DECOMPOSABLE MODEL SPACES AND A TOPOLOGICAL APPROACH TO CURVATURE

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Abstract. This research investigates the properties of k-plane constant sectional curvature and k-plane constant vector curvature, traditionally studied when $k = 2$, in finite-dimensional model spaces. We introduce (m, k)-plane constant vector curvature, an extension of k-plane constant vector curvature, and generalize several results. In two of our central theorems, we prove that the sets of k-plane and (m, k)-plane constant vector curvature values are connected, compact subsets of the real numbers. Next, we explore the relationship between decomposability and curvature. In particular, we demonstrate several connections between the k-plane constant vector curvature values of the component spaces and the $(k + 1)$-plane constant vector curvature values of the composite space. As a corollary, we prove every decomposable model space with a positive-definite inner product is k-cvc(ϵ) for some integer $k \geq 2$ and $\epsilon \in \mathbb{R}$. We also provide two examples to illustrate our theorems. These results allows us to easily construct model spaces with prescribed curvature values.

1. Introduction

Differential geometry uses concepts from analysis and algebra to study the properties of Hausdorff topological spaces called *manifolds*. Manifolds locally resemble Euclidean space, so we can utilize calculus and linear algebra to characterize its tangent space at any point. A classic example of a 2-dimensional manifold is a sphere, such as the surface of the Earth. To an observer on Earth, its surface appears linear, but an observer in space knows the Earth's surface is curved. However, Gauss proved that an observer on the surface of a two-dimensional manifold can determine its curvature without changing perspectives simply by computing distances and angles. Riemann later extended the concept of curvature to any finite-dimensional manifold.

To locally describe a manifold and its curvature, we construct a *model space* and investigate its properties. Though our research concerns model spaces, not manifolds,we sometimes comment on the latter. To study curvature, we need a notion of distance in the tangent space, which is itself a vector space. The most common tool is called an *inner product*.

Definition 1.1. *An* **inner product** $\langle \cdot, \cdot \rangle$ *on a real, finite-dimensional vector space* V *is a function* $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ *which assigns a scalar to two vectors and is:*

(1) *Symmetric:* $\langle x, y \rangle = \langle y, x \rangle$ *for all* $x, y \in V$,
(2) *Bilinear:* $\langle x + y, z \rangle = \langle x, y \rangle + \langle x, z \rangle$ *and* $\langle cx, y \rangle = c\langle x, y \rangle$ *for all* $c \in \mathbb{R}$ *and* $x, y, z \in V$ *(symmetry implies linearity in the second slot),*
(3) *Non-degenerate: for all* $x \in V \setminus \{0\}$ *there exists* $y \in V$ *such that* $\langle x, y \rangle \neq 0$.

In lieu of condition (3), we call an inner product *positive-definite* if $\langle x, x \rangle \geq 0$ for all $x \in V$, and $\langle x, x \rangle = 0$ if and only if $x = 0$. Every positive-definite inner product is non-degenerate since $\langle x, x\rangle > 0$ for all nonzero x. Unless otherwise stated, we suppose all inner products are positive-definite. We also exclusively use V to denote a vector space and n to designate the dimension of V (i.e. $\dim(V) = n$). For brevity, we often omit these statements in our theorems, though they are included when necessary. Now, to compute the curvature of a manifold at a point, we use an *algebraic curvature tensor*, often abbreviated as an *ACT*.

Definition 1.2. *An* **algebraic curvature tensor (ACT)** *R is a function from ordered quadruples of tangent vectors to scalars,*

$$R : V \times V \times V \times V \to \mathbb{R},$$

which satisfies the following properties for all $x, y, z, w \in V$:

(1) *Multilinearity: $R(x + v, y, z, w) = R(x, y, z, w) + R(v, y, z, w)$ for all $v \in V$ and $R(cx, y, z, w) = cR(x, y, z, w)$ for all $c \in \mathbb{R}$ (linearity is similar in the other slots),*
(2) *Skew symmetry: $R(x, y, z, w) = -R(y, x, z, w)$,*
(3) *Interchange symmetry: $R(x, y, z, w) = R(z, w, x, y)$, and*
(4) *First Bianchi Identity: $R(x, y, z, w) + R(z, x, y, w) + R(y, z, x, w) = 0$.*

If $\{e_1, \ldots, e_n\}$ is a basis for V, we regularly shorten $R(e_i, e_j, e_k, e_l)$ to R_{ijkl}. The set of algebraic curvature tensors is denoted $\mathcal{A}(V)$. In addition to conditions $(1) - (4)$ above, which concern the components of R, any tensor in $\mathcal{A}(V)$ satisfies the following additive and multiplicative conditions.

Proposition 1.3. *If $R, S \in \mathcal{A}(V)$, $x, y, z, w \in V$, and $\lambda \in \mathbb{R}$, then*

(1) $(R + S)(x, y, z, w) = R(x, y, z, w) + S(x, y, z, w)$,
(2) $(\lambda R)(x, y, z, w) = \lambda R(x, y, z, w)$.

Since the trivial ACT $R \equiv 0$, meaning every curvature entry is zero, is an element of $\mathcal{A}(V)$, Proposition 1.3 shows $\mathcal{A}(V)$ is a vector space [11]. Now we can introduce a *model space* to help us study the local curvature of a manifold.

Definition 1.4. *A* **model space** *$\mathcal{M} = (V, \langle\cdot,\cdot\rangle, R)$ is a vector space V, a non-degenerate inner product $\langle\cdot,\cdot\rangle$ on V, and an algebraic curvature tensor R.*

Given a manifold, a metric, and a point on the manifold, we can construct a model space from the tangent space, metric, and ACT at that point. A common curvature measurement is the *sectional curvature*, a function that takes a non-degenerate 2-plane π and returns a scalar $\kappa(\pi)$. We say a 2-plane is *non-degenerate* if the inner product restricted to the 2-plane is non-degenerate. Since this is always true for a positive-definite inner product, we assume all 2-planes are non-degenerate.

Definition 1.5. *Let $\mathcal{M}$ be a model space, R be an ACT, $x, y \in V$ be vectors, and $\pi = \mathrm{span}\{x, y\}$ be a non-degenerate 2-plane. The* **sectional curvature** *is a function $\kappa : V \times V \to \mathbb{R}$ that assigns a scalar to pairs of vectors, given by*

$$\kappa(\pi) = \frac{R(x, y, y, x)}{\langle x, x\rangle\langle y, y\rangle - \langle x, y\rangle^2}.$$

Importantly, the sectional curvature is basis-independent, so $\kappa(\pi)$ is the same no matter the choice of x and y. Also, notice that if x and y are orthonormal, the denominator in the expression for $\kappa(\pi)$ is one, so $\kappa(\pi) = R(x, y, y, x)$. Thus, it is much easier to work with the sectional curvature on an orthonormal basis.

Since we assume the inner product is positive-definite, any linearly independent set of 2 vectors determines a non-degenerate 2-plane. However, more care is necessary with non-degenerate inner products. We refer the reader to [13] for a thorough examination of curvature in this alternate setting. Note that we exclusively use π to designate a 2-plane. Also, if $\pi = \operatorname{span}\{x, y\}$, we often shorten $\kappa(\operatorname{span}\{x, y\})$ to $\kappa(x, y)$. With this understanding of sectional curvature, we can introduce two standard notions of constant curvature.

Definition 1.6. *A model space $\mathcal{M}$ is* **constant sectional curvature** ϵ*, denoted csc(ϵ), if $\kappa(\pi) = \epsilon$ for all non-degenerate 2-planes π.*

Constant sectional curvature means all 2-planes have the same sectional curvature value. Since this is a strong condition, constant sectional curvature is uncommon. A more useful property, called *constant vector curvature*, concerns whether each vector in V is contained in a 2-plane with a given sectional curvature [16].

Definition 1.7. *A model space $\mathcal{M}$ is* **constant vector curvature** ϵ*, denoted cvc(ϵ), if for all $v \in V \setminus \{0\}$ there exists a non-degenerate 2-plane π containing v such that $\kappa(\pi) = \epsilon$.*

One can show that if a model space is csc(ϵ), then it is cvc(ϵ) [15]. While these conditions are well understood for 3-dimensional model spaces [13], little is known about model spaces of arbitrary finite dimension. Until recently, research about curvature of higher-dimensional model spaces only used 2-planes. In [4], Calle considers k-planes for any model space of finite dimension to study curvature. Chen first defines the scalar curvature of a k-plane in [6].

Definition 1.8. *Let $\mathcal{M} = (V, \langle\cdot,\cdot\rangle, R)$ be a model space with $\{e_1, \ldots, e_n\}$ an orthonormal basis for V. Let $\{f_1, \ldots, f_k\}$ be an orthonormal basis for some subspace $L \subseteq V$. Define the model space $\mathcal{M}_L = (L, \langle\cdot,\cdot\rangle_L, R_L)$, where $\langle\cdot,\cdot\rangle_L = \langle\cdot,\cdot\rangle|_L$ and $R_L = R|_L \in \mathcal{A}(V)$ are the restrictions of $\langle\cdot,\cdot\rangle$ and R, respectively, to L. The* **k-plane scalar curvature of** *L is the function $\mathcal{K}_{R_L} : L \to \mathbb{R}$ given by*

$$\mathcal{K}_{R_L}(L) = \sum_{j>i=1}^{k} \kappa(f_i, f_j).$$

Although we actually evaluate $\mathcal{K}(L)$ with respect to $\mathcal{M}_L$, for brevity we usually discuss $\mathcal{K}(L)$ in terms of $\mathcal{M}$. If R and L are clear from the context, we omit the subscript and simply write $K(L)$. Also, if $L = \operatorname{span}\{e_1, \ldots, e_k\}$, we often shorten $\mathcal{K}(\operatorname{span}\{e_1, \ldots, e_k\})$ to $\mathcal{K}(e_1, \ldots, e_k)$. Based on [4], we now introduce a generalization of Definition 1.6.

Definition 1.9. *A model space $\mathcal{M}$ is* **k-plane constant sectional curvature** ϵ*, denoted k-csc(ϵ), if $\mathcal{K}(L) = \epsilon$ for all non-degenerate k-planes L.*

Similar as above, since the the inner product is positive-definite, we may suppose that all k-planes are non-degenerate. Analogous to 2-planes, because k-plane constant sectional curvature is uncommon, we introduce a looser, more useful property called *k-plane constant vector curvature* [4].

Definition 1.10. *A model space $\mathcal{M}$ is* **k-plane constant vector curvature** ϵ*, denoted k-cvc(ϵ), if for all $v \in V$, there exists a non-degenerate k-plane L containing v such that $\mathcal{K}(L) = \epsilon$.*

As with csc and cvc, one can show that if a model space $\mathcal{M}$ is k-csc(ϵ), then $\mathcal{M}$ is k-cvc(ϵ) [4]. Note that we use notation cvc and 2-cvc interchangeably. For a fixed integer k, $\mathcal{C}_k$ denotes the set of all k-cvc values of $\mathcal{M}$. Also, if we say $\mathcal{M}$ is k-cvc($[\epsilon, \delta]$), we mean $\mathcal{M}$ is *at least* k-cvc($[\epsilon, \delta]$). If $\mathcal{C}_k$ is *exactly* $[\epsilon, \delta]$, we say so explicitly. Unless otherwise stated, we assume $2 \leq k \leq n-1$. We do not consider $k = 1$ since κ is not defined for 1-dimensional subspaces, and hence neither is $\mathcal{K}$. Our next definition, from [4], shows why the case $k = n$ is uninteresting.

Definition 1.11. *Let $\mathcal{M} = (V, \langle \cdot, \cdot \rangle, R)$ be a model space with $\{e_1, \ldots, e_n\}$ an orthonormal basis for V. The* **scalar curvature** *(or* **Ricci scalar***) is*

$$\tau = \sum_{i,j} \mathcal{E}_i \mathcal{E}_j R_{ijji}$$

where $\mathcal{E}_r = \langle e_r, e_r \rangle = \pm 1$.

Since we assume the inner product is positive-definite, $\mathcal{E}_i = \mathcal{E}_j = 1$ for all i and j. Hence, τ is the sum of all possible R_{ijji} entries on an orthonormal basis. This means $\langle e_i, e_j \rangle = 1$ if and only if $i = j$ and is otherwise zero. If $\mathcal{M} = (V, \langle \cdot, \cdot \rangle, R)$, then the n-plane scalar curvature of $V = \operatorname{span}\{e_1, \ldots, e_n\}$ is

$$\mathcal{K}(V) = \sum_{j>i=1}^{n} \kappa(e_i, e_j) = \sum_{j>i=1}^{n} R_{ijji}.$$

So, in general, $\mathcal{K}(V) = \tau/2$. Because every model spaces is exactly n-cvc($\tau/2$), we do not consider the case $k = n$.

