

# An Examination of Linear Combinations of Skew-Adjoint Build Algebraic Curvature Tensors

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## 1 Introduction

Algebraic curvature tensors come in a variety of different forms, and there are a variety of different properties that are of interest. Something one might wish to do develop invariants to distinguish between different algebraic curvature tensors. A new potential invariant, the signature, will be examined. Furthermore, linear combinations of skew-adjoint build algebraic curvature tensors will be thoroughly examined so as to provide some insight into the possible forms of algebraic curvature tensors. The structure group, a known invariant of algebraic curvature tensors, is also investigated. There are many different ways to express algebraic curvature tensors, so it is useful to develop invariants to distinguish between them.

**Definition 1.0.1.** *On any vector space  $V$ , an inner product is a map  $\phi : V \times V \rightarrow \mathbb{R}$  with the following properties:*

- 1) *multilinear in both entries.*
- 2)  $\phi(u, v) = \phi(v, u)$  for all  $u, v \in V$ .
- 3)  $\phi(v, v) \neq 0$  if  $v \neq 0$  (positive definite).

**Definition 1.0.2.** *On any vector space  $V$ , algebraic curvature tensor is a map  $R : V \times V \times V \times V \rightarrow \mathbb{R}$  with the following properties:*

- 1) *multilinear in all four entries.*
- 2)  $R(x, y, z, w) = -R(y, x, z, w)$ .
- 3)  $R(x, y, z, w) = R(z, w, x, y)$ .
- 4)  $R(x, y, z, w) + R(x, w, y, z) + R(x, z, w, y) = 0$ .

**Definition 1.0.3.** *The kernel of an algebraic curvature tensor  $R$  is the set of all  $x \in V$  such that  $R(x, y, z, w) = 0$  for any  $y, z, w \in V$ .*

We can use properties 2 and 3 of algebraic curvature tensors to show that if  $R(x, y, z, w) = 0$  for any  $y, z, w \in V$ , then  $R(x, y, z, w) = R(y, x, z, w) = R(y, z, x, w) = R(y, z, w, x) = 0$  [2].

Given a manifold  $M$  and a point  $p$  on  $M$ , one can extract the tangent space  $V$  at  $p$ , an algebraic curvature tensor defined on  $V$ , and, provided the manifold had a metric, an inner product  $\phi$ . The tuple  $(V, \phi, R)$  is called a model space, and the pair  $(V, R)$  is called a weak model space. Understanding of the model space at a point helps understand the manifold at the point, hence a proper understanding of algebraic curvature tensors defined on a vector space with an inner product is important to a proper understanding of manifolds. One important property a model space, or a weak model space, can have is decomposability.

**Definition 1.0.4.** A model space  $(V, \phi, R)$  is decomposable if there exist sub-vector spaces  $V_1$  and  $V_2$  such that  $V_1 \cap V_2 = \{0\}$ , and  $(V_1, \phi, R_1) \oplus (V_2, \phi, R_2)$  is isomorphic to  $(V, \phi, R)$ , where  $(V_i, \phi, R_i)$  is the model space where  $\phi$  and  $R$  have been restricted to  $V_i$ . Similarly, a weak model space  $(V, R)$  is decomposable if there exist sub-vector spaces  $V_1$  and  $V_2$  such that  $V_1 \cap V_2 = \{0\}$ , and  $(V_1, R_1) \oplus (V_2, R_2)$  is isomorphic to  $(V, R)$ .

From now on we fix some vector space  $V$  of dimension  $n$  and an inner product  $\phi$  acting on  $V$ . The set of all possible algebraic curvature tensors acting on  $V$  is denoted by  $\mathbb{A}(V)$ . There is a special subgroup of algebraic curvature tensors, called canonical algebraic curvature tensors, which are known to be a spanning set of  $\mathbb{A}(V)$ . Canonical algebraic curvature tensors are associated with matrices, and there are two basic flavors. Firstly we have those defined with self-adjoint matrices.

**Definition 1.0.5.** Let  $A$  be an  $n \times n$  self-adjoint matrix. The canonical algebraic curvature tensor  $R_A^S : V \times V \times V \times V \rightarrow \mathbb{R}$  is defined as follows:  $R_A^S(x, y, z, w) = \phi(Ax, w)\phi(Ay, z) - \phi(Ax, z)\phi(Ay, w)$ .

Remember that we have fixed  $V$  to be  $n$  dimensional, and  $\phi$  is an inner product on  $V$  so the expression  $\phi(Ax, w)$  is well defined. The second flavor of canonical algebraic curvature tensors are defined with skew-adjoint matrices.

**Definition 1.0.6.** Let  $A$  be an  $n \times n$  skew-adjoint matrix. The canonical algebraic curvature tensor  $R_A : V \times V \times V \times V \rightarrow \mathbb{R}$  is defined as follows:  $R_A(x, y, z, w) = \phi(Ax, w)\phi(Ay, z) - \phi(Ax, z)\phi(Ay, w) - 2\phi(Ax, y)\phi(Az, w)$ .

One key property of canonical algebraic curvature tensors is:  $R_{\alpha A}(x, y, z, w) = \alpha^2 R_A(x, y, z, w)$ [4]. One can easily use the definition to expand the left hand side:

$$\begin{aligned} R_{\alpha A}(x, y, z, w) &= \phi(\alpha Ax, w)\phi(\alpha Ay, z) - \phi(\alpha Ax, z)\phi(\alpha Ay, w) - 2\phi(\alpha Ax, y)\phi(\alpha Az, w) \\ &= \alpha^2 \phi(Ax, w)\phi(Ay, z) - \alpha^2 \phi(Ax, z)\phi(Ay, w) - 2\alpha^2 \phi(Ax, y)\phi(Az, w) \\ &= \alpha^2 R_A. \end{aligned}$$

It should be noted that both flavors form spanning sets of  $\mathbb{A}(V)$ [4]. Hence the notion of the least number of each type of canonical curvature tensor required to express any curvature tensor  $R$  should be of interest.

**Definition 1.0.7.** For any  $R \in \mathbb{A}(V)$ ,  $\nu(R) := \min[k | \sum_{i=1}^k \alpha_i R_{A_i}^S = R]$ , where each  $\alpha_i$  is an element of  $V$ , and  $R_{A_i}^S$  is a self-adjoint build canonical algebraic curvature tensor.

