332)} + \epsilon R_{\psi(2332)} = \delta R_{\tau(2332)} \iff \delta\epsilon = T_{12}^2 - T_{11}T_{13} \tag{16}$$

$$R_{\varphi(1231)} + \epsilon R_{\psi(1231)} = \delta R_{\tau(1231)} \iff 0 = 0 \tag{17}$$

$$R_{\varphi(2132)} + \epsilon R_{\psi(2132)} = \delta R_{\tau(2132)} \iff 1+\epsilon\lambda^2 = \delta(T_{11}^2) \tag{18}$$

$$R_{\varphi(3123)} + \epsilon R_{\psi(3123)} = \delta R_{\tau(3123)} \iff \delta\epsilon\lambda = T_{11}T_{12} \tag{19}$$

It follows from Equation 15 that

$$T_{11} = \sqrt{\delta(1+\epsilon\lambda^2)}$$

From Equation 19 we then have

$$T_{12} = \frac{\delta\epsilon\lambda}{T_{11}} = \frac{\delta\epsilon\lambda}{\sqrt{\delta(1+\epsilon\lambda^2)}}$$

Finally, from Equation 16 we have

$$T_{13} = \frac{T_{12}^2 - \delta\epsilon}{T_{11}} = \frac{\frac{\lambda^2}{\delta(1+\epsilon\lambda^2)} - \delta\epsilon}{\sqrt{\delta(1+\epsilon\lambda^2)}} = \frac{\lambda^2 - \epsilon(1+\epsilon\lambda^2)}{\delta(1+\epsilon\lambda^2)\sqrt{\delta(1+\epsilon\lambda^2)}} = \frac{-\epsilon}{\delta(1+\epsilon\lambda^2)\sqrt{\delta(1+\epsilon\lambda^2)}}$$

This gives us

$$[T]_{\mathcal{F}} = \frac{1}{\sqrt{\delta(1+\epsilon\lambda^2)}} \begin{pmatrix} \delta(1+\epsilon\lambda^2) & \delta\epsilon\lambda & \frac{-\epsilon}{\delta(1+\epsilon\lambda^2)} \\ 0 & \delta(1+\epsilon\lambda^2) & \delta\epsilon\lambda \\ 0 & 0 & \delta(1+\epsilon\lambda^2) \end{pmatrix}$$

So since $\sqrt{\delta(1+\epsilon\lambda^2)}$ is in the denominator of all entries of the matrix, clearly solutions exist if and only if $\delta(1+\epsilon\lambda^2) \neq 0$. And since the solutions are complex if $\delta(1+\epsilon\lambda^2) < 0$ we see that real solutions exist if and only if

$$\delta(1+\epsilon\lambda^2) > 0.$$

$\square$

7 If $\tilde{T}$ is invertible and $\tilde{\Psi}$ is not invertible

Theorem 7.1. *If $[D_\varphi]_{\mathcal{F}} = \pm sip_3$, $\tilde{\Psi}$ is not invertible, and $\tilde{T}$ is invertible, a solution to $R_\varphi + \epsilon R_\psi = \delta R_\tau$ exists if and only if $\delta = 1$ and*

$$[T]_{\mathcal{F}} = \pm \begin{pmatrix} 1 & 0 & -\epsilon \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Proof. We note that $\tilde{\Psi}$ not being invertible is equivalent to $[\Psi]_{\mathcal{F}} = J(0,3)$, which is to say $\lambda = 0$. Therefore, the system of Equations 5 - 10 becomes:

$$R_{\varphi(1221)} + \epsilon R_{\psi(1221)} = \delta R_{\tau(1221)} \iff 0 = T_{31}T_{22} - T_{21}^2 \tag{20}$$

$$R_{\varphi(1331)} + \epsilon R_{\psi(1331)} = \delta R_{\tau(1331)} \iff \delta = T_{11}^2 - T_{31}T_{13} \tag{21}$$

$$R_{\varphi(2332)} + \epsilon R_{\psi(2332)} = \delta R_{\tau(2332)} \iff -\epsilon = \delta(T_{22}T_{13} - T_{12}^2) \tag{22}$$

$$R_{\varphi(1231)} + \epsilon R_{\psi(1231)} = \delta R_{\tau(1231)} \iff 0 = T_{12}T_{31} - T_{21}T_{11} \tag{23}$$

$$R_{\varphi(2132)} + \epsilon R_{\psi(2132)} = \delta R_{\tau(2132)} \iff 1 = \delta(T_{11}T_{22} - T_{21}T_{12}) \tag{24}$$

$$R_{\varphi(3123)} + \epsilon R_{\psi(3123)} = \delta R_{\tau(3123)} \iff 0 = T_{21}T_{13} - T_{11}T_{12} \tag{25}$$