In the following sections we explore these curvature properties. In Section 2, we study k-plane constant sectional curvature and propose future work on an open conjecture. Next, in Section 3, we introduce an extension of k-cvc called (m, k)-plane constant vector curvature and generalize several results. Section 4 contains our central theorems on the topological properties of the sets $\mathcal{C}_k$ and $\mathcal{C}_k^m$. In particular, we prove $\mathcal{C}_k$ and $\mathcal{C}_k^m$ are connected, compact subsets of $\mathbb{R}$, that is, real intervals of the form $[a, b]$. Next, in Section 5, we investigate the k-plane curvature properties of decomposable model spaces. We demonstrate several relationships between the k-plane constant vector curvature values of the component spaces and the $(k+1)$-plane constant vector curvature values of the composite space. As an important corollary, we show every decomposable model space with a positive-definite inner product is k-cvc(ϵ) for some integer $k \geq 2$ and $\epsilon \in \mathbb{R}$. Finally, we provide two examples in Section 6 to demonstrate (m, k)-cvc and our decomposability theorems. Our research allows us to easily construct model spaces with prescribed curvature values from simpler model spaces.

2. k-Plane Constant Sectional Curvature

This section presents some basic extensions of work in [3, 4] regarding k-plane constant sectional curvature and discusses a difficult open question.

Proposition 2.1. *If the model spaces $\mathcal{M}_1 = (V, \langle \cdot, \cdot \rangle, R_1)$ and $\mathcal{M}_2 = (V, \langle \cdot, \cdot \rangle, R_2)$ are k-csc(ϵ) and k-csc(δ), respectively, then $\mathcal{M} = (V, \langle \cdot, \cdot \rangle, R_1 + R_2)$ has k-csc($\epsilon + \delta$).*

Proof. Define $\mathcal{M}_1$, $\mathcal{M}_2$, and $\mathcal{M}$ as above. Let $v \in V$. By supposition, for any k-plane L containing v, $\mathcal{K}_{R_1}(L) = \epsilon$ and $\mathcal{K}_{R_2}(L) = \delta$. Then

$$\mathcal{K}_R(L) = \sum_{j>i=1}^{k} (R_1 + R_2)_{ijji} = \sum_{j>i=1}^{k} (R_1)_{ijji} + (R_2)_{ijji},$$

where the last equality follows from Proposition 1.3(1). Thus,

$$\mathcal{K}_R(L) = \sum_{j>i=1}^{k} (R_1)_{ijji} + \sum_{j>i=1}^{k} (R_2)_{ijji} = \epsilon + \delta.$$

Since $\mathcal{K}_R(L) = \epsilon + \delta$ for all k-planes L, we conclude that $\mathcal{M}$ is k-csc$(\epsilon + \delta)$. □

Our next proposition is also straightforward. These results may prove useful in more complex arguments or help resolve the open question of this section.

Proposition 2.2. *If the model space $\mathcal{M}' = (V, \langle\cdot,\cdot\rangle, R)$ is k-csc(ϵ), then the model space $\mathcal{M} = (V, \langle\cdot,\cdot\rangle, cR)$ is k-csc($c\epsilon$).*

Proof. Define $\mathcal{M}'$ and $\mathcal{M}$ as above. Let $v \in V$. By supposition, for any k-plane L containing v, $\mathcal{K}_R(L) = \epsilon$. Then by Proposition 1.3(2),

$$\mathcal{K}_{cR}(L) = \sum_{j>i=1}^{k} (cR)_{ijji} = c \sum_{j>i=1}^{k} R_{ijji} = c\epsilon$$

Since $\mathcal{K}_{cR}(L) = c\epsilon$ for all k-planes L, we conclude that $\mathcal{M}_2$ is k-csc$(c\epsilon)$. □

Now we transition to an open question. We know that if a model space is csc(0), then the curvature tensor R is identically zero [1]. This is intuitive: if $\kappa(\pi) = 0$ for all 2-planes π, necessarily $R \equiv 0$ since every sectional curvature value is a rational multiple of some R_{ijkl} component [1]. In [4], Calle generalizes this argument to include when a model space is k-csc(0).

Theorem 2.3. *Let $\mathcal{M} = (V, \langle\cdot,\cdot\rangle, R)$ be a model space and set $2 \leq k \leq n-2$. If $\mathcal{M}$ is k-csc(0), then $R \equiv 0$.*

Note an important omission in the hypotheses. While csc(0) implies $R \equiv 0$ for 3-dimensional model spaces, Calle's argument does not hold when $k = n-1$. As a partial result, one could impose a suitable additional constraint on the model space to force $R \equiv 0$. However, this merely circumvents the original question: does $(n-1)$-csc(0) imply $R \equiv 0$? Calle conjectures that there is an $(n-1)$-csc(0) model space with a nontrivial ACT, and we concur. One possible counterexample we investigated, at the suggestion of Calle, is a 3-plane embedded in $\mathbb{R}^4$.

Conjecture 2.4. *Let $\mathcal{M} = (V, \langle\cdot,\cdot\rangle, R)$ be the model space where $V = \mathbb{R}^4$. If $R_{1221} = R_{3443} = 1$, $R_{1331} = R_{2442} = -1$, and all other components of R are zero on some orthonormal basis $\{e_1, e_2, e_3, e_4\}$ for V, then $\mathcal{M}$ is 3-csc(0).*

Now we outline the evidence for Conjecture 2.4. For any 3-plane $L \subset \mathbb{R}^4$, we find an expression for $\mathcal{K}(L)$ and attempt to prove it must be zero, showing $\mathcal{M}$ is 3-csc(0). We know there is an orthonormal basis $\{f_1, f_2, f_3\}$ for L, where each f_i

is a linear combination of the vectors e_1, e_2, e_3, e_4. In all generality,

$$f_1 = a_{11}e_1 + a_{21}e_2 + a_{31}e_3 + a_{41}e_4,$$
$$f_2 = a_{12}e_1 + a_{22}e_2 + a_{32}e_3 + a_{42}e_4,$$
$$f_3 = a_{13}e_1 + a_{23}e_2 + a_{33}e_3 + a_{43}e_4,$$

where $a_{ij} \in \mathbb{R}$ for $i = 1, 2, 3, 4$ and $j = 1, 2, 3$. Because $\{f_1, f_2, f_3\}$ is an orthonormal basis for L, we can place restrictions on the coefficients a_{ij}. Since f_1, f_2, f_3 are unit vectors, the inner product of any vector with itself is one:

$$\langle f_1, f_1\rangle = a_{11}^2 + a_{21}^2 + a_{31}^2 + a_{41}^2 = 1, \tag{2.1}$$
$$\langle f_2, f_2\rangle = a_{12}^2 + a_{22}^2 + a_{32}^2 + a_{42}^2 = 1, \tag{2.2}$$
$$\langle f_3, f_3\rangle = a_{13}^2 + a_{23}^2 + a_{33}^2 + a_{43}^2 = 1. \tag{2.3}$$

Since the f_i are pairwise orthogonal, the inner product of any distinct two is zero:

$$\langle f_1, f_2\rangle = a_{11}a_{12} + a_{21}a_{22} + a_{31}a_{31} + a_{41}a_{42} = 0, \tag{2.4}$$
$$\langle f_1, f_3\rangle = a_{11}a_{13} + a_{21}a_{23} + a_{31}a_{33} + a_{41}a_{43} = 0, \tag{2.5}$$
$$\langle f_2, f_3\rangle = a_{12}a_{13} + a_{22}a_{23} + a_{32}a_{33} + a_{42}a_{43} = 0. \tag{2.6}$$

Substituting into $\mathcal{K}(L)$, expanding using basic properties of ACTs, and simplifying,

$$\begin{aligned}\mathcal{K}(L) &= R(f_1, f_2, f_2, f_1) + R(f_1, f_3, f_3, f_1) + R(f_2, f_3, f_3, f_2)\\ &= \begin{vmatrix} a_{11} & a_{21} \\ a_{12} & a_{22}\end{vmatrix}^2 - \begin{vmatrix} a_{11} & a_{31} \\ a_{12} & a_{32}\end{vmatrix}^2 - \begin{vmatrix} a_{21} & a_{41} \\ a_{22} & a_{42}\end{vmatrix}^2 + \begin{vmatrix} a_{31} & a_{41} \\ a_{32} & a_{42}\end{vmatrix}^2 \\ &+ \begin{vmatrix} a_{11} & a_{21} \\ a_{13} & a_{23}\end{vmatrix}^2 - \begin{vmatrix} a_{11} & a_{31} \\ a_{13} & a_{33}\end{vmatrix}^2 - \begin{vmatrix} a_{21} & a_{41} \\ a_{23} & a_{43}\end{vmatrix}^2 + \begin{vmatrix} a_{31} & a_{41} \\ a_{33} & a_{43}\end{vmatrix}^2 \\ &+ \begin{vmatrix} a_{12} & a_{22} \\ a_{13} & a_{23}\end{vmatrix}^2 - \begin{vmatrix} a_{12} & a_{32} \\ a_{13} & a_{33}\end{vmatrix}^2 - \begin{vmatrix} a_{22} & a_{42} \\ a_{23} & a_{43}\end{vmatrix}^2 + \begin{vmatrix} a_{32} & a_{42} \\ a_{33} & a_{43}\end{vmatrix}^2 .\end{aligned} \tag{2.7}$$

Interestingly, we can express $\mathcal{K}(L)$ as the sum of squares of determinants of simple 2×2 matrices. Together, Equations 2.1-2.7 give a system of 7 nonlinear equations in 12 variables. Though we do not explicitly solve for the a_{ij}, we investigate possible solutions using Matlab's fsolve function (in version 2018a). Our testing strongly suggests this system has a solution if and only if $\mathcal{K}(L) = 0$, proving Conjecture 2.4. Due to the mounting evidence that $(n-1)$-csc(0) does not imply $R \equiv 0$, we suggest further investigations to confirm the existence of a counterexample.

3. (m, k)-Plane Constant Vector Curvature

Much like Calle generalizes cvc to k-cvc, we extend k-plane constant vector curvature to (m, k)-plane constant vector curvature. Instead of requiring that each vector is contained in a k-plane with some k-plane scalar curvature value, we examine the possibility that an m-plane is contained in such a k-plane. By considering different curvature measurements, we can better understand the model space and hence the local properties of the manifold it represents.

Definition 3.1. *A model space $\mathcal{M}$ is (m, k)*-**plane constant vector curvature** *ϵ, denoted (m, k)-cvc(ϵ), if for all m-planes P, there exists a non-degenerate k-plane L containing P such that $\mathcal{K}(L) = \epsilon$.*

Analogous to standard k-plane scalar curvature, given a model space $\mathcal{M}$ and integers m and k, we let $\mathcal{C}_k^m$ denote the set of all (m,k)-plane constant vector curvature values of $\mathcal{M}$. In particular, setting $m=1$, we have $\mathcal{C}_k = \mathcal{C}_k^1$. We do not consider the case $m=k$, since if every k-plane is contained in a k-plane (i.e. itself) with curvature ϵ, then every k-plane has curvature ϵ. Now, similar to the previous section, we present some elementary extensions of work in [4].

Proposition 3.2. *If the model space $\mathcal{M} = (V, \langle\cdot,\cdot\rangle, R)$ is k-csc(ϵ), then the model space $\mathcal{M}_c = (V, \langle\cdot,\cdot\rangle, cR)$ is (m,k)-cvc(ϵ) for all $1 \le m \le k-1 < n$.*

Proof. Define $\mathcal{M}$ and $\mathcal{M}_c$ as above. Let $1 \le m \le k-1 < n$ and suppose P is an m-plane with orthonormal basis $\{e_1, \dots, e_m\}$. Extend $\{e_1, \dots, e_m\}$ to an orthonormal basis $\{e_1, \dots, e_n\}$ for V. Choose a k-plane L with orthonormal basis $\{e_1, \dots, e_k\}$. Since $\mathcal{M}$ is k-csc(ϵ), we have $\mathcal{K}(L) = \epsilon$. Such an L exists for all m-planes P because $m \le k-1$. We need only adjust the orthonormal basis for P, and hence for V and L. Thus, $\mathcal{M}_c$ is (m,k)-cvc(ϵ). □

The next result is a similar statement to Proposition 2.1.

