A METHOD TO FINDING INVARIANTS USING SECTIONAL CURVATURE CSUSB REU, 2019

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ABSTRACT. This research discovers a new method to finding invariant subspaces. Invariant subspaces are important for finding the shape of matrices within the structure group. This research also discovers an example of a finite structure group in the 3-dimensional case. This method uses what is known about sectional curvature and links it to the idea of invariant subspaces.

1. INTRODUCTION

Definition 1.1. An **algebraic curvature tensor** R is a function on a vector space V defined as $R : V \times V \times V \times V \to \mathbb{R}$ satisfying, for all $x, y, z, w, v \in V$ and $c \in \mathbb{R}$, the following properties:

(1) Multilinearity:
$$R(cx + y, z, w, v) = cR(x, z, w, v) + R(y, z, w, v),$$
(2) Antisymmetric in the first two slot:
$$R(x, y, z, w) = -R(y, x, z, w),$$
(3) Symmetric in the (1,2)-(3,4) slots:
$$R(x, y, z, w) = R(z, w, x, y),$$
(4) Bianchi Identity:
$$R(x, y, z, w) + R(y, z, x, w) + R(z, x, y, w) = 0.$$

In the case that we evaluate $R(e_i, e_j, e_k, e_l)$, then we write R_{ijkl}, where $\{e_1, e_2, \ldots, e_n\}$ is a basis. The use of this function R is important in the study of Differential Geometry.

Definition 1.2. In a vector space V, a **symmetric bilinear form** $\varphi : V \times V \to \mathbb{R}$ where, for all $x, y, x_1, x_2 \in V$ and $c \in \mathbb{R}$, φ is:

(1) Symmetric: $\varphi(x, y) = \varphi(y, x)$,
(2) Linear in the first slot: $\varphi(cx_1 + x_2, y) = c\varphi(x_1, y) + \varphi(x_2, y)$.

Definition 1.3. An **inner product** or **metric** $\langle \cdot, \cdot \rangle$ on a vector space V is a symmetric bilinear form.

(1) **positive definite** if $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ if and only if $x = 0$,
(2) **non-degenerate** for all $x \neq 0 \in V$, there exists $y \in V$ such that $\langle x, y \rangle \neq 0$.

Remark 1.4. Inner products may be represented as $\langle x, y \rangle$ or $\varphi(x, y)$ in this paper. For the purposes of this paper, it is assumed that $\langle \cdot, \cdot \rangle$ is a positive-definite inner product.

Definition 1.5. Let φ be a summetric bilinear form. A **canonical algebraic curvature tensor** R_φ is an algebraic curvature such that, for all $x, y, z, w \in V$,

$$R_\varphi(x, y, z, w) = \varphi(x, w)\varphi(y, z) - \varphi(x, z)\varphi(y, w).$$

Definition 1.6. A **model space** is defined as $\mathfrak{M} := (V, \varphi, R)$, where V is a vector space, φ is a metric, and R is an algebraic curvature tensor.

Definition 1.7. The **structure group** of a model space is, for all $x, y, z, w \in V$,

$$G_{\mathfrak{M}} = \{A \in GL(V) : R(x, y, z, w) = R(Ax, Ay, Az, Aw) \text{ and } \varphi(x, y) = \varphi(Ax, Ay)\}$$

Definition 1.8. Given a model space, the **sectional curvature** κ of a 2-plane $\pi = span\{x, y\}$ is a function that inputs a 2-plane and outputs a real value, defined by

$$\kappa(\pi) = \kappa(\langle x, y\rangle) = \frac{R(x, y, y, x)}{\langle x, x\rangle\langle y, y\rangle - \langle x, y\rangle^2}.$$

Sectional curvature is independent of the basis used. Thus if $\pi = span\{v, w\}$ rather than $span\{x, y\}$, then $\kappa(\langle x, y\rangle) = \kappa(\langle v, w\rangle)$.

Definition 1.9. An **invariant subspace** of a model space is a subspace W of the vector space V such that for any element $A \in G_{\mathfrak{M}}$, W will map to itself. It may also be expressed as:

If W is an invariant subspace, where $W \subseteq V$, then for any $A \in G_{\mathfrak{M}}$, $A : W \to W$.

Definition 1.10. If W is a subspace of V, then $W^\perp = \{v \in V : v \perp W\}$. $W^\perp$ is referred to as the **perp space** of W.

