An Algorithm to find Linking Matrices from Crushtaceans

Madeline Brown[1] and Rolland Trapp[2]

[1]Scripps College, Claremont, California, United States
[2]California State University San Bernardino, San Bernardino, California, United States

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Abstract

Hyperbolic links are a topic of interest in knot theory. Two special types of hyperbolic links known as *nested links* and *fully augmented links* (FALs), are of particular interest. Both of these are used to study specific topological invariants related to their complements. Deriving the *knot complement* involves a process called *cellular decomposition*. A further result of deriving the knot complement is a special type of graph called the *crushtacean*, used to represent the link as a 2D projection. Moreover, a *linking matrix* can be derived from the original link, providing a useful model of the structure.

It is also possible to reconstruct a link from its crushtacean. However, the process is long and does not lend itself well to computation. We describe an algorithm to quickly determine the linking matrix of a nested link or FAL from its crushtacean. This algorithm omits several steps previously required, and lends itself well to computational methods. Further, we outline an interesting relationship between linking matrices and their associated links, determining a way to distinguish links with homeomorphic complements.

1 Introduction & Background

A **knot** is a closed loop that exists in S^3. One way to imagine this is to tie a knot in a piece of string and then glue the ends together. This is a knot because there is no way to untangle it. An **unknot**, also known as a **trivial knot** is a circle. A knot contains one component while a **link** contains more than one component.

1.1 Fully Augmented Links & Nested Links

Hyperbolic links exist in S^3 and their complements have a unique hyperbolic metric. A link complement is the space surrounding the link , and a unique hyperbolic metric means that homeomorphic hyperbolic knot complements are isometric [1]. Fully augmented links (FALs) and nested links are two types of hyperbolic links that can be formed from link diagrams. Let $\mathscr{L}$ be a link with alternating crossings in each twist region. To form a FAL, place an unknotted component, known as a crossing circle, around each twist region. Then, remove all full twists such that only half-twists or no twists remain. If the number of twists in the region is odd, the result is a half twist, and if there are an even number of twists, then no twists remain. In FALs, each crossing circle is punctured twice by one or two knot circles [2]. An example of an FAL formed from a link can be seen in Figure 1.

Nested links are formed in the same manner; however, more than two knot circles can exist. By drawing multiple crossing circles nested inside each other, each crossing circle is still only punctured twice [3]. Please refer to Figure 2 for an example.

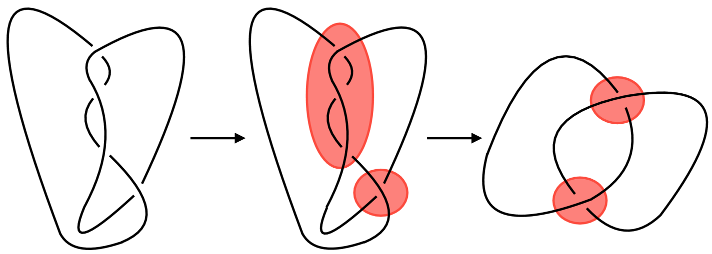

Figure 1: All full twists are removed from $\mathscr{L}$ resulting in two half twists.

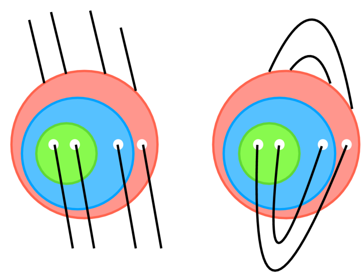

Figure 2: The smallest crossing circle "nests" two strands. The medium sized crossing circle nests the smallest crossing circle and another strand. The largest crossing circle nests the medium sized crossing circle and one more strand. The image on the right is an example of the strands connected as knot circles.