Proposition 3.3. *If $\mathcal{M}_1 = (V, \langle\cdot,\cdot\rangle, R_1)$ is k-csc(ϵ) and $\mathcal{M}_2 = (V, \langle\cdot,\cdot\rangle, R_2)$ is (m,k)-cvc(δ), then $\mathcal{M} = (V, \langle\cdot,\cdot\rangle, R_1 + R_2)$ is (m,k)-cvc($\epsilon+\delta$).*

Proof. Define $\mathcal{M}_1$, $\mathcal{M}_2$, and $\mathcal{M}$ as above. Let $1 \le m \le k-1 < n$ and suppose P is an m-plane. Since $\mathcal{M}_2$ is (m,k)-cvc(δ), there is a k-plane L containing P such that $\mathcal{K}_{R_2}(L) = \delta$. Because $\mathcal{M}_1$ is k-csc(ϵ), we have $\mathcal{K}_{R_1}(L) = \epsilon$. Hence,

$$\mathcal{K}_R(L) = \sum_{j>i=1}^{k} (R_1 + R_2)_{ijji} = \sum_{j>i=1}^{k} (R_1)_{ijji} + (R_2)_{ijji},$$

where the last equality follows from Proposition 1.3(1). Thus,

$$\mathcal{K}_R(L) = \sum_{j>i=1}^{k} (R_1)_{ijji} + \sum_{j>i=1}^{k} (R_2)_{ijji} = \epsilon + \delta.$$

Since P is arbitrary, we conclude that $\mathcal{M}$ is (m,k)-cvc($\epsilon+\delta$). □

This section's final result provides another way to build a model space with a desired k-cvc value from known examples.

Proposition 3.4. *If the model space $\mathcal{M} = (V, \langle\cdot,\cdot\rangle, R)$ has (m,k)-cvc(ϵ), then the model space $\mathcal{M}_c = (V, \langle\cdot,\cdot\rangle, cR)$ has (m,k)-cvc($c\epsilon$).*

Proof. Define $\mathcal{M}$ and $\mathcal{M}_c$ as above and let P be an m-plane. By supposition, there is a k-plane L containing P such that $\mathcal{K}_{R_1}(L) = \epsilon$. Then

$$\mathcal{K}_{cR}(L) = \sum_{j>i=1}^{k} (cR)_{ijji} = c \sum_{j>i=1}^{k} R_{ijji} = c\epsilon$$

where we use Proposition 1.3(2). Since P is arbitrary, $\mathcal{M}_c$ is (m,k)-cvc($c\epsilon$). □

With this understanding of (m,k)-plane constant vector curvature, we can ask and answer more difficult questions in Section 4, results that include statements about cvc and k-cvc by default.

4. Topological Properties of $\mathcal{C}_k$ and $\mathcal{C}_k^m$

Given a model space $\mathcal{M}$, $\mathcal{C}_k$ and $\mathcal{C}_k^m$, respectively, are the set of k-cvc values and (m,k)-cvc values of $\mathcal{M}$. Our goal in this section is to determine the topological invariants of $\mathcal{C}_k$ and $\mathcal{C}_k^m$. Recall that the *special orthogonal group* in dimension n, denoted $\mathrm{SO}(n)$, is the set of orthogonal $n \times n$ matrices with determinant 1, and that the *orthogonal complement* of a subspace $U \subseteq V$, denoted $U^{\perp}$, is the set of all vectors in V orthogonal to every vector in U. Our next proof utilizes that $\mathrm{SO}(n)$ is path-connected [17], so for the reader unfamiliar with this result, we provide a more direct, detailed proof in the appendix.

Theorem 4.1. *If $\mathcal{M} = (V, \langle \cdot, \cdot \rangle, R)$ is a model space with $n = \dim(V) \geq 3$, then $\mathcal{C}_k^m$ is connected for all $m, k \in \mathbb{Z}$ such that $1 \leq m < k \leq n$. In particular, $\mathcal{C}_k$ is connected for all $2 \leq k \leq n$.*

Proof. Suppose $\mathcal{M}$ is as above and let $1 \leq m < k \leq n$. Choose an m-plane P and let $\mathcal{B}_P \subset V$ be an orthonormal basis for P. Suppose $\mathcal{M}$ is (m,k)-cvc(ϵ) and (m,k)-cvc(δ) for $\epsilon, \delta \in \mathbb{R}$. By definition, there are k-planes $L_0, L_1 \subset V$ containing P such that $\mathcal{K}(L_0) = \epsilon$ and $\mathcal{K}(L_1) = \delta$. We can find orthonormal bases $\mathcal{B}_0$ and $\mathcal{B}_1$, respectively, for L_0 and L_1 so that $\mathcal{B}_P \subseteq \mathcal{B}_0$ and $\mathcal{B}_P \subseteq \mathcal{B}_1$. Extend $\mathcal{B}_0$ and $\mathcal{B}_1$ to orthonormal bases $\mathcal{V}_0$ and $\mathcal{V}_1$ for V. Consider $\mathcal{V}_0^{\perp} = \mathcal{V}_0 \setminus \mathcal{B}_P$ and $\mathcal{V}_1^{\perp} = \mathcal{V}_1 \setminus \mathcal{B}_P$, which have $n - m$ elements.

If $m = n-1$ and $k = n$, then $\mathcal{C}_k^m$ is connected since $\mathcal{M}$ is exactly n-cvc$(\tau/2)$ and V trivially contains P. If $m \leq n-1$, then $n - m \geq 2$, so $\mathrm{SO}(n-m)$ is path-connected. Then there is a continuous deformation of orthonormal bases from $\mathcal{V}_0^{\perp}$ to $\mathcal{V}_1^{\perp}$. Restricting to the first $k-m$ vectors yields such a deformation from $\mathcal{B}_0 \setminus \mathcal{B}_P$ to $\mathcal{B}_1 \setminus \mathcal{B}_P$. But the vectors in $\mathcal{B}_P$ are pairwise orthogonal to the vectors in $\mathcal{B}_0 \setminus \mathcal{B}_P$ and $\mathcal{B}_1 \setminus \mathcal{B}_P$, so adding the m vectors in $\mathcal{B}_P$ to any intermediate basis gives an orthogonal, linearly independent set of k unit vectors. Hence, this rotation in the orthogonal complement of P is a continuous deformation of orthonormal bases from $\mathcal{B}_0$ to $\mathcal{B}_1$, and the span of the basis vectors defines a path from L_0 to L_1. Since P is in each intermediate basis, the space of k-planes that contains P is path-connected, and hence connected. Because the map $L \mapsto \mathcal{K}(L)$ is continuous, the image set is connected. Since ϵ and δ are arbitrary, $\mathcal{C}_k^m$ is connected. In particular, letting $m = 1$, $\mathcal{C}_k$ is connected for all $2 \leq k \leq n$. □

The space of all k-dimensional subspaces of V is called the *Grassmannian*, denoted $\mathrm{Gr}_k(V)$. Because $\mathrm{Gr}_k(V)$ is compact [14] and the map $L \mapsto \mathcal{K}(L)$ is continuous, the set $\mathcal{S}_k^m$ of (m,k)-plane scalar curvature values of $\mathcal{M}$ is compact, and hence bounded. Then $\mathcal{C}_k^m \subseteq \mathcal{S}_k^m$ is bounded, but this does not imply $\mathcal{C}_k^m$ is closed. Rather, we can give a careful proof of this fact.

Theorem 4.2. *For any model space $\mathcal{M} = (V, \langle \cdot, \cdot \rangle, R)$, $\mathcal{C}_k^m$ is closed for all $m, k \in \mathbb{Z}$ such that $1 \leq m < k \leq n$. In particular, $\mathcal{C}_k$ is closed for all $2 \leq k \leq n$.*

Proof. Consider the continuous function $\mathcal{K} : \mathrm{Gr}_k(V) \to \mathbb{R}$ given by $L \mapsto \mathcal{K}(L)$. We prove that if $a \in \mathbb{R}$ is a limit point of $\mathcal{C}_k^m$, then $a \in \mathcal{C}_k^m$. Suppose $(a_i) \subset \mathcal{C}_k^m$ is a sequence with limit point a. Since $\mathcal{C}_k^m$ is bounded, a is finite. Choose m-plane P spanned by the vectors $v_1, \ldots, v_m \in V$. By definition, there is a k-plane L_i containing P such that $\mathcal{K}(L_i) = a_i$. Construct the sequence $(L_i) \subset \mathrm{Gr}_k(V)$. Since $\mathrm{Gr}_k(V)$ is compact, a is a limit point of $\mathcal{C}_k$, and $\mathcal{K}$ is continuous, there is a subsequence (L_{i_j}) converging to $L \in \mathrm{Gr}_k(V)$ such that $\mathcal{K}(L) = a$. Every L_{i_j}

contains $v_1, \ldots, v_m$, so L also contains $v_1, \ldots, v_m$ and hence P. Since P is arbitrary, $a \in \mathcal{C}_k^m$. Therefore, $\mathcal{C}_k^m$ is closed. In particular, letting $m = 1$, $\mathcal{C}_k$ is closed. □

This completes the topological classification of $\mathcal{C}_k^m$, and by extension, $\mathcal{C}_k$. Since $\mathcal{C}_k^m$ is a connected, compact subset of $\mathbb{R}$, $\mathcal{C}_k^m$ is precisely $[a, b]$ for some $a, b \in R$. For example, if $(0,1) \subseteq \mathcal{C}_k^m$, then necessarily $[0,1] \subseteq \mathcal{C}_k^m$. An important next step is to develop sufficient conditions to demonstrate certain $\epsilon \in \mathbb{R}$ are not in $\mathcal{C}_k^m$.

5. Decomposability and Curvature

Given a model space $\mathcal{M} = (V, \langle \cdot, \cdot \rangle, R)$, one could ask whether $\mathcal{M}$ decomposes as the *direct sum* (denoted $\oplus$) of two or more model spaces. If so, V, $\langle \cdot, \cdot \rangle$, and R must factor into orthogonal components. In this section, we investigate the relationship between decomposability and curvature, leveraging that $\mathcal{C}_k$ is connected. Note that these results only concern k-cvc, not (m, k)-cvc. Our end goal is to be able to build model spaces with desired curvature values using direct sum decompositions.

Definition 5.1. *We say a model space $\mathcal{M} = (V, \langle \cdot, \cdot \rangle, R)$ is* **decomposable***, written $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$, if $V = V_1 \oplus V_2$, $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_1 \oplus \langle \cdot, \cdot \rangle_2$, $R = R_1 \oplus R_2$, and the following hold for all vectors $v_1 \in V_1$ and $v_2 \in V_2$:*

(1) $\langle v_1, v_2 \rangle = \langle v_1, 0 \rangle_1 + \langle 0, v_2 \rangle_2 = 0$,
(2) $R(v_1, v_2, \cdot, \cdot) = R_1(v_1, 0, \cdot, \cdot) + R_2(0, v_2, \cdot, \cdot) = 0$.

Condition (1) say v_1 is orthogonal to v_2, or, equivalently, V_1 is orthogonal to V_2. The vectors in condition (2) are unbiased in input, meaning $R = 0$ even if v_1 or v_2 is in another slot [8]. Note that $\mathcal{M}_1$ and $\mathcal{M}_2$ are themselves model spaces. We let ${}_1\mathcal{C}_k$ and ${}_2\mathcal{C}_k$, respectively, denote the set of k-cvc values of $\mathcal{M}_1$ and $\mathcal{M}_2$. We also call $\mathcal{M}$ decomposable if it breaks into three or more model spaces, i.e., $\mathcal{M} = \oplus_{i=1}^{j} \mathcal{M}_i$ and $j \geq 3$. We say $\mathcal{M}_i$ is a *component space* and $\mathcal{M}$ is the *composite space*. Since distinct V_i are orthogonal, we can uniquely write any $v \in V$ as $v = a_1 v_1 + \cdots + a_j v_j$ for $a_i \in \mathbb{R}$ and unit vectors $v_i \in V_i$.

Recall that if $\mathcal{M} = (V, \langle \cdot, \cdot \rangle, R)$ is a model space, $\langle \cdot, \cdot \rangle$ is positive-definite, and $\{e_1, \ldots, e_n\}$ is an orthonormal basis for V, then $\tau = \sum_{i,j} R_{ijji}$ is the *scalar curvature.* Since $\mathcal{K}(V) = \tau/2$, every model space is exactly n-cvc$(\tau/2)$ where $n = \dim(V)$. This small observation has important implications for the k-cvc values of decomposable model spaces. With this background, we can now examine how decomposabilty impacts $\mathcal{C}_k$, beginning with $k = 2$.