**Definition 1.0.8.** For any  $R \in \mathbb{A}(V)$ ,  $\eta(R) := \min[k | \sum_{i=1}^k \alpha_i R_{A_i} = R]$ , where each  $\alpha_i$  is an element of  $V$ , and  $R_{A_i}$  is a skew-adjoint build canonical algebraic curvature tensor.

We can also consider  $\nu$  and  $\eta$  as functions of  $n$ , the dimension of the vector space, instead of individual algebraic curvature tensors.

**Definition 1.0.9.** For any vector space  $V$  of  $n$ ,  $\eta(n) := \max_{R \in \mathbb{A}(V)} [\eta(R)]$ .

This paper will primarily focus on this skew-adjoint build of canonical algebraic curvature tensor, as the self-adjoint build is fairly well understood. This paper will also serve to highlight some of the similarities and differences between the two builds. For instance, in [4] Gilkey shows that  $\lfloor \frac{n}{2} \rfloor \leq \nu(n) \leq \frac{n(n+1)}{2}$ , whereas in [5] Lopez shows that  $\eta(n) \leq \frac{n^2(n^2-1)}{12} - \binom{n}{2}$ . Clearly there is a big discrepancy between the two builds here;  $\frac{n(n+1)}{2}$  is much less than  $\frac{n^2(n^2-1)}{12} - \binom{n}{2}$  for large  $n$ .

## 1.1 Notation

Throughout this paper  $R$  will always be an algebraic curvature tensor,  $R_A$  will always be a canonical algebraic curvature tensor with the skew-adjoint build (so  $A$  is assumed to be skew-adjoint), and  $R_A^S$  will always be a canonical algebraic curvature tensor with the self-adjoint build (so  $A$  is assumed to be self-adjoint). Occasionally  $R_{ijkl}$  will be used to denote  $R(e_i, e_j, e_k, e_l)$  where each  $e_s$  is a basis vector of  $V$ .

There is one more useful piece of notation to introduce which simplifies an important definition. We will sometimes replace individual entries of a matrix with  $2 \times 2$  blocks when all of the other entries are 0. For instance:

$$A = \begin{bmatrix} \alpha_1 & 0 & \dots & 0 \\ 0 & \alpha_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \alpha_{\frac{n}{2}} \end{bmatrix}$$

where  $\alpha_i$  is the  $2 \times 2$  block:  $\begin{bmatrix} 0 & a_i \\ -a_i & 0 \end{bmatrix}$ . When we write down such a matrix we really mean that  $A = \alpha_1 \oplus \alpha_2 \oplus \dots \oplus \alpha_{\frac{n}{2}}$ . This brings us to the definition.

**Definition 1.0.10.** A square skew-adjoint matrix  $A$  is called block diagonalizable if there exists a basis in which the only non-zero entries of  $A$  are the  $i, i+1$ , and  $j, j-1$  entries, where  $i$  must be odd and  $j$  must be even.

Note that if  $A$  is block diagonal, then

$$A = \begin{bmatrix} \alpha_1 & 0 & \dots & 0 \\ 0 & \alpha_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \alpha_{\frac{n}{2}} \end{bmatrix} \quad \text{or if } A \text{ has odd dimensions} \quad A = \begin{bmatrix} \alpha_1 & 0 & \dots & 0 & 0 \\ 0 & \alpha_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \alpha_{\frac{n}{2}} & 0 \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix}$$

for some  $\alpha_i$ s.

## 2 Signature Conjecture

In this section we will investigate the extent to which the expression of any algebraic curvature tensor  $R$  as a linear combination of canonical algebraic curvature tensors with the skew-adjoint build is unique. Our first remark has to be that any linear combination of skew-adjoint build curvature tensors is definitely not unique. The polarization identity:

$$2R_A + 2R_B = R_{A-B} + R_{A+B} = R$$

provides an easy example of two linear combinations of different skew-adjoint build curvature tensors that both equal the same algebraic curvature tensor. One can come up with many other such examples.

However; there might still be something we can say on this subject. The *signature* of a linear combination of skew-adjoint build curvature tensors:  $\sum_{i=1}^m \alpha_i R_{A_i}$  is the ordered triple  $(p, q, s)$  where  $p$  is the number of positive  $\alpha_i$ s,  $q$  is the number of negative  $\alpha_i$ s, and  $s$  is the number of  $\alpha_i$ s equal to zero. In [7], Ragosta, proposes the following conjecture for  $R_A^S$ :

**Conjecture 2.0.1.** *If  $R = \sum_{i=1}^m \alpha_i R_{A_i}^S$ ,  $\nu(R) = m$ , and the rank of each  $R_{A_i}^S$  is greater than 3, then any other linear combination of  $m$  canonical algebraic curvature tensors with the self-adjoint build that equals  $R$  must preserve the signature. Put more plainly, if  $\sum_{i=1}^m \pm R_{A_i}^S = \sum_{i=1}^m \pm R_{B_i}^S$  is a minimal expression, and the rank of each  $A_i$  and  $B_i$  is greater than 3, then both sums have the same number of positive and negative terms.*

We can adapt this to the skew-adjoint build by replacing each self-adjoint matrix a skew-adjoint one, and getting rid of the rank greater than 3 restriction:

**Conjecture 2.0.2.** *If  $R = \sum_{i=1}^m \alpha_i R_{A_i}$  and  $\eta(R) = m$ , then any other linear combination of  $m$  skew-adjoint build curvature tensors that equals  $R$  must preserve the signature.*

Note that minimality is a very important assumption; the signature of  $\sum_{i=1}^m \alpha_i R_{A_i}$  can only equal the signature of  $\sum_{i=1}^h \beta_i R_{B_i}$  if  $m = h$ . The rest of the section is devoted to proving the signature conjecture for the skew-adjoint build when  $\eta(R) = 2$ . We start with a couple lemmas.