1.2 Cell Decomposition

We can determine the shape of the hyperbolic link complement through a process called **cell decomposition** that takes place in S^3. Because the complement is a complete hyperbolic manifold, it can be thought of as different polyhedra glued together in accordance with the structure of the manifold. This gluing is known as a **gluing pattern**. The complement has 0, 1, and 2-cells. 0-cells correspond to knot circles and 1-cells are intersections of 2-cells, which are crossing circles and the projection plane. The first step of the process is to cut along the planar 2-cells, splitting them into two isometric regions. This step is known more mathematically as "the pita bread slice," or "butterflied." One can think of it as slicing the crossing circle and then flattening each half. The next step is to "pinch" the crossing circles along their diameter such that the diameter has shrunken to an ideal point. Next, shrink the 1-cells (shown as black edges in Figure 3). We now have the link complement. To obtain the graphical representation of a FAL or nested link, we must obtain the dual to the nerve, also known as the crushtacean [4]. A **crushtacean** is a simple, trivalent, planar graph that contains painted and unpainted edges [5]. For this paper, painted edges will be drawn in color and unpainted edges in black. To determine the crushtacean, place a vertex in the center of each shaded region. Connect vertices on the boundary of the face to the vertex in the center, painting the edge in accordance with the vertices on the boundary [4].

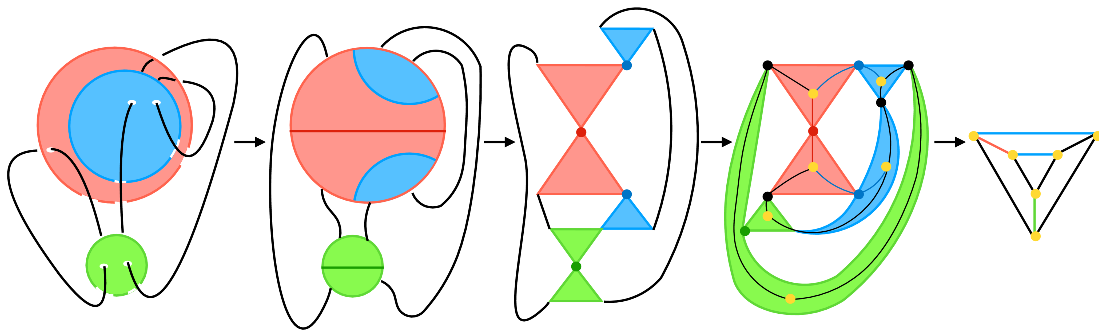

Figure 3: Cellular decomposition from a link to its complement. The final step shows how the crushtacean can be obtained.

Another example of a crushtacean can be seen in Figure 4. Painted edges of a crushtacean correspond to crossing circles and unpainted edges correspond to knot circles. Painted edges are either in the form of perfect matchings, balanced trees, or balanced forests.

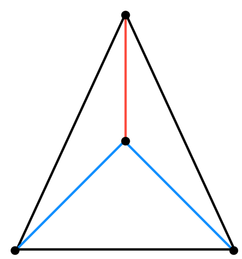

Figure 4: A crushtacean with painted and unpainted edges.

1.3 Balanced Trees

A balanced tree admits an involution, φ that fixes an edge. Let Γ be a crushtacean with a balanced tree, t. Let ε be a painted edge in the tree. Then $\varphi^2(\varepsilon) = \varepsilon$ is an involution. Let v be an endpoint of ε. $\varphi(v)$ and v are known as **mirrored vertices**. Each tree consists of one **balanced edge**, β such that $\varphi(\beta) = \beta$. An example of a tree with these labelings can be found in Figure 5. The tree excluding the balanced edge, $(t \setminus \beta)$ contains two isomorphic subtrees denoted τ and $\varphi(\tau)$ where v and ε exist in τ and $\varphi(v)$ and $\varphi(\varepsilon)$ exist in $\varphi(\tau)$. In other words τ contains the preimage of the vertices and edges while $\varphi(\tau)$ contains the image of the vertices and edges. Each endpoint of the balanced edge is considered to be the root of its corresponding subtree.