Proposition 5.2. *Suppose a model space $\mathcal{M} = (V, \langle \cdot, \cdot \rangle, R)$ decomposes as $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$. If there exists $\epsilon \in {}_1\mathcal{C}_2$ such that $-\epsilon \in {}_2\mathcal{C}_2$, then $\mathcal{M}$ is cvc(0).*

Proof. Suppose $\mathcal{M}$, $\mathcal{M}_1$, and $\mathcal{M}_2$ are as above. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis for V. Suppose without loss of generality that $v \in V$ is normalized. We need a 2-plane containing v with sectional curvature zero. There are three cases: $v \in V_1$, $v \in V_2$, or v is a linear combination of vectors from V_1 and V_2. If $v \in V_1$, take any unit vector $u_2 \in V_2$. Then $\kappa(v, u_2) = 0$. Similarly, if $v \in V_2$, take any unit vector $u_1 \in V_2$. Then $\kappa(v, u_2) = 0$.

Now, suppose $v = \frac{v_1 + v_2}{\sqrt{2}}$ for unit vectors $v_1 \in V_1$, $v_2 \in V_2$. Choose another unit vector $w \in V$ so that w is a linear combination of vectors from V_1 and V_2, but w is not a scalar multiple of v. Write $w = \frac{w_1 + w_2}{\sqrt{2}}$ for unit vectors $w_1 \in V_1$, $w_2 \in V_2$.

Define the 2-plane $\pi_{12} = \text{span}\{v, w\}$. Since $\{v, w\}$ is an orthonormal basis for π_{12},

$$\begin{aligned}\kappa(\pi_{12}) &= R(v, w, w, v) \\ &= R\left(\frac{v_1 + v_2}{\sqrt{2}}, \frac{w_1 + w_2}{\sqrt{2}}, \frac{w_1 + w_2}{\sqrt{2}}, \frac{v_1 + v_2}{\sqrt{2}}\right) \\ &= \frac{1}{4}R(v_1 + v_2, w_1 + w_2, w_1 + w_2, v_1 + v_2) \\ &= \frac{1}{4}R(v_1, w_1, w_1, v_1) + \frac{1}{2}R(v_1, w_1, w_2, v_2) + \frac{1}{4}R(v_2, w_2, w_2, v_2),\end{aligned}$$

which follows from Definition 5.1(2). Similarly, since $R(v_1, w_1, w_2, v_2)$ has inputs from V_1 and V_2, $R(v_1, w_1, w_2, v_2) = 0$. Hence,

$$\kappa(\pi_{12}) = \frac{1}{4}R(v_1, w_1, w_1, v_1) + \frac{1}{4}R(v_2, w_2, w_2, v_2). \tag{5.1}$$

Since $\mathcal{M}_1$ is cvc(ϵ), there is a 2-plane π_1 containing v_1 such that $\kappa(\pi_1) = \epsilon$. Choose w_1 orthogonal to v_1 such that $\text{span}\{v_1, w_1\} = \pi_1$. Then

$$\kappa(v_1, w_1) = R(v_1, w_1, w_1, v_1) = \epsilon.$$

Similarly, since $\mathcal{M}_2$ is cvc($-\epsilon$), there is a 2-plane π_2 containing v_2 such that $\kappa(\pi_2) = -\epsilon$. Choose w_2 orthogonal to v_2 such that $\text{span}\{v_2, w_2\} = \pi_2$. Then

$$\kappa(v_2, w_2) = R(v_2, w_2, w_2, v_2) = -\epsilon.$$

Therefore, the two terms in Equation 5.1 cancel, so $\kappa(\pi_{12}) = 0$. □

Note that we only prove $\mathcal{M}$ is 2-cvc(0). In general, $\mathcal{M}$ could have other 2-cvc values. Also, we need that there is some $\epsilon \in {}_1\mathcal{C}_2$ so that $-\epsilon \in {}_2\mathcal{C}_2$ to guarantee the two terms in Equation 5.1 cancel out. This proposition is a nice introduction to our decomposability results about general k-cvc. Our first theorem shows how k-cvc values can "lift" from a component space to the composite space. Given a nonzero vector v, note that we write $\hat{v} = \frac{v}{\|v\|}$ for the associated unit vector of v.

Theorem 5.3. *If a model space $\mathcal{M} = (V, \langle\cdot,\cdot\rangle, R)$ decomposes as $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$ and $\mathcal{M}_1$ is k-cvc(ϵ) for an integer $2 \leq k \leq n-1$ and $\epsilon \in \mathbb{R}$, then $\mathcal{M}$ is $(k+1)$-cvc(ϵ).*

Proof. Suppose $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$, where $\mathcal{M}$, $\mathcal{M}_1$, and $\mathcal{M}_2$ are as above. By definition, $V = V_1 \oplus V_2$. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis for V and let $v \in V$. We can uniquely write $v = av_1 + bv_2$ for scalars a, b and unit vectors $v_1 \in V_1$, $v_2 \in V_2$. We consider three cases: $v \in V_1$, $v \in V_2$, or v is a linear combination of vectors in V_1 and V_2.

Suppose $a \neq 0$ and $b = 0$, meaning $v \in V_1$. Since $\mathcal{M}_1$ is k-cvc(ϵ), there is a k-plane $\widetilde{L} \subseteq V_1$ containing v such that $\mathcal{K}(\widetilde{L}) = \epsilon$. Set $f_1 = \hat{v}$. We can find unit vectors $f_2, \ldots, f_k \in V_1$, pairwise orthogonal to f_1 and each other, so that $\widetilde{L} = \text{span}\{f_1, \ldots, f_k\}$. Take any unit vector $f_{k+1} \in V_2$, which by definition is pairwise orthogonal to the vectors $f_1, \ldots, f_k$. Consider the $(k+1)$-plane $L = \text{span}\{f_1, \ldots, f_{k+1}\}$. Then, because $\mathcal{K}(\widetilde{L}) = \epsilon$,

$$\mathcal{K}(L) = \sum_{j>i=1}^{k+1} \kappa(f_i, f_j) = \sum_{j>i=1}^{k} \kappa(f_i, f_j) + \sum_{i=1}^{k} \kappa(f_i, f_{k+1}) = \epsilon + \sum_{i=1}^{k} \kappa(f_i, f_{k+1}).$$

Since $\{f_1, \ldots, f_{k+1}\}$ is an orthonormal basis for L, $\kappa(f_i, f_{k+1}) = R(f_i, f_{k+1}, f_{k+1}, f_i)$ for $i = 1, \ldots, k$. But $f_i \in V_1$ for all i and $f_{k+1} \in V_2$, so $R(f_i, f_{k+1}, f_{k+1}, f_i) = 0$ by Definition 5.1(3). Hence, each summand is zero, which shows

$$\mathcal{K}(L) = \epsilon + \sum_{i=1}^{k} R(f_i, f_{k+1}, f_{k+1}, f_i) = \epsilon,$$

so L is a $(k+1)$-plane containing v such that $\mathcal{K}(L) = \epsilon$. This construction works for any nonzero vector in V_1, so $\mathcal{M}$ is $(k+1)$-cvc(ϵ) in this case.

Next, suppose $a = 0$ and $b \neq 0$, meaning $v \in V_2$. Since $\mathcal{M}_1$ is k-cvc(ϵ), there is a k-plane $\widetilde{\mathcal{L}} \subseteq V_1$, with orthonormal basis $\{f_1, \ldots, f_k\}$, such that $\mathcal{K}(\widetilde{\mathcal{L}}) = \epsilon$. Set $f_{k+1} = \hat{v}$, and note that f_{k+1} is pairwise orthogonal to the vectors $f_1, \ldots, f_k$. Then $\mathcal{L} = \operatorname{span}\{f_1, \ldots, f_{k+1}\}$ is a $(k+1)$-plane containing v. So, by a similar method as above, $\mathcal{K}(\mathcal{L}) = \epsilon$. Hence, $\mathcal{M}$ is $(k+1)$-cvc(ϵ) in this case as well.

Now, suppose $a \neq 0 \neq b$. Clearly, $v \in \operatorname{span}\{v_1, v_2\}$ since $v = av_1 + bv_2$. Set $f_1 = v_1$ and $f_{k+1} = v_2$. Because $\mathcal{M}_1$ is k-cvc(ϵ), there is a k-plane $\widetilde{\mathfrak{L}} \subseteq V_1$, with orthonormal basis $\{f_1, \ldots, f_k\}$, so that $\mathcal{K}(\widetilde{\mathfrak{L}}) = \epsilon$. While $v \notin \mathfrak{L}$, f_{k+1} is pairwise orthogonal to the vectors $f_1, \ldots, f_k$ and $v \in \operatorname{span}\{f_1, f_{k+1}\}$, so $\mathfrak{L} = \operatorname{span}\{f_1, \ldots, f_{k+1}\}$ is a $(k+1)$-plane containing v. Similarly as before, $\mathcal{K}(\mathfrak{L}) = \epsilon$, so $\mathcal{M}$ is also $(k+1)$-cvc(ϵ) in this case. Therefore, $\mathcal{M}$ is $(k+1)$-cvc$([\epsilon, \delta])$ by Theorem 4.1. □

The labeling of $\mathcal{M}_1$ and $\mathcal{M}_2$ is arbitrary, so Theorem 5.3 is equally valid if we replace $\mathcal{M}_1$ with $\mathcal{M}_2$. Now we generalize Theorem 5.3 to include any finite direct sum decomposition $\mathcal{M} = \oplus_{i=1}^{j} \mathcal{M}_i$ with $j \geq 3$.

Corollary 5.4. *If a model space $\mathcal{M} = (V, \langle \cdot, \cdot \rangle, R)$ decomposes as $\mathcal{M} = \oplus_{i=1}^{j} \mathcal{M}_i$, where some $\mathcal{M}_i$ is k-cvc(ϵ) for $2 \leq k \leq \dim(V_i)$ and $\epsilon \in \mathbb{R}$, then $\mathcal{M}$ is $(k+1)$-cvc(ϵ).*

Proof. We use a similar argument as Theorem 5.3. Suppose $\mathcal{M}$ decomposes as above. Let $v \in V$, which we can uniquely write as $v = a_1 v_1 + \cdots + a_j v_j$ for scalars a_i and unit vectors $v_i \in V_i$. We consider three cases: $v \in V_1$, $v \in V_1^{\perp}$, or v is a linear combination of vectors in V_1 and $V_1^{\perp}$.

First, suppose $a_1 \neq 0$ and $a_i = 0$ for all $i \geq 2$, meaning $v \in V_1$. By the same argument as the first case of Theorem 5.3, replacing V_2 with $V_1^{\perp}$, $\mathcal{M}$ is $(k+1)$-cvc(ϵ_1) in this case. Next, if $a_1 = 0$ and $a_i \neq 0$ for some $i \geq 2$, meaning $v \in V_1^{\perp}$, make the same exchange in the second case of Theorem 5.3. This proves $\mathcal{M}$ is $(k+1)$-cvc(ϵ_1) in this case. Now, suppose $a_1 \neq 0$ and $a_i \neq 0$ for some $i \geq 2$. Define the vector $w = v - a_1 v_1$, and consider its normal vector $\hat{w}$. Since $v = a_1 v_1 + w$, $v \in \operatorname{span}\{v_1, w\} = \operatorname{span}\{v_1, \hat{w}\}$. Use the same argument as the third case of Theorem 5.3, replacing v_2 with $\hat{w}$. This shows $\mathcal{M}$ is $(k+1)$-cvc(ϵ_1) in this case too, so $\mathcal{M}$ is $(k+1)$-cvc(ϵ_1) in general. □

As an application of Theorem 5.3, consider the case when $\dim(V_1) = n-1$ and $\dim(V_2) = 1$ (or vice versa). Since $\mathcal{M}$ is uniquely n-cvc$(\tau/2)$, where τ is the scalar curvature, $\tau/2$ is the only possible $(n-1)$-cvc value of $\mathcal{M}_1$. We now present an important corollary.