Now recall that we have

$$[T]_{\mathcal{F}} = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{12} \\ T_{31} & T_{21} & T_{11} \end{pmatrix}$$

Switching the first and the second rows of T gives us

$$[T_{\text{new}}]_{\mathcal{F}} = \begin{pmatrix} T_{21} & T_{22} & T_{12} \\ T_{11} & T_{12} & T_{13} \\ T_{31} & T_{21} & T_{11} \end{pmatrix}$$

and we then have

$$\begin{aligned} \det(T_{\text{new}}) &= T_{21}T_{12}T_{11} - T_{21}^2T_{13} - T_{22}T_{11}^2 + T_{22}T_{13}T_{31} + T_{12}T_{11}T_{21} - T_{12}^2T_{31} \\ &= T_{21}(T_{11}T_{12} - T_{13}T_{21}) - T_{22}(T_{11}^2 - T_{13}T_{31}) + T_{12}(T_{11}T_{21} - T_{12}T_{31}) \\ &= -\delta T_{22} \end{aligned}$$

Where the last line follows from our system of equations. Now, since T is invertible, we know that $\det(T) \neq 0$. Therefore we have $-\det(T_{\text{new}}) \neq 0$, which gives $T_{22} \neq 0$. This fact allows us to divide by T_{22}. Therefore, Equation 20 gives us

$$T_{31} = \frac{T_{21}^2}{T_{22}}, \tag{26}$$

while Equation 22 becomes

$$T_{13} = \frac{-\epsilon\delta + T_{12}^2}{T_{22}}, \tag{27}$$

and Equation 24 becomes

$$T_{11} = \frac{\delta + T_{12}T_{21}}{T_{22}}. \tag{28}$$

We can then plug Equation 27 and Equation 28 into Equation 25 as follows:

$$\begin{aligned}
0 &= T_{21}\left(\frac{-\epsilon\delta + T_{12}^2}{T_{22}}\right) - T_{12}\left(\frac{\delta + T_{12}T_{21}}{T_{22}}\right), \\
0 &= -\epsilon\delta T_{21} + T_{12}^2 T_{21} - \delta T_{12} - T_{12}^2 T_{21}, \\
0 &= -\epsilon\delta T_{21} - \delta T_{12},
\end{aligned}$$

which gives us

$$T_{12} = -\epsilon T_{21}. \tag{29}$$

Equation 23 then becomes

$$\begin{aligned}
0 &= -\epsilon T_{21}\left(\frac{T_{21}^2}{T_{22}}\right) - T_{21}\left(\frac{\delta + T_{12}T_{21}}{T_{22}}\right), \\
0 &= -\epsilon T_{21}^3 - \delta T_{21} - T_{12}T_{21}^3.
\end{aligned}$$

Applying Equation 29 here yields

$$\begin{aligned}
0 &= \epsilon T_{21}^3 - \delta T_{21} + \epsilon T_{21}^3, \\
0 &= -\delta T_{21}, \text{ so} \\
0 &= T_{21}.
\end{aligned}$$

And by Equations 29 and 26 this means we also have $T_{12} = 0$ and $T_{31} = 0$, respectively.

Equation 21 then becomes $\delta = T_{11}^2$, which means that $\delta = 1$ and thus $T_{11} = \sigma$ where $\sigma = \pm 1$. Revisiting our system of equations, we now have

$$R_{\varphi(1221)} + \epsilon R_{\psi(1221)} = \delta R_{\tau(1221)} \iff 0 = 0 \tag{30}$$

$$R_{\varphi(1331)} + \epsilon R_{\psi(1331)} = \delta R_{\tau(1331)} \iff 1 = \sigma^2 \tag{31}$$

$$R_{\varphi(2332)} + \epsilon R_{\psi(2332)} = \delta R_{\tau(2332)} \iff -\epsilon = T_{22}T_{13} \tag{32}$$

$$R_{\varphi(1231)} + \epsilon R_{\psi(1231)} = \delta R_{\tau(1231)} \iff 0 = 0 \tag{33}$$

$$R_{\varphi(2132)} + \epsilon R_{\psi(2132)} = \delta R_{\tau(2132)} \iff 1 = \sigma T_{22} \tag{34}$$

$$R_{\varphi(3123)} + \epsilon R_{\psi(3123)} = \delta R_{\tau(3123)} \iff 0 = 0 \tag{35}$$