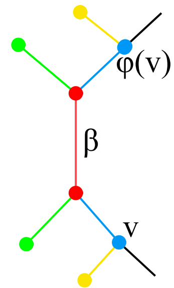

Figure 5: $\varphi(v)$ and v are mirrored vertices. β is the balanced edge. Each pair of colors corresponds to a pair of mirrored vertices.

If all the trees in the crushtacean are of length one, this is known as a **perfect matching**. In perfect matchings, vertices of painted edges do not connect to other painted edges. If there are multiple balanced trees, or a combination of balanced trees and perfect matchings, this is known as a **balanced forest**.

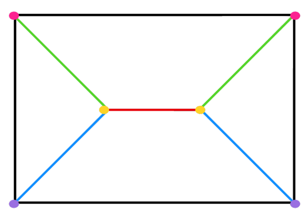

Figure 6: The red edge is the balanced. The pairs of colored vertices represent mirrored vertices.

The significance of crushtaceans is that they are graphical representations of fully augmented links (FALs) and nested links. Painted edges in the crushtacean correspond to crossing circles in links, and unpainted edges correspond to knot circles. Perfect matchings coincide with FALs and balanced trees/forests coincide with nested links. In FALs, each crossing circle is punctured twice by one or two knot circles. Each crossing circle is represented by one painted edge in the crushtacean. For nested links, more than two knot circles can exist, however each crossing circle is still only punctured twice. This is drawn as multiple crossing circles nested inside each other. In the crushtacean, the balanced edge represents the crossing circle containing all other crossing circles within the tree. The other nested crossing circles correspond to two painted edges of the same color in the balanced tree. Different ”clusters” of nested crossing circles correspond to different trees in the crushtacean.

Theorem 1.1. *The number of crossing circles is $\frac{v}{2}$ in nested links and FALs.*

Proof. Suppose Γ is a crushtacean with a total of v vertices and E edges, where p is the number of painted edges and u is the number of unpainted edges. Since FALs can correspond to perfect matchings and nested links can correspond to balanced trees and forests, let us divide the proof into two parts.

Perfect Matchings: Let c be the number of crossing circles for Γ. Because of the trivalent nature of crushtaceans, each vertex has three edges. In a perfect matching, painted edges do not share vertices, so each vertex can only have one painted edge. Because painted edges correspond to crossing circles, this means $p = c$ and $c = \frac{1}{3}E$. In a finite graph, the sum of degrees is equal to twice the number of edges. Since the crushtacean is trivalent, there will be 3 edges per vertex, so $E = \frac{3v}{2}$. Using substitution, we can see that $c = \frac{v}{2}$, and $u = v$.

Balanced Trees and Balanced Forests: The nature of a balanced tree means that two branches are connected by a balanced edge, and a single edge joins the same vertices. In order for this to happen and not result in a multiedge, every edge in the tree that is not a balanced edge needs two painted edges of the same color, one in front and one behind the balanced edge. This results in each crossing circle corresponding to two painted edges of the same color, and one balanced edge that has a distinct color.

In a tree, t with v_t vertices and E_t edges, $E_t = v_t - 1$, so t has $v_t - 1$ edges. The balanced edge corresponds to one crossing circle, so there are $v_t - 2$ edges left in the tree. (This means the number of painted edges for a single tree is $v_t - 2$). Thus, there are $\frac{v_t-2}{2}$ more crossing circles, so in total, $c = 1 + \frac{v_t-2}{2} = \frac{v_t}{2}$.

For a balanced forest, let T be the total number of trees. Each tree has $v_t - 1$ edges, so there are $v - T$ painted edges for the forest. Since each balanced edge in a tree corresponds to 1 crossing circle, there are $v_t - 2$ edges left in each tree, and $v - 2T$ edges left in the forest. For the total number of crossing circles, $c = T + \frac{v-2T}{2} = \frac{v}{2}$.