Corollary 5.5. *Suppose $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$. Let τ_i be the scalar curvature of $\mathcal{M}_i$ and set $\dim(V_i) = n_i$. Then $\mathcal{M}$ is (n_1+1)-cvc$(\tau_1/2)$ and (n_2+1)-cvc$(\tau_2/2)$.*

Proof. Suppose $\mathcal{M}$ decomposes into $\mathcal{M}_1$ and $\mathcal{M}_2$. Clearly, $\mathcal{M}_1$ and $\mathcal{M}_2$ are n_1-cvc($\tau_1/2$) and n_2-cvc($\tau_2/2$), respectively. Then $\mathcal{M}$ is (n_1+1)-cvc($\tau_1/2$) and (n_2+1)-cvc($\tau_2/2$) by Theorem 5.3. □

Corollary 5.5 has a significant consequence: *every decomposable model space with a positive-definite inner product is k-cvc(ϵ) for some integer $k \geq 2$ and $\epsilon \in \mathbb{R}$.* This is a valuable extension of the result that every three-dimensional model space equipped with a positive-definite inner product is cvc(ϵ) for a unique value ϵ [18]. This result significantly increases the number of known k-cvc model spaces.

Our next theorem uses a similar argument as Theorem 5.3. Notice the proof of Theorem 5.3 is independent of ${}_2\mathcal{C}_k$, the k-cvc values of $\mathcal{M}_2$. Hence, if $\mathcal{M}_2$ is k-cvc(δ) for the same k as $\mathcal{M}_1$, the range of k-cvc values "lift" from the component spaces to the composite space.

Theorem 5.6. *Suppose a model space $\mathcal{M} = (V, \langle\cdot,\cdot\rangle, R)$ decomposes as $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$. If $\mathcal{M}_1$ is k-cvc(ϵ) and $\mathcal{M}_2$ is k-cvc(δ) for some integer $2 \leq k \leq \min(\dim(V_1), \dim(V_2))$ and $\epsilon, \delta \in \mathbb{R}$, then $\mathcal{M}$ is $(k+1)$-cvc($[\epsilon, \delta]$).*

Proof. Suppose $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$, where $\mathcal{M}$, $\mathcal{M}_1$, and $\mathcal{M}_2$ are as above. By Theorem 5.3, $\mathcal{M}$ is $(k+1)$-cvc(ϵ) and $(k+1)$-cvc(δ). Therefore, using the convention that $[\epsilon, \delta] = \epsilon$ if $\epsilon = \delta$, $\mathcal{M}$ is $(k+1)$-cvc($[\epsilon, \delta]$) by Theorem 4.1. □

This result illustrates an even stronger relationship between the $(k+1)$-cvc values of a decomposable model space and the k-cvc values of its component spaces. Naturally, Theorem 5.6 generalizes to any finite direct sum decomposition.

Corollary 5.7. *Suppose a model space $\mathcal{M} = (V, \langle\cdot,\cdot\rangle, R)$ decomposes as $\mathcal{M} = \oplus_{i=1}^{j} \mathcal{M}_i$, where each $\mathcal{M}_i$ is k-cvc(ϵ_i) for some integer $2 \leq k \leq \min\{\dim(V_i)\}_{i=1}^{j}$ and $\epsilon_i \in \mathbb{R}$. Let ϵ_m and ϵ_M, respectively, be the minimum and maximum of the set $\{\epsilon_i\}_{i=1}^{j}$. Then $\mathcal{M}$ is $(k+1)$-cvc($[\epsilon_m, \epsilon_M]$).*

Proof. Suppose $\mathcal{M}$ decomposes as above. Define $\epsilon_m = \min\{\epsilon_i\}_{i=1}^{j}$ and $\epsilon_M = \max\{\epsilon_i\}_{i=1}^{j}$. Applying Corollary 5.4 to $\mathcal{M}_1$, $\mathcal{M}$ is $(k+1)$-cvc(ϵ_1). Repeat this argument separately for every $\mathcal{M}_i$, replacing V_1 with V_i and $V_1^{\perp}$ with $V_i^{\perp}$. Then $\mathcal{M}$ is $(k+1)$-cvc(ϵ_i) for all i. In particular, $\mathcal{M}$ is $(k+1)$-cvc(ϵ_m) and $(k+1)$-cvc(ϵ_M), so $\mathcal{M}$ is $(k+1)$-cvc($[\epsilon_m, \epsilon_M]$) by Theorem 4.1. □

As an application of Corollary 5.7, consider when $\mathcal{M} = \oplus_{i=1}^{j} \mathcal{M}_i$ and $\dim(V_i) = k$ for all i and some fixed k. Then, letting $\tau_m = \min\{\tau_i\}_{i=1}^{j}$ and $\tau_M = \max\{\tau_i\}_{i=1}^{j}$, $\mathcal{M}$ is $(k+1)$-cvc($[\tau_m/2, \tau_M/2]$). Now, our next theorem provides further insight into the connection between the k-cvc values of a decomposable model space and its components in the special case of zero k-plane scalar curvature.

Theorem 5.8. *Suppose a model space $\mathcal{M} = (V, \langle\cdot,\cdot\rangle, R)$ decomposes as $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$. If $\mathcal{M}_1$ is i-cvc(ϵ) and $\mathcal{M}_2$ is j-cvc(δ) for integers i and j, where $2 \leq i, j \leq \max(\dim(V_1), \dim(V_2))$, and $\epsilon, \delta \in \mathbb{R}$, then $\mathcal{M}$ is $(i+j)$-cvc($\epsilon + \delta$).*

Proof. Suppose $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$, where $\mathcal{M}$, $\mathcal{M}_1$, and $\mathcal{M}_2$ are as above. In particular, $V = V_1 \oplus V_2$. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis for V and let $v \in V$. We can uniquely write $v = av_1 + bv_2$ for scalars a, b and unit vectors $v_1 \in V_1$, $v_2 \in V_2$. Set $k = i + j$. We consider three cases: $v \in V_1$, $v \in V_2$, or v is a linear combination of vectors in V_1 and V_2.

Suppose $a \neq 0$ and $b = 0$, meaning $v \in V_1$. Since $\mathcal{M}_1$ is i-cvc(ϵ), there is an i-plane $I \subseteq V_1$ containing v such that $\mathcal{K}(I) = \epsilon$. Set $f_1 = \hat{v}$. We can find unit vectors $f_2, \ldots, f_i \in V_1$, pairwise orthogonal to f_1 and each other, so that $I = \operatorname{span}\{f_1, \ldots, f_i\}$. Since $\mathcal{M}_2$ is j-cvc(δ), there is a j-plane $J \subseteq V_2$ such that $\mathcal{K}(J) = \delta$. As before, we can find an orthonormal set of vectors in V_2, say $\{f_{i+1}, \ldots, f_k\}$, such that $J = \operatorname{span}\{f_{i+1}, \ldots, f_k\}$. By definition, the vectors $f_{i+1}, \ldots, f_k$ are pairwise orthogonal to $f_1, \ldots, f_i$. Consider the k-plane $L = \operatorname{span}\{f_1, \ldots, f_k\}$. We have

$$\mathcal{K}(L) = \sum_{t>s=1}^{k} \kappa(f_s, f_t) = \sum_{t>s=1}^{i} \kappa(f_s, f_t) + \sum_{t>i+1}^{k} \sum_{s=1}^{i} \kappa(f_s, f_t) + \sum_{t>s=i+1}^{k} \kappa(f_s, f_t).$$

Let us examine the three summations in the rightmost expression. The first summation is $\mathcal{K}(I)$, so this term is ϵ. Next, because $\{f_1, \ldots, f_k\}$ is an orthonormal basis for I, we know $\kappa(f_s, f_t) = R(f_s, f_t, f_t, f_s)$ for $s = 1, \ldots, i$ and $t = i+1, \ldots, k$. But $f_s \in V_1$ and $f_t \in V_2$ for all s and t, so each $R(f_s, f_t, f_t, f_s) = 0$. Thus, the second summation is also zero. The third summation is the expression for $\mathcal{K}(J)$, and hence equals δ. Since L is a k-plane containing v and $\mathcal{K}(L) = \epsilon + \delta$, and this construction works for any nonzero vector in V_1, $\mathcal{M}$ is k-cvc($\epsilon + \delta$) in this case.

Next, suppose $a = 0$ and $b \neq 0$, meaning $v \in V_2$. Since $\mathcal{M}_1$ is i-cvc(ϵ), there is an i-plane $\mathcal{I} \subseteq V_1$, with some orthonormal basis $\{f_1, \ldots, f_i\}$, such that $\mathcal{K}(\mathcal{I}) = \epsilon$. Set $f_{i+1} = \hat{v}$. Since $\mathcal{M}_2$ is j-cvc(δ), there is a j-plane $\mathcal{J} \subseteq V_2$ containing v, with orthonormal basis $\{f_{i+1}, \ldots, f_k\}$, such that $\mathcal{K}(\mathcal{J}) = \delta$. Then $\mathcal{L} = \operatorname{span}\{f_1, \ldots, f_k\}$ is a k-plane containing v, and by a similar method as before, $\mathcal{K}(\mathcal{L}) = \epsilon + \delta$. Hence, $\mathcal{M}$ is k-cvc($\epsilon + \delta$) in this case as well.

Now, suppose $a \neq 0 \neq b$. Clearly, $v \in \operatorname{span}\{v_1, v_2\}$ since $v = av_1 + bv_2$. Set $f_1 = v_1$ and $f_{i+1} = v_2$. Because $\mathcal{M}_1$ is i-cvc(ϵ), there is an i-plane $I' \subseteq V_1$ containing v_1, with orthonormal basis $\{f_1, \ldots, f_i\}$, such that $\mathcal{K}(I') = \epsilon$. Similarly, there is a j-plane $J' \subseteq V_2$ containing v_2, with orthonormal basis $\{f_{i+1}, \ldots, f_k\}$, such that $\mathcal{K}(J') = \delta$. Then $\mathfrak{L} = \operatorname{span}\{f_1, \ldots, f_k\}$ is a k-plane containing v, so $\mathcal{K}(\mathfrak{L}) = \epsilon + \delta$ by a similar method as before. Hence, $\mathcal{M}$ is k-cvc($\epsilon + \delta$) in this case too. Since $k = i + j$, $\mathcal{M}$ is $(i + j)$-cvc($\epsilon + \delta$). □

We can make a few observations. First, suppose neither $\mathcal{M}_1$ nor $\mathcal{M}_2$ is p-cvc(0) for any $p \geq 2$. If $\mathcal{M}_1$ is i-cvc(ϵ) and $\mathcal{M}_2$ is j-cvc($-\epsilon$) for integers $2 \leq i, j \leq \max(\dim(V_1), \dim(V_2))$ and $\epsilon \in \mathbb{R}$, then $\mathcal{M}_1 \oplus \mathcal{M}_2$ would still be $(i + j)$-cvc(0). Second, let $\mathcal{M} = (V, \langle\cdot,\cdot\rangle, R)$ and define the model space $-\mathcal{M} = (V, \langle\cdot,\cdot\rangle, -R)$. That is, the ACT entries of $-\mathcal{M}$ are the negative of those of $\mathcal{M}$. Then according to the previous observation, if $\mathcal{M}$ is k-cvc(ϵ) for any k and any ϵ, then $-\mathcal{M}$ must be k-cvc($-\epsilon$), so the direct sum $\mathcal{M} \oplus (-\mathcal{M})$ is k-cvc(0).

As with Theorem 5.6, we can easily generalize Theorem 5.8 to include any finite direct sum decomposition of a model space.

Corollary 5.9. *Suppose a model space $\mathcal{M} = (V, \langle\cdot,\cdot\rangle, R)$ decomposes as $\mathcal{M} = \oplus_{i=1}^{j} \mathcal{M}_i$, where each $\mathcal{M}_i$ is k_i-cvc(ϵ_i) for some integer $2 \leq k_i \leq \max\{\dim(V_i)\}_{i=1}^{j}$ and $\epsilon_i \in \mathbb{R}$. Define the scalars $k = \sum_{i=1}^{j} k_i$ and $\epsilon = \sum_{i=1}^{j} \epsilon_i$. Then $\mathcal{M}$ is k-cvc(ϵ).*

Proof. This follows from $j-1$ applications of Theorem 5.8. Suppose $\mathcal{M} = \oplus_{i=1}^{j} \mathcal{M}_i$, where each $\mathcal{M}_i$ is k_i-cvc(ϵ_i). Let $k = \sum_{i=1}^{j} k_i$ and $\epsilon = \sum_{i=1}^{j} \epsilon_i$. Applying Theorem 5.8 to $\mathcal{M}_1$ and $\mathcal{M}_2$ proves $\mathcal{M}_1 \oplus \mathcal{M}_2$ is $(k_1 + k_2)$-cvc($\epsilon_1 + \epsilon_2$). Now, since $k_1 + k_2 \leq$

$\dim(V_1) + \dim(V_2)$, we can apply Theorem 5.8 to $\mathcal{M}_1 \oplus \mathcal{M}_2$ and $\mathcal{M}_3$. This shows $\mathcal{M}_1 \oplus \mathcal{M}_2 \oplus M_3$ is $(k_1 + k_2 + k_3)$-cvc$(\epsilon_1 + \epsilon_2 + \epsilon_3)$. Applying this process recursively a total of $j - 1$ times (note j is finite), we conclude that $\mathcal{M}$ is k-cvc(ϵ). $\square$

Having established these general results, we are ready to consider some examples.