**Lemma 2.1.** *Let  $A$  and  $B$  be the following non zero block diagonalized skew-adjoint matrices:*

$$A = \begin{bmatrix} \alpha_1 & 0 & \dots & 0 \\ 0 & \alpha_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \alpha_{\frac{n}{2}} \end{bmatrix} \quad B = \begin{bmatrix} \beta_1 & 0 & \dots & 0 \\ 0 & \beta_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \beta_{\frac{n}{2}} \end{bmatrix}$$

where  $\alpha_i$  is the  $2 \times 2$  block:  $\begin{bmatrix} 0 & a_i \\ -a_i & 0 \end{bmatrix}$ , and  $\beta_i$  is the  $2 \times 2$  block:  $\begin{bmatrix} 0 & b_i \\ -b_i & 0 \end{bmatrix}$ . If  $a_i b_j = b_i a_j$  for all  $i$  and  $j$ , and there exists at least one  $i$  such that both  $a_i$

and  $b_i$  are non zero, then  $A$  and  $B$  are multiples of each other. Additionally,  $A = \frac{a_i}{b_i}B$  for any  $i$  where both  $a_i$  and  $b_i$  are non zero.

*Proof.* Pick any  $i$  such that  $a_i$  and  $b_i$  are non zero. Since  $a_i b_j = b_i a_j$  for all  $j$ , we can divide by  $a_i b_i$  so  $\frac{b_j}{b_i} = \frac{a_j}{a_i}$ . Now

$$\frac{a_i}{b_i} \beta_j = \begin{bmatrix} 0 & b_j \frac{a_i}{b_i} \\ -b_j \frac{a_i}{b_i} & 0 \end{bmatrix} = \begin{bmatrix} 0 & a_i \frac{b_j}{b_i} \\ -a_i \frac{b_j}{b_i} & 0 \end{bmatrix} = \begin{bmatrix} 0 & a_i \frac{a_j}{a_i} \\ -a_i \frac{a_j}{a_i} & 0 \end{bmatrix} = \alpha_j$$

Hence  $A$  and  $B$  are multiples, and  $A = \frac{a_i}{b_i}B$ .  $\square$

From [3] we have the following lemma:

**Lemma 2.2.** *if  $\sum_{i=1}^m R_{A_i} = R_B$ , where all  $A_i$  and  $B$  are skew adjoint, then all the  $A_i$ s are simultaneously block diagonalizable.*

We now propose the following theorem which will be instrumental to proving the signature conjecture for  $\eta(R) = 2$ , and has even more far reaching implications.

**Theorem 2.3.** *If  $\sum_{i=1}^m R_{A_i} = R_B$ , where each  $A_i$  and  $B$  are skew adjoint matrices, then there exists real numbers  $c_i$  such that  $A_i = c_i B$ , or in other words all the matrices involved are multiples of one another.*

*Proof.* We know that all the  $A_i$ s and  $B$  are simultaneously block diagonalizable, so let's start by picking a basis,  $\{e_1, \dots, e_n\}$  such that all the  $A_i$ s and  $B$  are block diagonalized. We can now write down each matrix:

$$A_i = \begin{bmatrix} \alpha_{i_1} & 0 & \dots & 0 \\ 0 & \alpha_{i_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \alpha_{i_{\frac{n}{2}}} \end{bmatrix} \quad B = \begin{bmatrix} \beta_1 & 0 & \dots & 0 \\ 0 & \beta_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \beta_{\frac{n}{2}} \end{bmatrix}$$

where  $\alpha_{i_j}$  is the  $2 \times 2$  block:  $\begin{bmatrix} 0 & a_{i_j} \\ -a_{i_j} & 0 \end{bmatrix}$ , and  $\beta_j$  is the  $2 \times 2$  block:  $\begin{bmatrix} 0 & b_j \\ -b_j & 0 \end{bmatrix}$ . Note that if  $n$  is odd there are only  $\frac{n-1}{2}$  blocks and an extra row and column of zeros in each matrix. Since all the matrices are block diagonalizable, we can re-order the basis so that the  $n^{th}$  row and column are identically zero in all the matrices.

Clearly any number of the  $A_i$ s could be zero without changing the overall sum, but in that case  $A_i = c_i B$  where  $c_i = 0$ , so from now on we will assume that each  $A_i \neq 0$ . The only issue with this assumption would be if every  $A_i = 0$ , but in that case  $B = 0$ , so  $A_i = B$  for every  $i$  and we may proceed with our assumption. Now we will consider  $\sum_{i=1}^m R_{A_i}$ . We are assuming that the sum equals  $R_B$  for some . Since algebraic curvature tensors can be uniquely determined by their action on a basis, we will look at  $R_B(e_k, e_l, e_l, e_k)$ . We may assume that  $k < l$  since  $R_B(e_k, e_l, e_l, e_k) = -R_B(e_k, e_l, e_l, e_k)$  and  $R_B(e_k, e_k, e_k, e_k) = 0$ .

$R_B(e_k, e_l, e_l, e_k) = \sum_{i=1}^m R_{A_i}(e_k, e_l, e_l, e_k)$ , and  $A_i(e_k, e_l, e_l, e_k) = 0$  for all  $k$  and  $l$  except when  $l$  is even and  $k = l - 1$ , in which case  $A_i(e_k, e_l, e_l, e_k) = 3a_{i\frac{l}{2}}^2$ . In which case we conclude that  $b_{\frac{l}{2}}^2 = \sum_{i=1}^m a_{i\frac{l}{2}}^2$ .

Going through the other cases, we find that the only other permutation of basis vectors  $e_k, e_l, e_r, e_s$  such that  $R_B(e_k, e_l, e_r, e_s)$  is non zero is when  $l$  and  $s$  are distinct and even,  $k = l - 1$ , and  $r = s - 1$ . In this case  $R_B(e_{l-1}, e_l, e_{s-1}, e_s) = -2 \sum_{i=1}^m a_{i\frac{l}{2}} a_{i\frac{s}{2}}$ . But  $R_B(e_{l-1}, e_l, e_{s-1}, e_s) = -2b_{\frac{l}{2}} b_{\frac{s}{2}}$ , so  $b_{\frac{l}{2}} b_{\frac{s}{2}} = \sum_{i=1}^m a_{i\frac{l}{2}} a_{i\frac{s}{2}}$ .