Since $c = \frac{v}{2}$ for perfect matches, balanced trees, and balanced forests, the number of crossing circles is the same for nested links and FALs. □

Corollary 1.1.1. *The number of knot circles is less than or equal to the number of crossing circles for FALs and nested links.*

Proof. Suppose Γ is a crushtacean with v vertices and E edges. Let p be the number of painted edges, u be the number of unpainted edges, c be the number of crossing circles, and k be the number of knot circles.

FALs: A single knot circle uses at least two edges because in order for a knot circle to use only one edge, two edges must have the same vertices. In a perfect matching, this would result in a multiedge; however, crushtaceans do not contain multiedges. This means a knot circle must use at least two edges in a perfect matching. Thus, the maximum number of knot circles for a single crushtacean is $\frac{v}{2}$. Because $c = \frac{v}{2}$, the number of knot circles, k cannot be greater than the number of crossing circles. It is possible for $k < c$, so k must be less than or equal to c.

Nested Links: Suppose the crushtacean is spanned by a balanced forest, F with T trees. Let ℓ be a leaf with two unpainted edges. Only one of the unpainted edges can connect to the pair of mirrored vertices, since both connecting would result in a multiedge. This means in a leaf, there is at most one edge that connects two mirrored vertices. For the other unpainted edge, we must consider the structure of the balanced edge. A balanced edge has two mirrored vertices, but in order to not result in a multiedge, these two vertices are connected by two different edges. These two distinct edges correspond to the same knot circle by definition of mirrored vertices. Let $d = u - k$ be the difference between the number of unpainted edges and the number of knot circles. For the maximal case, where each mirrored vertex is connected by one knot edge, and the balanced edge is connected by two knot edges, $d = 1$ so $k = u - d = \frac{v}{2} + 1 - 1 = \frac{v}{2}$. For forests, this would be extended to $k = \frac{v}{2} + T - T = \frac{v}{2}$. For every edge that corresponds to the same knot circle in the tree, subtract 1. This means if some mirrored vertices are not connected by a knot edge, more than two knot edges can correspond to the same knot circle. Thus, since $c = \frac{v}{2}$, this means $k \leq c$. □

2 Linking Matrix Algorithm (LMA) Determines the Linking Number from the Crushtacean

In this section, we will introduce an algorithm to determine the linking number matrix for FALs and nested links directly from the crushtacean. We will also use the linking matrix to distinguish links with homeomorephic complements.

2.1 Linking Numbers for Nested Links

Let Γ be a crushtacean containing a balanced forest. Let t be a balanced tree in the crushtacean. Orient the balanced tree by choosing which subtree to call τ and $\varphi(\tau)$. Recall that $\varphi^2(\tau) = \tau$ is an involution, where τ contains the preimage of edges and vertices and $\varphi(\tau)$ contains the image. The chosen orientation for one balanced tree should be the same for all trees in the forest. An example of an oriented crushtacean with one balanced tree is shown in Figure 7.

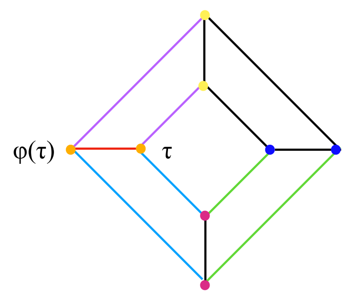

Figure 7: The vertices corresponding to the balanced edge, shown in orange are a pair of mirrored vertices as well as the pink, dark blue, and yellow vertices. The crushtacean is oriented such that knot edges entering the side of the balanced edge closer to the inner square first enter τ.

Each mirrored vertex will have the same knot edge going through it. Now, direct the unpainted edges keeping note of which subtree they go through. Let u be an unpainted edge. Due to the trivalent nature of the crushtacean, two mirrored or nonmirrored vertices will be connected by u. If the vertices are mirrored, this means that u connects the two subtrees. Choose whether to orient u from τ to $\varphi(\tau)$ or $\varphi(\tau)$ to τ. If the vertices are not mirrored, this means u connects two vertices of the same subtree. Choose which of these vertices u "starts" and "stops" at. There will be another unpainted edge connecting the respective mirrored vertices.