6. Examples

Now that we have topologically classified $\mathcal{C}_k$ and $\mathcal{C}_k^m$ and investigated the curvature values of decomposable model spaces, we can examine specific examples to illustrate these results. Our examples concern (m, k)-plane constant vector curvature (from Section 3) and decomposability (from Section 5). Since k-plane constant vector curvature is far more common than k-plane constant sectional curvature, we restrict our attention to the former.

To show a model space has (m, k)-cvc(ϵ), we choose an arbitrary m-plane P and construct a k-plane with curvature ϵ that contains P. Similarly, to show a decomposable model space has k-cvc(δ), we choose an arbitrary vector v and find a k-plane with curvature δ that contains v. We must construct the plane and the vector so that the curvature value is independent of the specific components of P and v. We now introduce a special type of ACT.

Definition 6.1. *Let V be a vector space. Given a symmetric, bilinear function $\phi : V \times V \to \mathbb{R}$, a* **canonical ACT** *with respect to ϕ is an ACT of the form*

$$R_\phi(x, y, z, w) = \phi(x, w)\phi(y, z) - \phi(x, z)\phi(y, w).$$

Interestingly, the set of canonical ACTs spans $\mathcal{A}(V)$, the set of all algebraic curvature tensors on V [9]. We exclusively use canonical ACTs in our two examples.

Now, recall that the kernel of a linear transformation is the set of vectors in the domain that map to the zero vector in the range. Analogously, we can define the *kernel* of an algebraic curvature tensor as the set of vectors that produce a zero curvature value regardless of the other inputs.

Definition 6.2. *The* **kernel** *of an algebraic curvature tensor R is the set*

$$\ker(R) = \{v \in V \,:\, R(v, y, z, w) = 0 \;\forall y, z, w \in V\}.$$

Note that vectors in the kernel are unbiased in input, meaning $R = 0$ even if v is in the second, third, or fourth slot [8]. The size of the kernel relative to V is often a helpful indicator of a manifold's curvature. In nearly all cases, the kernel of a symmetric, bilinear function ϕ and that of its associated canonical ACT R_ϕ are closely linked. The proof of the following proposition appears in [10].

Proposition 6.3. *If ϕ is a symmetric, bilinear function and rank$(\phi) \geq 2$, then*

$$\ker(\phi) = \{v \in V \,:\, \phi(v, w) = 0 \quad \forall w \in V\} = \ker(R_\phi).$$

The method we use in our first example, inspired by [4], is to decompose v into vectors from the eigenspaces of ϕ. Given a linear transformation $A : V \to V$, recall that $v \in V \setminus \{0\}$ is an *eigenvector* of A with *eigenvalue* $\lambda \in \mathbb{R}$ if $Av = \lambda v$. If λ_i is an eigenvalue, E_i denotes the *eigenspace* spanned by the associated eigenvectors. For a discussion of eigenspaces and related concepts, see [2]. Given a symmetric, bilinear function ϕ defined on a vector with a non-degenerate inner product, it is well known that there is a self-adjoint linear transformation $A : V \to V$ characterized by the equation $\phi(x,y) = \langle Ax, y\rangle$. Therefore, ϕ and A have the same eigenvalues with the same multiplicity. So, if f_i, f_j are orthonormal eigenvectors,

$$\phi(f_i, f_j) = \langle Af_i, f_j\rangle = \lambda_i\langle f_i, f_j\rangle.$$

Since ϕ is diagonalized, $\langle f_i, f_j\rangle$ can only be nonzero if $i = j$. With this understanding, we can introduce our next proposition, whose proof is in [4].

Proposition 6.4. *Let $\mathcal{M} = (V, \langle\cdot,\cdot\rangle, R)$ be a model space. If f_i, f_j are orthogonal unit vectors in the eigenspaces for λ_i, λ_j, respectively, then $\kappa(f_i, f_j) = \lambda_i\lambda_j$.*

Proposition 6.4 motivates us to use canonical ACTs in our first example. Knowing only the eigenvalues of ϕ (which are the same as A), we can easily calculate sectional curvature values, and hence k-plane scalar curvature values. From [5], we also have a helpful bound on the sectional curvature values in terms of the products of eigenvalues.

Proposition 6.5. *Let ϕ be a symmetric, bilinear function, and let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of ϕ, repeated according to multiplicity. Let m and M, respectively, be the minimum and maximum of the set $\{\lambda_i\lambda_j : i \neq j\}$. The set of sectional curvatures of R_ϕ is precisely the interval $[m, M]$.*

We are ready to introduce our example on (m,k)-plane constant vector curvature.

Example 6.6. Let $\mathcal{M} = (V, \langle\cdot,\cdot\rangle, R)$ be a model space with $\{e_1, \ldots, e_6\}$ an orthonormal basis for V, $\langle\cdot,\cdot\rangle$ a positive-definite inner product on V, and $R = R_\phi$, where ϕ is represented by the matrix

$$\phi = \begin{bmatrix} I_3 & 0_3 \\ 0_3 & 0_3 \end{bmatrix}.$$

Here, I_3 is the 3×3 identity matrix and 0_3 is the 3×3 zero matrix.

We defined the matrix in Example 6.6 so that $\phi(e_i, e_j)$ is the ij^{th} entry. Hence, the eigenvalues of ϕ are $\lambda_1 = 1$ and $\lambda_2 = 0$. The associated eigenspaces are $E_1 = \operatorname{span}\{e_1, e_2, e_3\}$ and $E_2 = \operatorname{span}\{e_4, e_5, e_6\}$, so $\dim(E_1) = \dim(E_2) = 3$. Since $\operatorname{rank}(\phi) = 3 \geq 2$, we know $E_2 = \ker(R)$ by Proposition 6.3. Also, E_1 and E_2 are orthogonal because $\{e_1, \ldots, e_6\}$ is an orthonormal basis for V. Therefore, we can write any $v \in V$ as $v = \alpha v_1 + \beta v_2$ for $\alpha, \beta \in \mathbb{R}$ and unit vectors $v_1, v_2 \in V$. Our approach to the proof of the following proposition is inspired by [4].

Proposition 6.7. *The model space $\mathcal{M}$ in Example 6.6 has the following properties:*

(1) $\mathcal{C}_2^3 \subseteq [0,1]$, $\mathcal{C}_3^4 \subseteq [0,3]$, *and* $\mathcal{C}_4^5 \subseteq [0,6]$,
(2) $\mathcal{M}$ *is* $(2,4)$-*cvc*(1),
(3) $\mathcal{M}$ *is* $(2,5)$-*cvc*$([1,3])$.
(4) $\mathcal{M}$ *is* $(m,6)$-*cvc*(3) *and only* $(m,6)$-*cvc*(3) *for* $1 \leq m \leq 5$.

(1) $\mathcal{C}_2^3 \subseteq [0,1]$, $\mathcal{C}_3^4 \subseteq [0,1]$, *and* $\mathcal{C}_4^5 \subseteq [0,6]$.

Proof. Suppose $\mathcal{M}$ is $(2,3)$-cvc(ϵ) for some $\epsilon \in \mathbb{R}$. We narrow down possible values for ϵ by a careful choice of 2-plane. Set $P = \operatorname{span}\{e_4, e_5\}$. Let L be a 3-plane containing P with orthonormal basis $\mathcal{B} = \{f_1, f_2, f_3\}$. Since L contains P, suppose without loss of generality that $f_1 = (e_4 + e_5)/\sqrt{2}$. Complete $\mathcal{B}$ with any two unit vectors $f_2, f_3 \in V$ orthogonal to f_1 and each other. Since $f_1 \in \ker(R)$,

$$\mathcal{K}(L) = \sum_{j>i=1}^{3} \kappa(f_i, f_j) = \kappa(f_2, f_3).$$

Set $\pi = \text{span}\{f_2, f_3\}$. By 6.5, $0 \leq \kappa(\pi) \leq 1$, meaning $0 \leq \mathcal{K}(L) \leq 1$. Therefore, if $\mathcal{M}$ is $(2,3)$-cvc(ϵ), then $\epsilon \in [0,1]$, so $\mathcal{C}_2^3 \subseteq [0,1]$.

A similar argument shows $\mathcal{C}_3^4 \subseteq [0,3]$. Consider $P' = \text{span}\{e_4, e_5, e_6\}$ and let $\widetilde{L}$ be a 4-plane with orthonormal basis $\{f_1', f_2', f_3', f_4'\}$. Set $f_1 = (e_4+e_5+e_6)/\sqrt{3}$ and proceed as before. Then

$$\mathcal{K}(\widetilde{L}) = \sum_{j>i=2}^{4} \kappa(f_i', f_j') = \kappa(f_2', f_3') + \kappa(f_2', f_4') + \kappa(f_3', f_4').$$

As before, the sectional curvatures are bounded by the products of eigenvalues, so $0 \leq \mathcal{K}(L) \leq 3$. Hence, $\mathcal{C}_3^4 \subseteq [0,3]$. By similar reasoning in the case of $(4,5)$-cvc, we obtain a summation with six nonzero terms, so $\mathcal{C}_4^5 \subseteq [0,6]$. □

(2) $\mathcal{M}$ *is* $(2,4)$-*cvc*(1).

Proof. Let P be any 2-plane. Choose an orthonormal basis $\{x,y\}$ for P. We must find a 4-plane L containing P such that $\mathcal{K}(L) = 1$. Since $y \in x^\perp$, y is a linear combination of vectors orthogonal to x. So, we can write $x = ax_1 + bx_2$ and $y = cy_1 + dy_2$ for $a,b,c,d \in \mathbb{R}$ and pairwise orthogonal unit vectors $x_i, y_i \in E_i$. Construct an orthonormal basis $\{f_1, f_2, f_3, f_4\}$ for L, where

$$f_1 = x_1,\ f_2 = x_2,\ f_3 = y_1,\ \text{and}\ f_4 = y_2.$$

If $a = 0$, set $f_1 = u_1$ for some unit vector $u_1 \in y_1^\perp$. If $c = 0$, set $f_3 = u_1'$ for some unit vector $u_1' \in x_1^\perp$. Similarly, if $b = 0$, set $f_2 = u_2$ for some unit vector $u_2 \in y_2^\perp$. If $d = 0$, set $f_4 = u_2'$ for some unit vector $u_2' \in x_2^\perp$. In any case, L contains P. Note that $f_3, f_4 \in \ker(R)$. So, by Proposition 6.4,

$$\mathcal{K}(L) = \sum_{j>i=1}^{4} \kappa(f_i, f_j) = \kappa(f_1, f_2) = \lambda_1^2 = 1.$$

Since P is arbitrary, we conclude that $\mathcal{M}$ is $(2,4)$-cvc(1). □

(3) $\mathcal{M}$ *is* $(2,5)$-*cvc*$([1,3])$.

Proof. Let P be any 2-plane. Choose an orthonormal basis $\{x,y\}$ for P. We must find a 5-plane L containing P such that $\mathcal{K}(L) = 1$. Write x and y as in part (2). Since $\dim(E_2) = 3$, there is a unit vector $w_2 \in E_2$ orthogonal to x_2 and y_2. Construct an orthonormal basis $\{f_1, f_2, f_3, f_4, f_5\}$ for L, where

$$f_1 = x_1,\ f_2 = x_2,\ f_3 = y_1,\ f_4 = y_2,\ \text{and}\ f_5 = w_2.$$

If $a = 0$, set $f_1 = u_1$ for some unit vector $u_1 \in y_1^\perp$. If $c = 0$, set $f_3 = u_1'$ for some unit vector $u_1' \in x_1^\perp$. Similarly, if $b = 0$, set $f_2 = u_2$ for some unit vector $u_2 \in y_2^\perp \cap w_2^\perp$. If $d = 0$, set $f_4 = u_2'$ for some unit vector $u_2' \in x_2^\perp \cap w_2^\perp$. In any case, L contains P. Note that $f_3, f_4, w_2 \in \ker(R)$. So, by Proposition 6.4,

$$\mathcal{K}(L) = \sum_{j>i=1}^{4} \kappa(f_i, f_j) = \kappa(f_1, f_2) = \lambda_1^2 = 1.$$

Since P is arbitrary, $\mathcal{M}$ is $(2,5)$-cvc(1).