We have already shown that  $b_{\frac{l}{2}}^2 = \sum_{i=1}^m a_{i\frac{l}{2}}^2$ , so we can now combine these two equations:

$$b_{\frac{l}{2}}^2 b_{\frac{s}{2}}^2 = \left( \sum_{i=1}^m a_{i\frac{l}{2}}^2 \right) \left( \sum_{i=1}^m a_{i\frac{s}{2}}^2 \right) = \left( \sum_{i=1}^m a_{i\frac{l}{2}} a_{i\frac{s}{2}} \right) \left( \sum_{i=1}^m a_{i\frac{l}{2}} a_{i\frac{s}{2}} \right) = (b_{\frac{l}{2}} b_{\frac{s}{2}})^2$$

If we expand, we can cancel all the terms of the form  $a_{i\frac{l}{2}}^2 a_{i\frac{s}{2}}^2$ , and move everything to the left side:

$$\sum_{i=1}^m \sum_{j=1, j \neq i}^m a_{i\frac{l}{2}}^2 a_{j\frac{s}{2}}^2 - \sum_{i=1}^m \sum_{j=1, j \neq i}^m a_{i\frac{l}{2}} a_{i\frac{s}{2}} a_{j\frac{l}{2}} a_{j\frac{s}{2}} = 0$$

Note that  $a_{i\frac{l}{2}}^2 a_{j\frac{s}{2}}^2 + a_{j\frac{l}{2}}^2 a_{i\frac{s}{2}}^2 - a_{i\frac{l}{2}} a_{i\frac{s}{2}} a_{j\frac{l}{2}} a_{j\frac{s}{2}} - a_{j\frac{l}{2}} a_{j\frac{s}{2}} a_{i\frac{l}{2}} a_{i\frac{s}{2}} = (a_{i\frac{l}{2}} a_{j\frac{s}{2}} - a_{j\frac{l}{2}} a_{i\frac{s}{2}})^2$ .

Hence:

$$\sum_{i=1}^m \sum_{j=1, j \neq i}^m a_{i\frac{l}{2}}^2 a_{j\frac{s}{2}}^2 - \sum_{i=1}^m \sum_{j=1, j \neq i}^m a_{i\frac{l}{2}} a_{i\frac{s}{2}} a_{j\frac{l}{2}} a_{j\frac{s}{2}} = \frac{1}{2} \sum_{i=1}^m \sum_{j=1, j \neq i}^m (a_{i\frac{l}{2}} a_{j\frac{s}{2}} - a_{j\frac{l}{2}} a_{i\frac{s}{2}})^2 = 0$$

Note that the factor of  $\frac{1}{2}$  after the first equality since we have counted each of the terms twice. This means that  $a_{i\frac{l}{2}} a_{j\frac{s}{2}} = a_{j\frac{l}{2}} a_{i\frac{s}{2}}$  for all  $i$  and  $j \neq i$ . Now let  $i = m$ , pick any  $j < m$ , and choose any  $l$  such that  $a_{m\frac{l}{2}} \neq 0$ . Such an  $l$  must exist since we have assumed that each  $A_i \neq 0$ . Since  $A_j \neq 0$ , we can also find an  $s$  such that  $a_{j\frac{s}{2}} \neq 0$ . We know that  $a_{m\frac{l}{2}} a_{j\frac{s}{2}} = a_{j\frac{l}{2}} a_{m\frac{s}{2}}$ , and we now know that  $a_{m\frac{l}{2}} a_{j\frac{s}{2}} \neq 0$ . Therefore  $a_{j\frac{l}{2}} \neq 0$ . We can now apply Lemma 2.1, which

tells us that  $A_j = \frac{a_{j\frac{l}{2}}}{a_{m\frac{l}{2}}} A_m$ . So we can now simplify the sum:

$$\sum_{j=1}^m R_{A_i} = \sum_{j=1}^m \left( \frac{a_{j\frac{l}{2}}}{a_{m\frac{l}{2}}} \right)^2 R_{A_m} = R_{A_m} \sum_{j=1}^m \left( \frac{a_{j\frac{l}{2}}}{a_{m\frac{l}{2}}} \right)^2 = B$$

Therefore  $R_{A_m}$  is a multiple of  $B$ , and since each  $R_{A_i}$  is a multiple of  $R_{A_m}$ , each  $R_{A_i}$  must also be a multiple of  $B$ .  $\square$

This theorem is extremely useful and lets us prove some interesting results.

**Corollary 2.3.1.** *If  $\sum_{i=1}^m R_{A_i} = R$ , and  $\eta(R) = m$  and  $m > 1$ , then there do not exist  $n \times n$  skew-adjoint matrices  $B$  and  $C_j$  for  $1 \leq j < m$  such that  $\sum_{i=1}^m R_{A_i} = R_B - \sum_{j=1}^{m-1} R_{C_j}$ .*

*Proof.* We will consider the  $m$  skew-adjoint  $n \times n$  matrices  $A_i$ , the algebraic curvature tensor  $R = \sum_{i=1}^m R_{A_i}$  with  $\eta(R) = m$ , and any other  $m$  skew-adjoint  $n \times n$  matrices  $B$  and  $C_j$ . We will assume, for a contradiction, that  $\sum_{i=1}^m R_{A_i} = R_B - \sum_{j=1}^{m-1} R_{C_j}$ . Therefore  $\sum_{i=1}^m R_{A_i} + \sum_{j=1}^{m-1} R_{C_j} = R_B$ , so we can apply Theorem 2.2 which tells us that all the matrices involved are multiples of  $B$ . Therefore  $\sum_{i=1}^m R_{A_i}$  is also a multiple of  $R_B$ . Since  $R = \sum_{i=1}^m R_{A_i}$  must be a multiple of  $R_B$ ,  $\eta(R) = 1$ . This is a contradiction, as  $\eta(R) = m > 1$ .  $\square$

**Corollary 2.3.2.** *Conjecture 2.0.2 is true when  $\eta(R) = 2$ , i.e. if  $A, B$ , are skew-adjoint  $n \times n$  matrices, and  $R_A + R_B = R$  is a minimal expression of  $R$ , then there do not exist skew-adjoint  $n \times n$  matrices  $C$  and  $D$  such that  $R_A + R_B = R_C - R_D$ .*

*Proof.* This is just a specific case of the previous corollary. Simply let  $m = 2$  and the result follows.  $\square$

### 3 The Kernel of $R_A \pm R_B$

We start this section by introducing some of the work that has been done on the kernel of canonical algebraic curvature tensors. In [4], Gilkey proves that  $\text{Ker}(R_A) = \text{Ker}(A)$ , and  $\text{Ker}(R_A^S) = \text{Ker}(A)$  if the rank of  $A \neq 1$ . Furthermore, in [9], Williams proves that  $\dim(\text{Ker}(R_A^S \pm R_B^S)) = 0, 1$ , or  $n$ .