2. General $G_{\mathfrak{M}}$ Facts

Some drawbacks about the algebraic curvature tensor R is that the value may depend on the basis used to reference V. We take the following example to illustrate.

Example 2.1. Suppose we have a 3-dimensional vector space V with an orthonormal basis $\{e_1, e_2, e_3\}$, and suppose that the following values of R are defined as, up to the curvature symmetries:

$$\begin{aligned} R_{1223} &= 2, \\ R_{2332} &= 3, \\ \text{anything else} &= 0. \end{aligned}$$

However, if we consider another basis $\{f_1, f_2, f_3\}$, where $f_1 = e_1 + e_2$, $f_2 = e_2$, and $f_3 = e_3$. Then if we were to calculate $R(f_1, f_2, f_2, f_3)$, then

$$\begin{aligned} R(f_1, f_2, f_2, f_3) &= R(e_1 + e_2, e_2, e_2, e_3), \\ &= R(e_1, e_2, e_2, e_3) + R(e_3, e_2, e_2, e_3), \\ &= R_{1223} + R_{2332}, \\ &= 2 + 3, \\ &= 5. \end{aligned}$$

We could write the change of basis of $\{e_i\}_{i=1}^3$ to $\{f_i\}_{i=1}^3$ as a linear transformation, or matrix, with

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and declare that this matrix A cannot be in the structure group. Thus we can see how the structure group may need to come into play. If we take the collection of matrices that preserve the algebraic curvature tensor R, and the inner product φ, what would they look like?

This leads us to what we know about invariant subspaces. Given that we know about matrices, we want to find out the spaces that map to themselves, without trying to find the matrices blindly. The following example illustrates what we may gather from invariant subspaces.

Example 2.2. If we have a 5-dimensional vector space V with the orthonormal basis $\{e_1, e_2, e_3, e_4, e_5\}$, and we know that $W = span\{e_1, e_2\}$ is an invariant subspace of V, then it is necessarily true that any $A \in G_{\mathfrak{M}}$ must look like

$$A = \left[\begin{array}{cc|ccc} a_1 & b_1 & c_1 & d_1 & e_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 \\ \hline 0 & 0 & c_3 & d_3 & f_3 \\ 0 & 0 & c_4 & d_4 & f_4 \\ 0 & 0 & c_5 & d_5 & f_5 \end{array}\right]$$

since A must send W to itself.

The upper-right corner of the matrix still has elements in terms of e_1 and e_2, however. But, what is known about perp spaces will come to be convenient for reducing that space even more.

It is known that when we have a non-degenerate vector space V, and an invariant subspace W, that $W^\perp$ is also an invariant subspace.

This is evident from the following theorem.

Theorem 2.3. *If W is an invariant subspace of V, and $w \in W^\perp$, then $Aw \in W^\perp$, for $A \in G_{\mathfrak{M}}$.*

Proof. Let $v \in W$ such that W is an invariant subspace of V. Let $w \in W^\perp$ and let $A \in G_{\mathfrak{M}}$.

Since W is an invariant subspace, and $v \in W$, then $Av \in W$, which implies $A^{-1}v \in W$. Recall $\varphi(v, w) = 0$ by the definition of $W^\perp$. We take Aw, and find that

$$\varphi(v, Aw) = A\varphi(A^{-1}v, w),$$

since $A^\perp \in W$, then

$$A\varphi(A^{-1}v, w) = 0,$$

$$\implies \varphi(v, Aw) = 0,$$

$$\implies Aw \in W^\perp.$$

□

Thus in example 2.2, we can further reduce A to

$$A = \left[\begin{array}{cc|ccc} a_1 & b_1 & 0 & 0 & 0 \\ a_2 & b_2 & 0 & 0 & 0 \\ \hline 0 & 0 & c_3 & d_3 & f_3 \\ 0 & 0 & c_4 & d_4 & f_4 \\ 0 & 0 & c_5 & d_5 & f_5 \end{array}\right].$$

Meanwhile, we also want to understand what this means for the structure group $G_{\mathfrak{M}}$. We see that if we have an invariant subspace in V, then our structure group elements become more defined.

It is known according to [2] that the $kerR$ is also an invariant subspace. Some studies use this fact to aid in their research, however that will not be the case in this paper, although it is important to mention regardless.

3. Sectional Curvature

So how exactly can we determine invariant subspaces then? This is what this research paper will focus on, and we begin with some facts from [3].