Next, we must find a 5-plane L containing P such that $\mathcal{K}(L) = 3$. Write x and y as above. Since $\dim(E_1) = 3$, there is a unit vector $w_1 \in E_1$ orthogonal to x_1 and y_1. Construct an orthonormal basis $\{f_1, f_2, f_3, f_4, f_5\}$ for L, where

$$f_1 = x_1,\ f_2 = x_2,\ f_3 = y_1,\ f_4 = y_2,\ \text{and } f_5 = w_1.$$

If $a = 0$, set $f_1 = u_1$ for some unit vector $u_1 \in y_1^\perp \cap w_1^\perp$. If $c = 0$, set $f_3 = u_1'$ for some unit vector $u_1' \in x_1^\perp \cap w_1^\perp$. Similarly, if $b = 0$, set $f_2 = u_2$ for some unit vector $u_2 \in y_2^\perp$. If $d = 0$, set $f_4 = u_2'$ for some unit vector $u_2' \in x_2^\perp$. In any case, L contains P. Note that $f_3, f_4 \in \ker(R)$. So, by Proposition 6.4,

$$\mathcal{K}(L) = \sum_{j>i=1}^{4} \kappa(f_i, f_j) = \kappa(f_1, f_2) + \kappa(f_1, f_5) + \kappa(f_1, f_5) = 3\lambda_1^2 = 3.$$

This shows $\mathcal{M}$ is $(2,5)$-cvc(3). Then $\mathcal{M}$ is $(2,5)$-cvc($[1,3]$) by Theorem 4.1. □

(4) *$\mathcal{M}$ is $(m,6)$-cvc(3) and only $(m,6)$-cvc(3) for $1 \le m \le 5$.*

Proof. Choose an integer $1 \le m \le 5$ and consider any m-plane P. We know

$$\mathcal{K}(V) = \sum_{j>i=1}^{6} \kappa(e_i, e_j) = \kappa(e_1, e_2) + \kappa(e_1, e_3) + \kappa(e_2, e_3) = 3\lambda_1 = 3.$$

Hence, $\mathcal{M}$ is 6-cvc(3) and only 6-cvc(3). Since every m-plane is trivially contained in V, the claim follows. □

We now present our second example. Given a model space, we assume $\langle\cdot,\cdot\rangle$ is symmetric and bilinear, so the ACT with respect to $\langle\cdot,\cdot\rangle$ is canonical with the associated matrix I_n. Hence, $\mathcal{M} = (V, \langle\cdot,\cdot\rangle, R_{\langle\cdot,\cdot\rangle})$ is csc(1). Letting $R_a = aR_{\langle\cdot,\cdot\rangle}$, Proposition 1.3 implies $\mathcal{M}_a = (V, \langle\cdot,\cdot\rangle, R_a)$ is csc(a), and hence cvc(a). Our next example using this construction to illustrate our results concerning decomposability.

Example 6.8. Let $\mathcal{M} = (V, \langle\cdot,\cdot\rangle, R)$ be a model space with $\{e_1, \ldots, e_6\}$ an orthonormal basis for V, $\langle\cdot,\cdot\rangle$ a positive-definite inner product on V, and $R = R_2 \oplus R_0 \oplus R_{-2}$. Define three model spaces $\mathcal{M}_i = (V_i, \langle\cdot,\cdot\rangle|_{V_i}, R_i)$ such that $\dim(V_i) = 2$ and $\mathcal{M} = \mathcal{M}_2 \oplus \mathcal{M}_0 \oplus \mathcal{M}_{-2}$.

When we group components of $\mathcal{M}$ using parentheses in the following proposition, for example, $\mathcal{M} = (\mathcal{M}_2 \oplus \mathcal{M}_0) \oplus \mathcal{M}_{-2}$, we view $\mathcal{M}_2 \oplus \mathcal{M}_0$ as a single model space. Also note that $\ker(R) = V_0$.

Proposition 6.9. *The model space $\mathcal{M}$ in Example 6.8 has the following properties:*

(1) *$\mathcal{M}$ is 3-cvc($[-2,2]$),*
(2) *$\mathcal{M}$ is 4-cvc($[-2,2]$),*
(3) *$\mathcal{M}$ is 5-cvc($[-2,2]$),*
(4) *$\mathcal{M}$ is 6-cvc(0).*

(1) *$\mathcal{M}$ is 3-cvc($[-2,2]$).*

Proof. Consider the decompositon $\mathcal{M} = \mathcal{M}_2 \oplus \mathcal{M}_0 \oplus \mathcal{M}_{-2}$. Since $\mathcal{M}_2$ is csc(2), $\mathcal{M}_2$ is 2-cvc(2). Similarly, $\mathcal{M}_0$ is 2-cvc(0) and $\mathcal{M}_{-2}$ is 2-cvc(-2), so $\mathcal{M}$ is 3-cvc($[-2,2]$) by Corollary 5.7. □

(2) *$\mathcal{M}$ is 4-cvc($[-2,2]$).*

Proof. View $\mathcal{M}$ as $\mathcal{M} = (\mathcal{M}_2 \oplus \mathcal{M}_0) \oplus \mathcal{M}_{-2}$. We claim $\mathcal{M}_2 \oplus \mathcal{M}_0$ is 2-cvc(0). To see this, we take an arbitrary $v \in V$ and verify the three possible cases. First, if $v \in V_2$, take $w \in V_0$. Then $\kappa(v, w) = R(v, w, w, v) = 0$ since v is orthogonal to w and $w \in \ker(R)$. Second, if $v \in V_2$, choose $w \in V_2$ orthogonal to v. This is possible since $\dim(V_0) = 2$. Then $\kappa(v, w) = R(v, w, w, v) = 0$ as before. Third, suppose $v = v_2 + v_0$ for $v_2 \in V_2$ and $v_0 \in V_0$. Take $w \in V_0$ orthogonal to v. Then $\kappa(v, w) = R(v_2 + v_0, w, w, v_2 + v_0) = 0$ since $w \in \ker(R)$. Also, $\mathcal{M}_{-2}$ is 2-cvc(-2), so $\mathcal{M}$ is 4-cvc(-2) by Theorem 5.8.

Now, view $\mathcal{M}$ as $\mathcal{M} = \mathcal{M}_2 \oplus (\mathcal{M}_0 \oplus \mathcal{M}_{-2})$. By a similar argument, replacing v_2 with v_{-2} and V_2 with V_{-2}, $\mathcal{M}_0 \oplus \mathcal{M}_{-2}$ is 2-cvc(0). Since $\mathcal{M}_2$ is 2-cvc(2), Theorem 5.8 tells us $\mathcal{M}$ is 4-cvc(0). Then $\mathcal{M}$ is 4-cvc($[-2, 2]$) by Theorem 4.1. □

(3) $\mathcal{M}$ *is* 5-*cvc*($[-2, 2]$).

Proof. Consider the decomposition $\mathcal{M}$ as $\mathcal{M} = (\mathcal{M}_2 \oplus \mathcal{M}_{-2}) \oplus \mathcal{M}_0$. Because $\mathcal{M}_2$ is 2-cvc(2) and $\mathcal{M}_{-2}$ is 2-cvc(-2), $\mathcal{M}_2 \oplus \mathcal{M}_{-2}$ is 3-cvc($[-2, 2]$) by Theorem 5.6. Since $\mathcal{M}_0$ is 2-cvc(0), then, in particular, $\mathcal{M}$ is 5-cvc(-2) and 5-cvc(2) by Theorem 5.8. Using Theorem 4.1, we conclude that $\mathcal{M}$ is 5-cvc($[-2, 2]$). □

(4) $\mathcal{M}$ *is* 6-*cvc*($[-2, 2]$).

Proof. View $\mathcal{M}$ as $\mathcal{M} = \mathcal{M}_2 \oplus \mathcal{M}_{-2} \oplus \mathcal{M}_0$. Since $\mathcal{M}_2$ is 2-cvc(2), $\mathcal{M}_0$ is 2-cvc(0), and $\mathcal{M}_{-2}$ is 2-cvc(-2), $\mathcal{M}$ is 6-cvc(0) by Corollary 5.9. □

7. Conclusion

In this research, we study k-plane constant vector curvature in finite-dimensional model spaces and generalize this definition to (m, k)-plane constant vector curvature. In particular, we fully classify the topological characteristics of the sets $\mathcal{C}_k$ and $\mathcal{C}_k^m$: both are connected, compact subsets of $\mathbb{R}$, that is, real intervals of the form $[a, b]$. We also consider the relationship between k-plane constant vector curvature and decomposable model spaces. We demonstrate several connections between the k-cvc values of the component spaces and the $(k + 1)$-cvc values of the composite space. As an important corollary, we show every decomposable model space with a positive-definite inner product is k-cvc(ϵ) for some integer $k \geq 2$ and $\epsilon \in \mathbb{R}$. We also give the first specific example of (m, k)-cvc and give a separate example to illustrate our theorems on decomposability. Our research allows us to easily construct model spaces with prescribed curvature values from simpler model spaces.

8. Open Questions

For every answered question, several new ones arise. Here are merely a few possibilities for future research related to k-plane curvature, (m, k)-plane curvature, and their connections to decomposable model spaces.

(1) Complete Calle's work: if a model space $\mathcal{M} = (V, \langle \cdot, \cdot \rangle, R)$ is $(n-1)$-csc(0), does this imply $R \equiv 0$? If not, provide a suitable counterexample. If the theorem does not hold in all generality, what are some restrictions on $\mathcal{M}$ (e.g. if $\mathcal{M}$ is weakly Einstein) that ensure R is trivial?

(2) While our paper introduces (m,k)-plane constant vector curvature, we do not investigate this property extensively. What results do and do not generalize from k-cvc? Are certain theorem only true when $m \geq 2$? A particularly intriguing, though difficult, case is $(n-2,n-1)$-cvc. For example, $(2,3)$-cvc in $\mathbb{R}^4$ is a natural extension of cvc (i.e. $(1,2)$-cvc) in $\mathbb{R}^3$.

(3) In our work, we prove that $\mathcal{M}$ is at least, not exactly, k-cvc($[\epsilon,\delta]$). Are there general methods to determine if a model space does not have k-cvc(γ) for certain γ? Generalize the arguments in [5] to find bounds on $\mathcal{C}_k$.

(4) Another approach to problem (3) is to search for converses (or partial converses) of the theorems and corollaries in Section 5. For example, if $\mathcal{M}$ decomposes into $\mathcal{M}_1$ and $\mathcal{M}_2$, do ${}_1\mathcal{C}_k$ and ${}_2\mathcal{C}_k$ completely determine the $(k+1)$-cvc values of $\mathcal{M}$?

    Conjecture 8.1. *Suppose a model space $\mathcal{M} = (V, \langle\cdot,\cdot\rangle, R)$ decomposes as $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$ and choose an integer $2 \leq k \leq \min(\dim(V_1), \dim(V_2))$. Let ϵ_m and ϵ_M, respectively, be the minimum and maximum of the set ${}_1\mathcal{C}_k \cup {}_2\mathcal{C}_k$. Then $\mathcal{C}_{k+1}$ is exactly $[\epsilon_m, \epsilon_M]$.*

    A possible proof is to show $[\epsilon_m, \epsilon_M]$ and $\mathcal{C}_{k+1}$ are subsets of one another. The containment $[\epsilon_m, \epsilon_M] \subseteq \mathcal{C}_{k+1}$ is a direct consequence of Corollary 5.7. However, the reverse inclusion is more difficult.

(5) A model space $\mathcal{M}$ has *extremal constant vector curvature*, denoted ecvc(ϵ), if ϵ is a bound (lower or upper) on the values in $\mathcal{C}_2$. Analogously, $\mathcal{M}$ has *k-plane extremal constant vector curvature*, written k-ecvc(ϵ), if ϵ is a bound (lower or upper) on the values in $\mathcal{C}_k$. Much is known about extremal constant vector curvature, yet no one has studied k-ecvc. What properties generalize from 2-planes to k-planes? How does this relate to the previous suggestion? For example, consider the following conjecture.

    Conjecture 8.2. *If a model space $\mathcal{M} = (V, \langle\cdot,\cdot\rangle, R)$ is k-ecvc(ϵ) and k-ecvc(δ), then $\epsilon = \delta$ and $\mathcal{M}$ is k-csc(ϵ).*

(6) In Proposition 5.2, we prove that if $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$ and there exists $\epsilon \in {}_1\mathcal{C}_2$ such that $-\epsilon \in {}_2\mathcal{C}_2$, then $\mathcal{M}$ is cvc(0). Is it true the $\mathcal{M}$ is only 2-cvc(0). Is there a similar result for k-cvc?