In this section we hope to achieve a similar result for  $\text{Ker}(R_A \pm R_B)$ . Hence the following theorems.

**Theorem 3.1.** *If  $A_k$ , for  $1 \leq k \leq m$  are skew-adjoint matrices and the algebraic curvature tensor  $R$  equals  $\sum_{k=1}^m R_{A_k}$ , then the kernel of  $R$  is  $\bigcap_k \text{Ker}(A_k)$*

*Proof.* Let  $a_{k_{ij}}$  represent the  $ij$ th entries of the matrix  $A_k$ . It is clear that  $\text{Ker}(R) \supset \bigcap_k \text{Ker}(R_{A_k})$ . Hence if  $\text{Ker}(R) = \{0\}$ , then  $\bigcap_k \text{Ker}(R_{A_k}) = \{0\}$  as well.

The other case, where  $\text{Ker}(R) \neq \{0\}$ , is more interesting. We will start by picking a basis,  $\{e_1, e_2, \dots, e_l\}$  of  $\text{Ker}(R)$ , then extend this basis to a basis of the whole space  $\{e_1, \dots, e_l, e_{l+1}, \dots, e_n\}$ , and write down the matrices  $A$  and  $B$  in this new basis. We will now consider some basis vector  $e_i \in \text{Ker}(R)$ , and any other basis vector  $e_j$ .

$$0 = R(e_i, e_j, e_j, e_i) = \sum_{k=1}^m R_{A_k}(e_i, e_j, e_j, e_i) = 3 \sum_{k=1}^m a_{k_{ij}}^2 \quad (1)$$

So  $a_{k_{ij}}^2 = 0$  for all  $k$ . Since  $e_j$  was an arbitrary basis vector this means that  $a_{k_{ij}} = 0$  for every  $j$  whenever  $e_i \in \text{Ker}(R)$ . Hence  $e_i \in \text{Ker}(R_{A_k})$  whenever  $e_i \in \text{Ker}(R)$ . Therefore  $\text{Ker}(R) = \bigcap_k \text{Ker}(R_{A_k})$ .  $\square$

Unfortunately this proof does not work in an expression of the form  $R_A - R_B$ , as the introduction of the minus sign means we would get  $a_{ij} = \pm b_{ij}$  instead of  $a_{ij} = b_{ij} = 0$  from Equation 1. Hence we will have to work a lot harder to prove the following theorem.

**Theorem 3.2.** *If  $A$  and  $B$  are skew-adjoint matrices and the algebraic curvature tensor  $R$  equals  $R_A - R_B$ , then either the kernel of  $R$  is  $\text{Ker}(R_A) \cap \text{Ker}(R_B)$ , or  $B = \pm A$  so  $R = 0$  and has kernel equal to  $V$ .*

*Proof.* As before, let  $a_{ij}$  and  $b_{ij}$  represent the  $ij$ th entries of the matrices  $A$  and  $B$ , and notice that if  $\text{Ker}(R) = \{0\}$ , then  $\text{Ker}(R_A) \cap \text{Ker}(R_B) = \{0\}$  as well. In the case that  $\text{Ker}(R) \neq \{0\}$ , pick a basis  $\{e_1, e_2, \dots, e_k\}$  of  $\text{Ker}(R)$ , extend this basis to a basis of the whole space  $\{e_1, \dots, e_k, e_{k+1}, \dots, e_n\}$ , and write down the matrices  $A$  and  $B$  in this new basis as before. Now let  $e_i$  be a specific basis vector of  $\text{Ker}(R)$ , and let  $e_j, e_k, e_l$  be any other distinct basis vectors.

$$0 = R(e_i, e_j, e_j, e_i) = R_A(e_i, e_j, e_j, e_i) - R_B(e_i, e_j, e_j, e_i) = 3a_{ij}^2 - 3b_{ij}^2$$

Hence  $a_{ij} = \pm b_{ij}$ . Since  $j$  was arbitrary, we also know that  $a_{ik} = \pm b_{ik}$  and  $a_{il} = \pm b_{il}$

$$0 = R(e_i, e_j, e_k, e_i) = R_A(e_i, e_j, e_k, e_i) - R_B(e_i, e_j, e_k, e_i) = 3a_{ij}a_{ik} - 3b_{ij}b_{ik}$$

Since  $a_{ij} = \pm b_{ij}$  and  $a_{ik} = \pm b_{ik}$ , this tells us that either  $a_{ij} = b_{ij}$  and  $a_{ik} = b_{ik}$  or  $a_{ij} = -b_{ij}$  and  $a_{ik} = -b_{ik}$ . If it happens that  $a_{ij} = -b_{ij}$ , without loss of generality we can just replace the matrix  $B$  with  $-B$ , since  $R_B = R_{-B}$ . Hence we will assume that  $a_{ij} = b_{ij}$ ,  $a_{ik} = b_{ik}$ , and  $a_{il} = b_{il}$  from now on.

Now there are 2 cases to consider. Either  $a_{ij} = b_{ij} = 0$  for all  $j$ , then we must have that  $e_i \in \text{Ker}(R_A)$  and  $e_i \in \text{Ker}(R_B)$ , or there exists a  $j$  such that  $a_{ij} \neq 0$ , in which case we will let  $j$  be such that  $a_{ij} \neq 0$  and continue the proof.

$$0 = R(e_j, e_i, e_k, e_j) = R_A(e_j, e_i, e_k, e_j) - R_B(e_j, e_i, e_k, e_j) = 3a_{ji}a_{jk} - 3b_{ji}b_{jk}$$

So  $a_{ji}a_{jk} = b_{ji}b_{jk}$ . We have chosen  $j$  such that  $a_{ij} \neq 0$ , and we know that  $a_{ij} = b_{ij}$  so we can divide:  $a_{jk} = b_{jk}$ .

$$0 = R(e_i, e_j, e_k, e_l) = a_{il}a_{jk} - a_{ik}a_{jl} - 2a_{ij}a_{kl} - b_{il}b_{jk} + b_{ik}b_{jl} + 2b_{ij}b_{kl}$$

$$0 = R(e_i, e_l, e_j, e_k) = -a_{ik}a_{jl} + a_{ij}a_{kl} - 2a_{il}a_{jk} + b_{ik}b_{jl} - b_{ij}b_{kl} + 2b_{il}b_{jk}$$

$$\text{Subtracting yields: } 0 = 3a_{il}a_{jk} - 3a_{ij}a_{kl} - 3b_{il}b_{jk} + 3b_{ij}b_{kl}$$