Lemma 3.1. *Let $\varphi \in S^2(V^*)$, and suppose $R_\varphi \neq 0$. If π is a 2-plane whose sectional curvature is extremal, then there exists an orthonormal basis of eigenvectors for π.*

Theorem 3.2 (Sectional Curvature is Bounded)**.** *Let $\varphi \in S^2(V^*)$, and let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of φ, repeated according to multiplicity. Let m and M be the* ***minimum and maximum*** *of the set $\{\lambda_i\lambda_j | i \neq j\}$, respectively. The set of sectional curvatures of R_φ is exactly the interval $[m, M]$.*

We can come up with an example to motivate the uses of sectional curvature. One things we can do are constructing a basis which is inspired by geometry.

Suppose we have a model space where $\dim V = 3$ and V has a unique maximal and minimal sectional curvature, made by two distinct 2-planes, called π_m and π_M respectively. When we visualize these planes, they must intersect at some line of intersection, call this line $\pi_{m\cap M}$. Because $\pi_{m\cap M}$ is 1-dimensional, then we can span that line with a vector $v_{m\cap M}$. This vector will be a basis vector for V, and we can obtain two other basis vectors for V by taking a vector to span $\pi_m^\perp$ and $\pi_M^\perp$, which each of the previous spaces must be 1-dimensional. And behold, we now have some specific method to constructing a basis that (although nonchalant) can aid in one's studies.

This was only to illustrate how to construct a basis which, until further research shows, may or may not be useful, but at least is some consistent basis for $\dim V = 3$ with extremal sectional curvature.

4. Results

Lemma 4.1 (The Structure Group Preserves Sectional Curvature)**.** *If $\mathfrak{M}$ is the model space $(V, \langle\cdot,\cdot\rangle, R)$ and V has a 2-plane π, then for any $A \in G_{\mathfrak{M}}$, $\kappa(A\pi) = \kappa(\pi)$.*

Proof. Suppose that $\pi = span\{v, w\}$ is a 2-plane. Then

$$\kappa(\pi) = \frac{R(v,w,w,v)}{\langle v,v\rangle\langle w,w\rangle - \langle v,w\rangle^2} .$$

Given $A \in G_{\mathfrak{M}}$, then

$$\kappa(A(\pi)) = \kappa(Av,Aw) = \frac{R(Av,Aw,Aw,Av)}{\langle Av,Av\rangle\langle Aw,Aw\rangle - \langle Av,Aw\rangle^2}.$$

Since $A \in G_{\mathfrak{M}}$, then $A^*R = R$ and $A^*\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle$. Thus

$$\kappa(A(\pi)) = \frac{R(Av,Aw,Aw,Av)}{\langle Av,Av\rangle\langle Aw,Aw\rangle - \langle Av,Aw\rangle^2} = \frac{R(v,w,w,v)}{\langle v,v\rangle\langle w,w\rangle - \langle v,w\rangle^2},$$

$$\implies \kappa(A(\pi)) = \kappa(\pi).$$

□

Theorem 4.2 (Unique Minimal and Maximal 2-planes are Invariant)**.** *Suppose* $\mathfrak{M} = (V,\langle\cdot,\cdot\rangle,R)$ *and that there is a unique* 2*-plane* π *such that* $\kappa(\pi)$ *is an extrema. Then* $A : \pi \to \pi$*, where* $A \in G_{\mathfrak{M}}$*.*

Proof. If we have only **one** 2-plane π such that $\kappa(\pi) = m$ or M, and that $A \in G_{\mathfrak{M}}$ preserves the sectional curvature, then A must map π to itself.

□

The consequences of this theorem are significant. If it happens to be that we have a model space with one 2-plane for each of the bounds, then we know then that we have an invariant subspace. It may be more obvious so in this example.

Example 4.3 (A Finite Structure Group)**.** Suppose we have a model space $\mathfrak{M} = (V,\langle\cdot,\cdot\rangle,R_\varphi)$ where $\dim V = 3$, and $\{e_1,e_2,e_3\}$ is an orthonormal basis for V, and that $\langle e_i,e_j\rangle = \delta_{ij}$, and that

$$\varphi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

We can identify the 2-planes which bound the sectional curvature κ. Here, we can identify that the set of eigenvalues of φ are $\{1,2,3\}$. According to [3], the set of sectional curvatures of R_φ are bounded as $[2,6]$. The 2-plane π_m where $\kappa(\pi_m) = 2$ must be spanned by the vectors $\{e_1,e_2\}$ since, by [3], and π_M where $\kappa(\pi_M) = 6$ is similarly spanned by $\{e_2,e_3\}$.