Equation 27 is now

$$T_{13} = \frac{-\epsilon}{T_{22}} = -\epsilon T_{11} = -\epsilon\sigma$$

and equation 34 gives us $T_{22} = \sigma$. So our matrix now looks like

$$[T]_{\mathcal{F}} = \begin{pmatrix} \sigma & 0 & -\epsilon\sigma \\ 0 & \sigma & 0 \\ 0 & 0 & \sigma \end{pmatrix} = \sigma \begin{pmatrix} 1 & 0 & -\epsilon \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

□

8 Summary

We can collect the theorems proved in the preceding three sections as follows:

Theorem 8.1. *Consider arbitrary $\tilde{\varphi}, \tilde{\psi}, \tilde{\tau} \in S^2(V)$, where $\dim(V) = 3$ and $\tilde{\varphi}$ is non-degenerate. Given the equation*

$$c_1 R_{\tilde{\varphi}} + c_2 R_{\tilde{\psi}} + c_3 R_{\tilde{\tau}} = 0$$

let $\varphi = \tilde{\varphi}$, $\psi = \sqrt{|\frac{c_2}{c_1}|}\tilde{\psi}$, $\tau = \sqrt{|\frac{c_3}{c_1}|}\tilde{\tau}$, $\epsilon = sign(c_2/c_1)$, *and* $\delta = sign(c_3/c_1)$. *Let* Ψ *and* T *be the associated endomorphisms for* ψ *and* τ, *respectively, and assume that* Ψ *with respect to a basis* $\mathcal{F}$ *has the Jordan normal form* $J(\lambda, 3)$, *where we also have* $[D_\varphi]_\mathcal{F} = \pm sip_3$.

- *If* $\tilde{T}$ *is not invertible, then the set* $\{R_{\tilde{\varphi}}, R_{\tilde{\psi}}, R_{\tilde{\tau}}\}$ *is not properly linearly dependent.*

- *If* $\tilde{T}$ *and* $\tilde{\Psi}$ *are both invertible, then a solution to* $R_\varphi + \epsilon R_\psi = \delta R_\tau$ *exists if and only if* $\delta(1+\epsilon\lambda^2) > 0$ *and*

$$[T]_\mathcal{F} = \frac{1}{\sqrt{\delta(1+\epsilon\lambda^2)}} \begin{pmatrix} \delta(1+\epsilon\lambda^2) & \delta\epsilon\lambda & \frac{-\epsilon}{\delta(1+\epsilon\lambda^2)} \\ 0 & \delta(1+\epsilon\lambda^2) & \delta\epsilon\lambda \\ 0 & 0 & \delta(1+\epsilon\lambda^2) \end{pmatrix}.$$

- *If* $\tilde{T}$ *is invertible and* $\tilde{\Psi}$ *is not invertible, then a solution to* $R_\varphi + \epsilon R_\psi = \delta R_\tau$ *exists if and only if* $\delta = 1$ *and*

$$[T]_\mathcal{F} = \pm \begin{pmatrix} 1 & 0 & -\epsilon \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Previous studies of the linear dependence of three canonical algebraic curvature tensors have shown that, if one of the three tensors is defined by a positive-definite symmetric bilinear form (and some specific eigenvalue relationships hold), then linear dependence of three canonical ACTs occurs if and only if the other two tensors are simultaneously diagonalizable with respect to the first (positive-definite) tensor. However, if our tensor is defined by a form which is not necessarily positive-definite, and is known only to be non-degenerate, then Theorem 8.1 shows that simultaneous diagonalization is not a necessary and sufficient condition for linear dependence.

9 Open Questions

- The most natural open problem following from this work is the complete characterization of the problem in dimension 3. That is, what are the conditions under which we have linear independence when Ψ has one of the other three possible Jordan types.
- Once a complete solution is found for the dimension 3 case, it would then be natural to also extend these results to greater dimensions.
- A more involved project would be to investigate whether computing norms of curvature tensors could help in studying questions of linear independence.

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