But we know that  $a_{ij} = b_{ij}$ ,  $a_{il} = b_{il}$ , and  $a_{jk} = b_{jk}$ . Hence  $0 = a_{ij}(a_{kl} - b_{kl})$ , but  $a_{ij} \neq 0$ , so  $a_{kl} = b_{kl}$ . Since  $k$  and  $l$  were arbitrary, and each of  $a_{ij} = b_{ij}$ ,  $a_{ik} = b_{ik}$ , and  $a_{il} = b_{il}$ , we can conclude that  $A = B$ . But we might have switched the sign of  $B$ , so  $A = \pm B$ .

Thus we have shown that either  $e_i \in \text{Ker}(R_A) \cap \text{Ker}(R_B)$ , or  $A = \pm B$  so  $R = R_A - R_B = 0$   $\square$



When  $m = 2$  in Theorem 3.1,  $\text{Ker}(R_A + R_B) = \text{Ker}(A) \cap \text{Ker}(B)$ . If we combine this result with Theorem 3.2, we see that if  $\text{Ker}(A) \cap \text{Ker}(B) = \{0\}$ , then  $\dim(\text{Ker}(R_A \pm R_B)) = 0, n$ . This contrasts with the self-adjoint case, where [9] proves that  $\dim(\text{Ker}(R_A^S \pm R_B^S)) = 0, 1, n$ .

## 4 Decomposability of $R_A \pm R_B$

In this section we want to study the decomposability of model spaces of the form  $(V, \phi, R_A \pm R_B)$ , however; the inclusion of an inner product adds an extra layer of complexity which detracts somewhat from the focus on canonical algebraic curvature tensors. Hence we will study the decomposability of weak model spaces of the form  $(V, R_A \pm R_B)$ .

It is easy to show that if an algebraic curvature tensor  $R$  has a non-trivial kernel, then the weak model space  $(V, R)$  is decomposable. In fact, it can at least be decomposed into  $\text{Ker}(R)$  and  $V \cap \text{Ker}(R) \cup 0$ , as well as any subspace of  $\text{Ker}(R)$ , and depending on  $R$  it could be decomposed into other subspaces.

Moving away from general algebraic curvature tensors, it is known that  $(V, R_A)$  can only be decomposed if  $A$  has non-trivial kernel [4]. Clearly if  $R_A$  and  $R_B$  share a common kernel, then  $R_A + R_B$  will have non-zero kernel by Theorem 3.1 and thus is decomposable. Thus we want to investigate the case where  $R_A$  and  $R_B$  do not necessarily share a common kernel.

**Theorem 4.1.** *If an algebraic curvature tensor  $R$  equals  $\sum_{k=1}^m R_{A_k}$  for some skew-adjoint linear maps  $A_k$ , and the weak model space  $(V, R)$  can be decomposed into  $(V_1, R)$  and  $(V_2, R)$ , then all the linear maps  $A_k$  preserve the vector spaces  $V_1$  and  $V_2$ .*

*Proof.* Let  $\{e_1, e_2, \dots, e_p\}$  be a basis of  $V_1$  and  $\{f_1, f_2, \dots, f_q\}$  be a basis of  $V_2$ .  $\{e_1, e_2, \dots, e_p, f_1, f_2, \dots, f_q\}$  is a basis of  $V$ . We will express each  $A_k$  in this basis. Now let us consider  $R(e_i, f_l, f_l, e_i)$ . Since  $R$  is decomposable,  $R(e_i, f_l, f_l, e_i) = 0$ ; however, it also equals  $\sum_{k=1}^m A_k(e_i, f_l, f_l, e_i) = 3 \sum_{k=1}^m a_{k_{il}}^2$ . Hence we conclude that  $a_{k_{il}} = 0$  whenever  $i \leq p$  and  $l > p$ . This means that each  $A_k$  must preserve  $V_1$  and  $V_2$ , since now  $A_k e_i = \sum_{l=1}^n a_{k_{li}} e_l$  and  $A_k f_i = \sum_{l=n+1}^m a_{k_{li}} f_l$ .  $\square$

**Theorem 4.2.** *If an algebraic curvature tensor,  $R$ , equals  $R_A - R_B$  for some skew-adjoint linear maps  $A$  and  $B$ , and the weak model space  $(V, R)$  can be decomposed into  $(V_1, R)$  and  $(V_2, R)$  where  $V_1$  has dimension  $n$  and  $V_2$  has dimension  $m$ , then there exists a basis of  $V$  such that the entries of  $A$  and  $B$  satisfy the following properties:*

- 1)  $a_{ik} = \delta b_{ik}$  for all  $i \leq n$  and  $k > n$ , where  $\delta$  is either 1 or  $-1$
- 2) given any  $i \leq n$ , either  $a_{ik} = b_{ik} = 0$  for all  $k > n$ , or  $a_{ij} = \delta b_{ij}$  for all  $j \leq n$
- 3)  $a_{ij} a_{kl} = b_{ij} b_{kl}$  for all  $i, j \leq n$  and  $k, l > n$

Note that it is still possible for  $A$  and  $B$  to preserve  $V_1$  and  $V_2$ , but this is not necessarily the case as it was for  $R = R_A + R_B$ .