(7) If we have a decomposable model space $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$, where $\mathcal{M}_1$ is k-cvc(ϵ) and $\mathcal{M}_2$ is k-cvc(δ), is $\mathcal{M}$ k-cvc(γ) for some $\gamma \in \mathbb{R}$? While we prove that $\mathcal{M}$ are $(k+1)$-cvc($[\epsilon,\delta]$), our argument provides little insight on the k-cvc values of $\mathcal{M}$.

(8) Calle proves in [4] that any model space $\mathcal{M}$ with $\dim(\ker(R)) \geq k-1$ is k-cvc(0). Equivalently, if $\dim(\ker(R)) \geq k-1$, then $\mathcal{M}$ is $(1,k)$-cvc(0). With this observation, we can try to extend Calle's argument to general (m,k)-plane curvature.

    Conjecture 8.3. *If $\mathcal{M} = (V, \langle\cdot,\cdot\rangle, R)$ is a model space with $\dim(\ker(R)) \geq k-1$, then $\mathcal{M}$ is (m,k)-cvc(0) for all $1 \leq m \leq k-1$.*

    In a similar manner, one could investigate how the size of the kernel of a decomposable model space impacts its curvature properties.

(9) If ψ is a bilinear function on V, we say ψ is anti-symmetric if $\psi(x,y) = -\psi(y,x)$ for all $x,y \in V$. In [4], Calle presents two examples of k-cvc model spaces using R_ϕ canonical ACTs, where ϕ is a symmetric, bilinear function

on V. What are some illustrative examples of k-cvc with an R_ψ curvature tensor? What differences arise, if any, compared to the symmetric case?

(10) Suppose we have canonical ACTs R_ϕ and R_ψ, where $\operatorname{rank}(\phi), \operatorname{rank}(\psi) \geq 2$ and ϕ, ψ are symmetric, bilinear functions. How do the results of Section 5 change if the model space $\mathcal{M} = (V, \langle\cdot,\cdot\rangle, R)$ is "almost" decomposable, that is, if $\mathcal{M}$ factors into $\mathcal{M}_1 = (V, \langle\cdot,\cdot\rangle, R_1)$ and $\mathcal{M}_2 = (V, \langle\cdot,\cdot\rangle, R_2)$, where $R = R_1 + R_2$? Let us consider a simple example. Since R_ϕ is canonical, the matrix corresponding to ϕ also represents R_ϕ. The same is true for ψ and R_ψ. If ϕ and ψ are diagonalized and there is no overlap between the entries, i.e. $\phi_{ii} \neq 0$ implies $\psi_{ii} = 0$ and vice versa, then R decomposes as $R_\phi \oplus R_\psi$. However, if there is overlap, e.g. $\phi_{11} = \psi_{11} \neq 0$, then $R \neq R_\phi \oplus R_\psi$, but one could say $\mathcal{M}$ is "almost" decomposable. Investigate this new case and compare to the results in Section 5.

(11) As mentioned in Section 5, every three-dimensional model space equipped with a positive-definite inner product is $\operatorname{cvc}(\epsilon)$ for a unique value ϵ [18]. (See [7] for a counterexample in the non-degenerate case.) By Corollary 5.5, we now know every decomposable model space is k-$\operatorname{cvc}(\epsilon)$ for some integer $k \geq 2$ and $\epsilon \in \mathbb{R}$. So, are all model spaces k-cvc for some $k \geq 3$ and $\delta \in \mathbb{R}$? Is this true only in the positive-definite case? If so, provide counterexamples.

(12) Investigate the k-cvc values of *quotient model spaces*. Given a model space $\mathcal{M} = (V, \langle\cdot,\cdot\rangle, R)$, we define the *quotient* of $\mathcal{M}$ by the kernel to be $\overline{\mathcal{M}} = (\overline{V}, \overline{\langle\cdot,\cdot\rangle}, \overline{R})$, where $\overline{V} = V/\ker(R)$ and $\pi^*\overline{R} = R$. It requires more care to ensure $\overline{\langle\cdot,\cdot\rangle}$ is well-defined. What is the relationship between the k-cvc values of $\mathcal{M}$ and those of $\overline{\mathcal{M}}$, if any?

These are but a few of the possible directions for future research.

9. Appendix

We present an alternate, more detailed proof of Theorem 4.1. Our approach is adapted from the proof that $\mathrm{SO}(n)$ is path-connected [17].

Theorem 4.1 (revisited). *If $\mathcal{M} = (V, \langle\cdot,\cdot\rangle, R)$ is a model space with $n = \dim(V) \geq 3$, then $\mathcal{C}_k^m$ is connected for all $m, k \in \mathbb{Z}$ such that $1 \leq m < k \leq n$. In particular, $\mathcal{C}_k$ is connected for all $2 \leq k \leq n$.*

Proof. Let $\mathcal{M}$ be a model space as above, and suppose $\mathcal{M}$ is (m,k)-$\operatorname{cvc}(\epsilon)$ and (m,k)-$\operatorname{cvc}(\delta)$ for scalars $\epsilon < \delta$. Take any m-plane P, and let $\mathcal{B}_P = \{v_1, \ldots, v_m\}$ be an orthonormal basis for P. By definition, there are k-planes $L_0, L_1 \subset V$ containing P so that $\mathcal{K}(L_0) = \epsilon$ and $\mathcal{K}(L_1) = \delta$. We want to show $\mathcal{M}$ is (m,k)-$\operatorname{cvc}([\epsilon,\delta])$.

We can find orthonormal bases for L_0 and L_1, respectively, say

$$\mathcal{B}_0 = \mathcal{B}_P \cup \{e_{m+1}, \ldots, e_k\} \text{ and } \mathcal{B}_1 = \mathcal{B}_P \cup \{\widetilde{e}_{m+1}, \ldots, \widetilde{e}_k\},$$

so that $\mathcal{B}_0$ spans L_0 and $\mathcal{B}_1$ spans L_1. Extend $\mathcal{B}_0$, $\mathcal{B}_1$ to orthonormal bases for V:

$$\mathcal{V}_0 = \{v_1, \ldots, v_m, e_{m+1}, \ldots, e_k, f_1, \ldots, f_j\},$$
$$\mathcal{V}_1 = \{v_1, \ldots, v_m, \widetilde{e}_{m+1}, \ldots, \widetilde{e}_k, \widetilde{f}_1, \ldots, \widetilde{f}_j\}.$$

We want a continuous deformation of orthonormal bases in which $v_1, \ldots, v_m$ always remain basis vectors. Our method is to carefully spin the vectors in $\mathcal{V}_0$ until the first k align with those of $\mathcal{V}_1$, which requires at most k rotations.

Let $A_t \in \mathrm{GL}_n(\mathbb{R})$, where $0 \le t \le 1$, be transition matrices. Here, $\mathrm{GL}_n(\mathbb{R})$ is the general linear group of degree n. Let A_1 denote the transition matrix from $\mathcal{V}_0$ to $\mathcal{V}_1$. We need a continuous path

$$\begin{aligned}\alpha : [0,1] &\to \mathrm{SO}(n)\\ t &\mapsto A_t\end{aligned}$$

such that $\alpha(0) = A_0 = I_n$ and $\alpha(1) = A_1$, where $\widetilde{e}_q = Ae_q$, $\widetilde{f}_r = Af_r$, and $v_s = A_t v_s$ for all $s = 1, \dots, m$ and all t. The first m rotations are trivial. For all s, define $\alpha_s : [0,1] \to \mathrm{SO}(n)$ by $\alpha_s(t) \equiv I_n$. Then $\alpha_s(0)v_s = v_s$ and $\alpha_s(1)v_s = v_s$.

Next, we find a path $\alpha_{m+1} : [0,1] \to \mathrm{SO}(n)$ so that $\alpha_{m+1}(0)e_{m+1} = e_{m+1}$ and $\alpha_{m+1}(1)e_{m+1} = \widetilde{e}_{m+1}$. Choose a new basis for V. Let u be a unit vector such that $\widetilde{e}_{m+1} \in \operatorname{span}\{e_{m+1}, u\}$ and call $U = \operatorname{span}\{e_{m+1}, u\}$. Arbitrarily complete an orthonormal basis for V, say $\mathcal{B} = \mathcal{B}_P \cup \{e_{m+1}, u, b_{m+3}, \dots, b_n\}$. Next, construct a rotation of U that leaves vectors in $U^\perp$ unaffected. Since e_{m+1} and $\widetilde{e}_{m+1}$ are both unit length, there is an angle θ so that

$$\widetilde{e}_{m+1} = \begin{bmatrix} I_m & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & I_{n-m-2} \end{bmatrix} e_{m+1}.$$

This matrix represent a counterclockwise rotation through θ from e_{m+1} to $\widetilde{e}_{m+1}$. Therefore, a suitable path α_{m+1} is

$$\alpha_2(t) = \begin{bmatrix} I_m & 0 & 0 & 0 \\ 0 & \cos(\theta t) & -\sin(\theta t) & 0 \\ 0 & \sin(\theta t) & \cos(\theta t) & 0 \\ 0 & 0 & 0 & I_{n-m-2} \end{bmatrix}.$$

One can verify that $\alpha_{m+1}(t) \in \mathrm{SO}(n)$, $\alpha_{m+1}(0) = I_n$, and $\alpha_{m+1}(t)e_{m+1} = \widetilde{e}_{m+1}$. The vectors in $\mathcal{B}_P$ are invariant under α_{m+1} because each one is orthogonal to $\alpha_m(e_{m+1}) = e_{m+1}$ and e_{m+1}. (This happens since $\mathcal{B}_P \subseteq U^\perp$, the complement of the subspace where the rotation occurs.) We get a new orthonormal basis for V:

$$\mathcal{B}_P \cup \{e_{m+1}, \alpha_{m+1}(1)e_{m+2}, \dots, \alpha_{m+1}(1)e_k, \dots, \alpha_{m+1}(1)f_1, \dots, \alpha_{m+1}(1)f_j\}.$$

Now, recursively apply this process. That is, find paths $\alpha_{m+2}, \dots, \alpha_k$ taking orthonormal bases to orthonormal bases that rotate the first k vectors of $\mathcal{V}_0$ until they align with those of $\mathcal{V}_1$. Since $\alpha_{i+1}(0) = \alpha_i(1)$ for $i = 1, \dots, k-2$, we can concatenate these paths to obtain a single path $\alpha : [0,1] \to \mathrm{SO}(n)$ given by

$$\alpha = \alpha_1 * \alpha_2 * \cdots \alpha_k,$$

where $\alpha(0) = \alpha_1(0) = A_0$ and $\alpha_1 = A_1$. Set $A_t = \alpha(t)$ for $0 \le t \le 1$. Since α_i leaves the vectors in $\mathcal{B}_P$ invariant, $v_1, \dots, v_m$ are in the orthonormal basis of every intermediate transition matrix A_t. Letting L_t be the plane spanned by the first k vectors in $\mathcal{V}_t$, this means L_t contains P. We have

$$\begin{aligned}\mathcal{K}(A_0L_0) &= \mathcal{K}(v_1, \dots, v_m, e_{m+1}, \dots, e_k) = \epsilon,\\ \mathcal{K}(A_tL_0) &= \mathcal{K}(A_tv_1, \dots, A_tv_m, A_te_{m+1}, \dots, A_te_k) = \mathcal{K}(L_t),\\ \mathcal{K}(A_1L_0) &= \mathcal{K}(v_1, \dots, v_m, \widetilde{e}_{m+1}, \dots, \widetilde{e}_k) = \mathcal{K}(L_1) = \delta.\end{aligned}$$

Since the map $L \mapsto \mathcal{K}(L)$ is continuous, $\mathrm{Gr}_k(V)$ is connected [12], and the continuous image of a connected set is connected, for all $\gamma \in [\epsilon, \delta]$, there exists $t \in [0,1]$

such that L_t contains P and $\mathcal{K}(L_t) = \gamma$ by the Intermediate Value Theorem. Hence, $\mathcal{M}$ is (m, k)-cvc($[\epsilon, \delta]$). Since ϵ and δ are arbitrary, $\mathcal{C}_k^m$ is connected. In particular, letting $m = 1$, $\mathcal{C}_k$ is connected for all $2 \leq k \leq n$. □

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