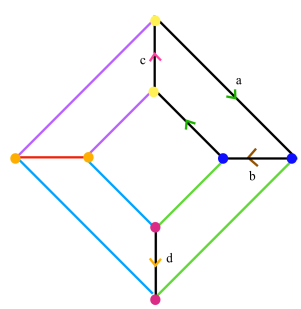

Figure 8: Directed knot edges. The knot circles go through one subtree, and then through the balanced edge to connected the mirrored edges.

2.1.1 Constructing the Matrix

The **linking number** is how many times knot circles are linked. The orientation of knot circles in a link determines if the linking number is positive or negative. To see what constitutes a positive or negative crossing, refer to Figure 9. Linking numbers are useful in determining whether two links are distinct, and more efficient than using **Reidemeister moves**. Reidemeister moves allow the projection of the knot to be changed which ultimately changes the linking number. [6].

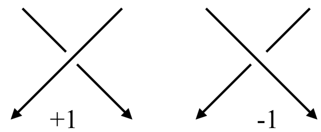

Figure 9: A positive orientation means that the understrand will point in the same direction as the overstrand when rotated clockwise. For a negative orientation, this direction is counterclockwise [6].

Because the orientation of FALs and nested links can be chosen, we will determine the linking number based on which subtree the knot edge enters first. Recall that because two subtrees in a balanced tree are isometric, two painted edges of the same color correspond to one crossing circle. Each painted edge the knot edge goes through determines the number of times it is linked with that crossing circle. If the path of a knot edge goes from τ to $\varphi(\tau)$, then it is linked $+1$ times. If the knot edge goes from $\varphi(\tau)$ to τ, then it is linked -1 times. Summing each linking number between the crossing circle and knot circle gives the net times the crossing circle is linked. Below is the linking matrix for the crushtacean from the previous section. The rows refer to knot circles and the columns refer to crossing circles.

$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} \text{Red} & \text{Purple} & \text{Blue} & \text{Green} \end{array} \\ \begin{bmatrix} 0 & -1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ -1 & -1 & 0 & 0 \\ -1 & 0 & -1 & 0 \end{bmatrix} \end{array}$$

Theorem 2.1. *For FALs, knot circles linking other knot circles need not be considered in the linking matrix.*

Proof. In a linking matrix, let m be the number of rows and n be the number of columns, where each knot circle is a row and each crossing circle is a column. By definition, crossing circles do not link other crossing circles, so they do not need to be included in the rows. While it is possible for a knot circle to link another knot circle, this can be simplified being unlinked. We can see from the Dehn twist that half twists can be chosen such that the separate knot components are unlinked. In other words, we can use the Dehn twist to invert the crossing, thereby changing the linking number by -1.

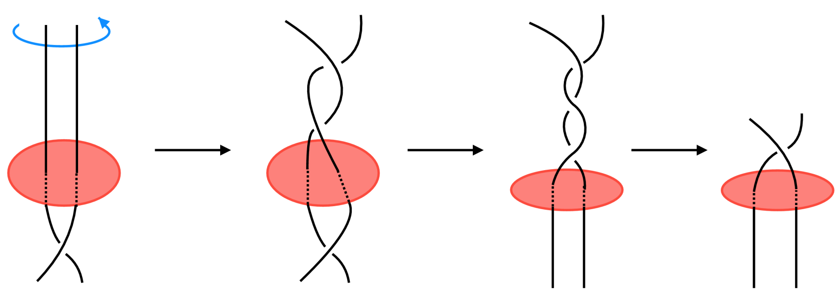

Figure 10: The Dehn twist is done on the two strands in the top half. Since there is a twist between the bottom two strands, this is isotopic to "sliding the crossing disk down" so that the bottom twist is in the top. A Reidemeister II move[2] is used to remove the unnecessary twist leaving us with the opposite crossing.