According to the Theorem 4.2, then when we consider $A \in G_{\mathfrak{M}}$, then we must preserve the sectional curvature, and map toward ourselves if there is only one 2-plane to produce an extrema.

If we consider the 2-plane π_m, then $\kappa(\pi_m) = \kappa(A\pi_m) = 2$. Thus $A\pi_m = \pi_m$. If we replace the π_m with the vectors that span the plane, $\{e_1,e_2\}$, then we see that A must leave that space invariant. Since $\{e_1,e_2\}$ is an invariant subspace, it it also true that $\{e_1,e_2\}^\perp = \{e_3\}$ is also an invariant subspace. Thus,

$$A = \left[\begin{array}{cc|c} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ \hline 0 & 0 & c_3 \end{array}\right].$$

Recall though that since we **also** have to consider the 2-plane which produces maximal sectional curvature, that A will further be restricted. Thus when π_M was spanned by $\{e_2, e_3\}$, then we have another invariant subspace, where also the perp space is an invariant. A is now restricted to the form

$$A = \left[\begin{array}{c|cc} a_1 & 0 & 0 \\ \hline 0 & b_2 & 0 \\ 0 & 0 & c_3 \end{array}\right].$$

Thus when we operate A with any basis vector e_i, then we have $Ae_1 = a_1 e_1, Ae_2 = b_2 e_2$, and $Ae_3 = c_3 e_3$. If we recall that $\langle e_i, e_j \rangle = \delta_{ij}$, then we see that for e_1 that

$$\begin{aligned} 1 &= \langle e_1, e_1 \rangle, \\ &= \langle Ae_1, Ae_1 \rangle, \\ &= \langle a_1 e_1, a_1 e_1 \rangle, \\ &= (a_1)^2 \langle e_1, e_1 \rangle, \\ &= (a_1)^2, \\ \implies a_1 &= \pm 1. \end{aligned}$$

We repeat for e_2 and e_3. Thus $a_1, b_2, c_3 = \pm 1$. Therefore, of all combinations, we see here that we have a structure group of 8 elements. Listed, they are

$$G_{\mathfrak{M}} = \{ \begin{bmatrix} 1&0&0\\0&1&0\\0&0&1 \end{bmatrix}, \begin{bmatrix} -1&0&0\\0&1&0\\0&0&1 \end{bmatrix}, \begin{bmatrix} 1&0&0\\0&-1&0\\0&0&1 \end{bmatrix}, \begin{bmatrix} -1&0&0\\0&-1&0\\0&0&1 \end{bmatrix}, \begin{bmatrix} 1&0&0\\0&-1&0\\0&0&-1 \end{bmatrix}, \begin{bmatrix} 1&0&0\\0&1&0\\0&0&-1 \end{bmatrix}, \begin{bmatrix} -1&0&0\\0&1&0\\0&0&-1 \end{bmatrix}, \begin{bmatrix} -1&0&0\\0&-1&0\\0&0&-1 \end{bmatrix} \}.$$

In fact, we have covered the structure groups for which the φ matrix is diagonal and has unique extremal 2-plane sectional curvature.

5. Open Questions

- Using the minimum and maximum sectional curvature allows for 3-dimensional cases to be solved with ease. Is there a way this method can be generalized for larger dimensional cases?
- What happens when we have multiple 2-planes which provide an extremal sectional curvature?
- In the case where $R_\tau = R_\varphi + R_\psi$, where φ is positive definite, is it true that when $R_\tau = R_\varphi + R_{\psi'}$ that $\psi = \pm\psi'$?
- In the model spaces of varying signatures, what can we observe when we have at least one extremal 2-plane spanned by 2 vectors e_i and e_j where $\langle e_i, e_j \rangle = -\langle e_i, e_j \rangle$?
- If we investigate the kernels of the model spaces, we also have an invariant subspace, discovered from [2]. Is there a way to connect this invariant subspace with our method of finding invaraint subspaces?
- Let α_i be a tensor of any type. If we have a model space $\mathfrak{M}_1 = (V, \alpha_1)$ and find the structure group $G_{\mathfrak{M}_1}$, and for the model space $\mathfrak{M}_2 = (V, \alpha_1, \alpha_2)$, what does the structure group look like for $G_{\mathfrak{M}_2}$. What does $G_{\mathfrak{M}_i}$ look like for $\mathfrak{M}_i$?

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