*Proof.* As before, let  $\{e_1, e_2, \dots, e_n\}$  be a basis of  $V_1$  and  $\{f_1, f_2, \dots, f_m\}$  be a basis of  $V_2$ . So  $\{e_1, e_2, \dots, e_n, f_1, f_2, \dots, f_m\}$  is a basis of  $V$ . Express  $A$  and  $B$  as matrices in this basis, and consider  $R(e_i, f_k, f_k, e_i)$ :

$$0 = R(e_i, f_k, f_k, e_i) = R_A(e_i, f_k, f_k, e_i) - R_B(e_i, f_k, f_k, e_i) = 3a_{ik}^2 - 3b_{ik}^2$$

Hence  $a_{ik} = \pm b_{ik}$  for all  $i \leq n$  and  $k > n$ . Now let's consider  $R(e_i, f_k, f_l, e_i)$ :

$$R(e_i, f_k, f_l, e_i) = 3a_{il}a_{ik} - 3b_{il}b_{ik} = 0$$

Since  $a_{il} = \pm b_{il}$  and  $a_{ik} = \pm b_{ik}$ ,  $a_{il}a_{ik} = b_{il}b_{ik}$  if and only if  $a_{ih} = \delta b_{ih}$  for all  $i \leq n$  and  $h > n$  where  $\delta$  is either 1 or  $-1$ . This proves 1). With this in mind we can now consider  $R(e_i, e_j, f_k, e_i)$ :

$$R(e_i, e_j, f_k, e_i) = 3a_{ik}a_{ij} - 3b_{ik}b_{ij} = 0$$

Since  $a_{ik} = \delta b_{ik}$ , either  $a_{ik} = 0$  for all  $k > n$ , or there exists one  $k$  with  $a_{ik} \neq 0$  so we can divide to get  $a_{ij} = \delta b_{ij}$  for all  $j \leq n$ . This proves 2).

To prove 3) we need to consider  $R(e_i, e_j, f_k, f_l)$ :

$$R(e_i, e_j, f_k, f_l) = a_{il}a_{jk} - a_{ik}a_{jl} - 2a_{ij}a_{kl} + b_{il}b_{jk} - b_{ik}b_{jl} - 2b_{ij}b_{kl} = 0$$

We know that  $a_{il}a_{jk} - b_{il}b_{jk} = 0$  and  $-a_{ik}a_{jl} + b_{ik}b_{jl} = 0$ , so  $a_{ij}a_{kl} = b_{ij}b_{kl}$  for all  $i, j \leq n$  and  $k, l > n$ .  $\square$

In the case that we are considering an expression of the form:  $R_A \pm R_B$ , theorem 4.2 gives us all the information about  $R_A - R_B$ , and theorem 4.1 tells us that in the  $R_A + R_B$  case both  $A$  and  $B$  must preserve the decomposable subspaces. However, we can actually say a little more about  $A$  and  $B$  in this case by considering  $R(e_i, e_j, f_k, f_l)$ :

$$\begin{aligned} R(e_i, e_j, f_k, f_l) &= R_A(e_i, e_j, f_k, f_l) + R_B(e_i, e_j, f_k, f_l) \\ &= a_{il}a_{jk} - a_{ik}a_{jl} - 2a_{ij}a_{kl} + b_{il}b_{jk} - b_{ik}b_{jl} - 2b_{ij}b_{kl} \end{aligned}$$

But we just showed that  $a_{il} = a_{ik} = a_{jk} = a_{jl} = b_{il} = b_{ik} = b_{jk} = b_{jl} = 0$ . Therefore we have that  $a_{ij}a_{kl} = -b_{ij}b_{kl}$  for all  $i, j \leq p$  and  $k, l > p$ , where  $p = \dim(V_1)$ . This gives us an easy way of constructing examples of two skew-adjoint matrices  $A$  and  $B$  such that neither  $R_A$  nor  $R_B$  are decomposable, but  $R_A + R_B$  is decomposable.

**Example 4.2.1.** We will be considering the following linear maps  $A$  and  $B$ , which have been expressed as matrices on the standard basis of  $\mathbb{R}^4$ :

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Note that  $b_{ij} = -a_{ij}$  if  $i, j > 2$  and  $b_{ij} = a_{ij}$  otherwise. Since  $A$  and  $B$  have a trivial kernel, [4] tells us that  $R_A$  and  $R_B$  are indecomposable. However,  $R_A + R_B$  decomposes into two dimensional subspaces; one spanned by  $e_1$  and  $e_2$  and the other by  $e_3$  and  $e_4$ .

## 5 Structure Group of $R_A + R_B$

We start this section with a definition:

**Definition 5.0.1.** *The structure group  $G_R$  of a algebraic curvature tensor  $R$  is the set of all invertible linear transformations  $g$  such that  $R(x, y, z, w) = R(gx, gy, gz, gw)$*

In [8], Ragosta gives an example of two self-adjoint build canonical algebraic curvature tensors and a linear transformation  $g$  such that  $g \in G_{R_C^S + R_D^S}$ ,  $g \notin G_{R_C^S}$  and  $g \notin G_{R_D^S}$ . This is an intriguing since it means that  $G_{R_C^S + R_D^S} \neq G_{R_C^S} \cap G_{R_D^S}$  as one might expect. We can transform Ragosta's example into the skew-adjoint setting by defining the matrices  $A$  and  $B$  as follows:

$$A = \begin{bmatrix} 0 & 0 & \frac{\lambda_1}{\sqrt{3}} \\ 0 & 0 & 0 \\ -\frac{\lambda_1}{\sqrt{3}} & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{\lambda_2}{\sqrt{3}} \\ 0 & -\frac{\lambda_2}{\sqrt{3}} & 0 \end{bmatrix}, \quad R = R_A + R_B$$

Now  $R_A + R_B = R_C^S + R_D^S$  from Ragosta's example. The following theorem presents itself:

**Theorem 5.1.** *The structure group of  $R = R_A + R_B$ ,  $G_R$ , is not equal to  $G_{R_A} \cap G_{R_B}$ , where the matrices  $A$  and  $B$  are as above.*

*Proof.* We can write down how the algebraic curvature tensor acts on a basis  $\{e_1, e_2, e_3\}$  of  $\mathbb{R}^3$ :

$$R_{1221} = 0, R_{1331} = \lambda_1^2, R_{2332} = \lambda_2^2, R_{1231} = R_{2132} = R_{3123} = 0$$

If we define  $g_-$  and  $g_+$  as follows (for  $\alpha \in [-1, 1]$ ), then one can check that both  $g_-$  and  $g_+ \in G_R$ .

$$g_- = \begin{bmatrix} \alpha & \frac{\lambda_1}{\lambda_2} \sqrt{1 - \alpha^2} & 0 \\ \frac{\lambda_2}{\lambda_1} \sqrt{1 - \alpha^2} & -\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad g_+ = \begin{bmatrix} \alpha & \frac{\lambda_1}{\lambda_2} \sqrt{1 - \alpha^2} & 0 \\ -\frac{\lambda_2}{\lambda_1} \sqrt{1 - \alpha^2} & \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now we can calculate what  $R_{g_-^* A g_-}$  and  $R_{g_+^* A g_+}$  do on the standard basis:

$$R_{g_-^* A g_-}(e_1, e_3, e_3, e_1) = \alpha^2 \lambda_1^2 \text{ and } R_{g_+^* A g_+}(e_1, e_3, e_3, e_1) = \alpha^2 \lambda_2^2$$

This means that  $g_-$  and  $g_+ \notin G_{R_A}$  so long as  $\alpha \in (-1, 1)$ . Similarly,  $g_-$  and  $g_+ \notin G_{R_B}$  so long as  $\alpha \in (-1, 1)$ .  $\square$

This begs the question: is this a global property of the curvature tensor  $R$ ? i.e. does any linear combination of algebraic curvature tensors that equals  $R$  have this property?