For FALs, a crossing only occurs at the crossing circle. As shown in the diagram, since a crossing can be changed from under to over or vice versa, this means that a crossing can be changed in an FAL such that it is connected to the same knot circle. □

[2]Reidemeister II moves allow for the addition or removal of two crossings [6].

Example 2.1. *The Borromean rings are an example of a FAL whose crossings can be changed via the Dehn twist.*

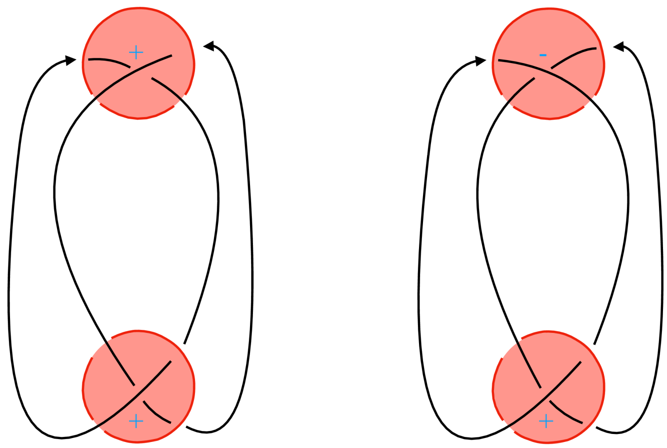

Figure 11: Using the Dehn twist, the linking number in the Borromean rings is changed so that the knot circles are not linked.

2.2 Constructing the Nested Link Directly from the Crushtacean

To construct the nested link directly from the crushtacean, utilize the following rotation system to determine if knot circles are flat or twisted.

1. Mark all pairs of mirrored vertices.
2. Starting with any edge from a pair of mirrored vertices, keep track of the counterclockwise rotation order.
3. Start with that same edge for the other mirrored vertex. If the counterclockwise rotations have the same order of edges, then the knot circles are twisted. If the orders are opposite, then the knot circles are flat.

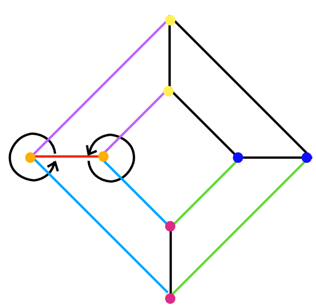

Figure 12: The counterclockwise rotation for the pair of mirrored vertices of the balanced edge is opposite, so the link is flat. This is true for all the pairs of mirrored vertices in this particular crushtacean.

The balanced edge will always be the outermost crossing circle in a balanced tree. Let the "inner square" be τ and the "outer square" be $\varphi(\tau)$. We can consider τ to be the "front" of the link which means the purple edge is on the right side of the balanced edge, and the blue edge is on the left. Looking at the endpoints of the blue edges, shown as pink vertices, let us apply the rotation system. Starting with the blue edge of τ, the rotation order is blue, an unpainted edge, and green. Looking at the blue edge of $\varphi(\tau)$, the rotation order is blue, green, and an unpainted edge. Thus, the knot edge connecting the blue edges of τ and φ is flat. Because the endpoints of the purple and green edges each have two unpainted edges, we can choose whether those knot edges are flat or twisted. Let us have them be flat. The corresponding link can be seen in 13.

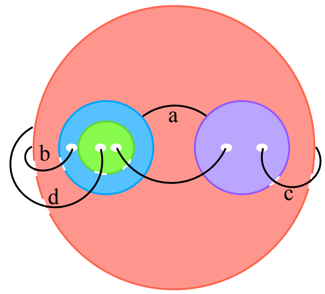

Figure 13: The corresponding nested link drawn from the crushtacean. The knot circles correspond to the directed knot edges, labelled in Figure 8.

2.3 Distinguishing Nested Links from Linking Numbers

Theorem 2.2. *Suppose L and L' are isotopic links with linking matrices M and M'. In other words, there is a homeomorphism $h = S^3 \to S^3$ with $h(L) = L'$. Then, M and M' are related by the following equivalence relations:*

1. *Multiply any row or column by -1.*
2. *Interchange any two rows or columns.*

Proof. The direction of the knot edge can be chosen to go from τ to $\varphi(\tau)$ or $\varphi(\tau)$ to (τ). This determines if the knot circle is oriented counterclockwise or clockwise, and determines if the linking numbers with the crossing circles it passes through are -1 or 1. If the orientation of the knot circle is reversed, then the linking numbers for each crossing circle it passes through are also reversed. Thus, we can multiply each row of the linking numbers matrix by -1 to represent reversed knot circle directions. If there are k rows in the matrix, then there are 2^k possible ways to write the rows of the matrix since there are two ways to express each row.

The order of rows and columns can be different since the actual linking number between a knot circle and crossing circle remains unchanged. If there are c number of columns and k number of rows, then there are $c!$ possible column orderings and $k!$ possible rows orderings. Thus, for each link, there are $k! \cdot c! \cdot 2^k$ possible ways to write the linking matrix. For nested links, crossing circles cannot link other crossing circles. Thus, each row represents a knot circle. While knot circles can link other knot circles, this may simplify to being unlinked; however, since this has not yet been proven, knot circles must also be considered in the columns of the matrix. □

Corollary 2.2.1. *If M and M' have a different number of zeros, then L and L' are not isotopic.*

Proof. This is because each 0 remains unchanged by the equivalence relations. There is no way to get from 0 to 1 or -1. □

Example 2.2. *The equivalence relations show that linking complements can be homeomorphic while the links themselves are not isotopic.*

Using the LMA, we can observe the following:

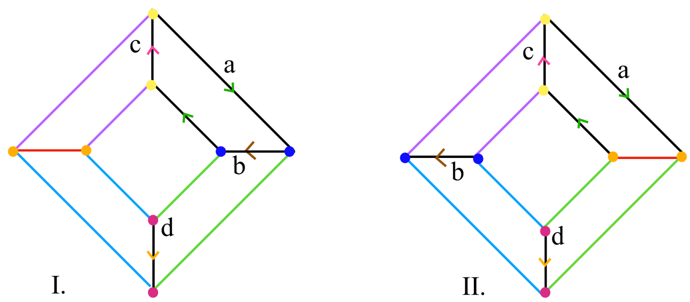

Figure 14: Depicted are two balanced trees who exhibit the same gluing pattern since the same edges are glued along the mirrored vertices. This means their link complements will be homeomorphic.

(I) has a linking matrix of:

$$\begin{array}{c c} & \begin{array}{cccc} \text{Red} & \text{Purple} & \text{Blue} & \text{Green} \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 0 & -1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ -1 & -1 & 0 & 0 \\ -1 & 0 & -1 & 0 \end{bmatrix} \end{array}$$

and (II) has a linking matrix of:

$$\begin{array}{c c} & \begin{array}{cccc} \text{Red} & \text{Purple} & \text{Blue} & \text{Green} \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

From the equivalence relations, we see that the number of zeroes in both matrices are different, hence their links cannot be isotopic.

2.4 Linking Numbers for FALs

If an FAL is flat, each crossing circle is punctured twice by two distinct knot circle. If it is twice punctured by the same knot circle, then it is unlinked. For twisted FALs, if a crossing circle is twice punctured by the same knot circle, it can be linked ± 2 times.

2.5 Distinguishing FALs from Linking Numbers

For FALs, each knot circle links a unique combination of two crossing circles. Since each crossing disk is twice punctured, by two distinct knot circles, there will be two ones in each column of the matrix with

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