## 6 $\eta_m(n)$

We have already introduced  $\eta(n)$  in the introduction. In this section we will investigate a slightly different concept,  $\eta_m(n)$ .

**Definition 6.0.1.**  $\eta_m(R) := \min[k | \sum_{i=1}^k \alpha_i R_{A_i} = R, \text{ and } \text{rank}(A_i) \geq m]$ .

**Definition 6.0.2.**  $\eta_m(n) := \max_{R \in \mathbb{A}} [\eta_m(R)]$ .

The following lemma is vital in establishing some interesting results about  $\eta_m(n)$ .

**Lemma 6.1.** *If  $D$  is a skew-adjoint matrix of rank  $2m$ , then there exist skew-adjoint matrices  $A$ ,  $B$ , and  $C$  each of rank  $2m+2$  such that  $R_D = R_A + R_B - R_C$*

*Proof.* Let  $D$  be any skew-adjoint matrix of rank  $2m$ . Since  $D$  is skew-adjoint, there exists a basis that block diagonalizes  $D$ , so we can write  $D$  as:

$$\begin{bmatrix} \delta_1 & 0 & \dots & 0 & \dots & 0 \\ 0 & \delta_2 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \delta_m & \dots & 0 \\ 0 & 0 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & \dots & 0 \end{bmatrix}$$

Where  $\delta_i$  is the  $2 \times 2$  block:  $\begin{bmatrix} 0 & d_i \\ -d_i & 0 \end{bmatrix}$ . Now if we consider the following matrices, where  $i$  is the  $2 \times 2$  block:  $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ :

$$A = \begin{bmatrix} \frac{5}{2}\delta_1 & 0 & \dots & 0 & \dots & 0 \\ 0 & \frac{5}{2}\delta_2 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \frac{5}{2}\delta_m & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 3i & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & \dots & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 5\delta_1 & 0 & \dots & 0 & \dots & 0 \\ 0 & 5\delta_2 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 5\delta_m & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 4i & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & \dots & 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} \frac{11}{2}\delta_1 & 0 & \dots & 0 & \dots & 0 \\ 0 & \frac{11}{2}\delta_2 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \frac{11}{2}\delta_m & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 5i & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & \dots & 0 & 0 \end{bmatrix}$$

Clearly,  $A$ ,  $B$ , and  $C$  are all of rank  $2m+2$ . We now claim that  $R_A + R_B - R_C = R_D$ . To prove this claim we just need to check that  $R_{Aijkl} + R_{Bijkl} - R_{Cijkl} = R_{Dijkl}$  for all  $i, j, k, l$ . Since all the matrices involved are block diagonal, there are not that many possible permutations of  $i, j, k, l$  which yield non zero entries, and one can check them all. We only include three sample calculations, one for each of the cases:

$$R_{A1221} + R_{B1221} - R_{C1221} = \frac{5^2}{2} d_1^2 + 25d_1^2 - \frac{11^2}{2} d_1^2 = d_1^2 \left[ \frac{25}{4} + \frac{100}{4} - \frac{121}{4} \right] = d_1^2 = R_{D1221}$$

$$R_{A1234} + R_{B1234} - R_{C1234} = \frac{5^2}{2} d_1 d_2 + 25d_1 d_2 - \frac{11^2}{2} d_1 d_2 = d_1 d_2 = R_{D1234}$$

$$R_{A12(2m)(2m+1)} + R_{B12(2m)(2m+1)} - R_{C12(2m)(2m+1)} = d_1 \left[ \frac{15}{2} + \frac{40}{2} - \frac{55}{2} \right] = 0 = R_{D12(2m)(2m+1)}$$

□

Note that this is just one example from an infinite number of possible  $A$ ,  $B$ , and  $C$ .

**Theorem 6.2.**  $\eta_{2m+2}(n) \leq 3^m \eta(n)$ .

*Proof.* The result follows from the above result: any canonical algebraic curvature tensor with the skew-adjoint build can be written in  $3^m$  rank  $2m+2$  skew-adjoint build tensors, so  $\eta_{2m+2}(n) \leq 3^m \eta(n)$ . □

This is interesting, because in [7], Ragosta proves that, for the self-adjoint build:  $\mu_m(n) \leq 2^m \mu(n)$ . This means that one can move up single dimensions in the self-adjoint case, something that is impossible in the skew-adjoint case because ever skew-adjoint matrix has an even rank, but using these estimates the skew-adjoint build seems to be more efficient, needing only  $3^m \eta(n)$  rank  $2m+2$  matrices, whereas in the self-adjoint case one might need up to  $2^{2m} \mu(n) = 4^m \mu(n)$ . The word only is a bit gratuitous here;  $3^m$  and  $4^m$  are much larger than  $\eta(n)$  and  $\mu(n)$  for large  $m$ , so it is highly unlikely that  $\eta_{2m+2}(n) = 3^m \eta(n)$  for large  $n$ .

**Corollary 6.2.1.** *The canonical algebraic curvature tensors with the skew-adjoint build in which the defining matrices all have rank  $\geq 2m$  form a spanning set of all algebraic curvature tensors.*

We can use this fact to improve upon the previous estimate for large  $n$ . Since every spanning set of a vector space must contain a basis, and we know that the skew-adjoint build curvature tensors of rank  $\geq 2m$  form a spanning set of  $\mathbb{A}(n)$ , we conclude that  $\eta_{2m}(n) \leq \dim(\mathbb{A}(n)) = \frac{n^2(n^2-2)}{12}$ .

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