Topological Realizations of Fully Augmented Links and Polyhedral Partners

Lisa Cenek

August 16, 2020

Abstract

We provide a new way of obtaining link complements from a polyhedral decomposition. Our topological approach shows that fully augmented link complements are homeomorphic to link complements in other 3-manifolds. We then extend our technique to find topological realizations of the more general class of polyhedral partners.

1 Introduction

A link L is a collection of closed loops embedded in a 3-manifold M. Links are typically studied in S^3, but in this paper, we discuss links in other 3-manifolds as well. If the complement $M \backslash L$ has a complete hyperbolic structure, then we say that L is hyperbolic.

A *fully augmented link* $\mathcal{F}$ is obtained from a link L in S^3 as follows, shown below in Figure 1. Begin with a link diagram of L, denoted $D(L)$. A twist region of $D(L)$ is a sequence of alternating crossings between two strands of the link. To augment $D(L)$, add an unknotted component, called a *crossing circle*, around each twist region. Then, to obtain $\mathcal{F}$, remove pairs of crossings from each twist region. This leaves either 0 or 1 crossings in each twist region, and we call the crossing circle either flat or twisted, respectively. The components of $\mathcal{F}$ that are not crossing circles are called *knot circles*. The fully augmented link is hyperbolic if and only if $D(L)$ has at least two twist regions and is nonsplittable, prime, and twist reduced (see [5]). Furthermore, any hyperbolic link can be obtained from some FAL via Dehn filling.

Complements of hyperbolic FALs have been well-studied, particularly for their geometric properties. In 2004, Agol and Thurston showed that the complement of a hyperbolic FAL can be decomposed into two identical,

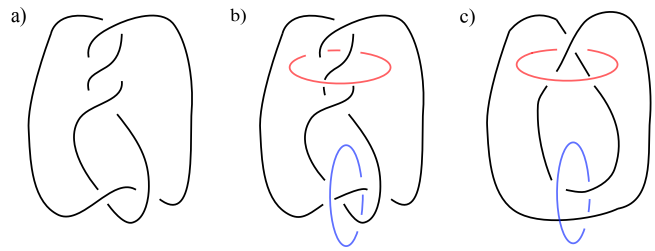

Figure 1: a) A link diagram; b) adding crossing circles to twist regions; c) removing pairs of crossings to obtain a fully augmented link with one twisted crossing circle (red) and one flat crossing circle (blue).

right-angled, totally geodesic, ideal polyhedra, P^+ and P^-, with faces that can be checkerboard-colored. See more in [5]. Conversely, given a pair of polyhedra $P^\pm$ with the properties just described, along with an FAL gluing pattern, one can obtain (up to half-twists of crossing circles) the complement of some FAL in S^3.

The decomposition of an FAL complement into P^+ and P^- is shown below in Figure 2. Part a) shows the Borromean Rings; the space surrounding the link is the link complement. In part b), we slice the link complement across the plane containing the black knot circle, which we call the plane of projection. In part c), the top halves of *crossing disks* (bounded by crossing circles) are flattened. Then, in part d), the top halves of crossing circles are shrunk down to vertices. In part e), black *knot arcs* are shrunk down to vertices. We call part e) the *cell decomposition* of $P^\pm$. In part f), we transform the cell decomposition to a circle packing on S^2. The circle packing gives rise to the two identical hyperbolic polyhedra, one above S^2 (P^+) and one below (P^-). See [5] for a more detailed description of the decomposition process.

Since the polyhedra $P^\pm$ can be checkerboard-colored, in the cell decomposition we choose to make the triangles that come from top halves of crossing disks shaded (blue and red triangles in part e) above), and all other remaining regions are left unshaded. An FAL gluing pattern on the polyhedra $P^\pm$ is an identification of each unshaded face on P^+ with its corresponding face (i.e. its reflection) on P^-, and an identification of each shaded triangle with another shaded triangle that shares a common vertex. The FAL gluing pattern is represented in the cell decomposition by the col-

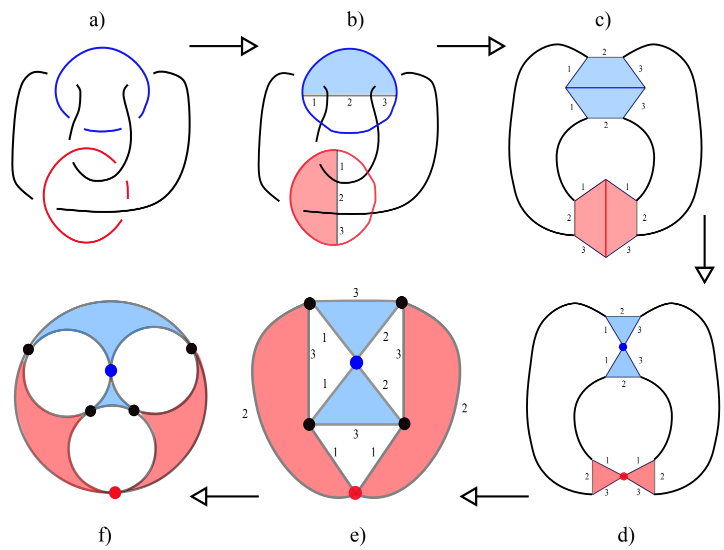

Figure 2: Polyhedral decomposition of the complement of the Borromean rings.

oring and edge numbering of the shaded triangles. Two triangles shaded in the same color are identified by gluing their edges with the same number. For any pair of identified triangles, there are two possible edge numberings that yield FALs, corresponding to a flat or a twisted crossing circle. A flat crossing circle is the result of identifying two triangles in P^+, whereas a twisted crossing circle is the result of identifying a triangle in P^+ with a triangle in P^-. The edge numbering shown on the blue triangles in Figure 2f yields a flat crossing circle in the link; swapping the numbers 1 and 2 on one of the blue triangles would lead to a twisted crossing circle. Other ways of numbering these edges do not yield FAL complements (but rather polyhedral partners, which we will discuss in Section 4).

Figure 2 shows a geometric approach to studying the polyhedral decomposition of an FAL complement $S^3 \backslash L$. The goal of this paper is to find topological realizations of these glued polyhedra. By a *topological realization*, we mean another 3-manifold M and a link L' in M such that $S^3 \backslash L$ and $M \backslash L'$ are homeomorphic. In Section 3, we provide an original approach –

an alternative to the geometric approach just described– of obtaining link complements from cell decompositions, and we show that our method yields not only $S^3 \backslash L$ but complements of links in other manifolds as well. We will describe the tools needed for our topological approach in Section 2. We will show in Section 4 that our approach can be applied to polyhedral partners, a generalization of FAL complements. In Section 5, we will use our approach to analyze the topological realizations of a particular class of polyhedral partners.

2 Handlebodies and Heegaard Splittings

In this section, we introduce the construction of different 3-manifolds that will arise in later sections as link complements.

We begin with an example. The manifold $S^2 \times S^1$ can be constructed from two solid tori H_1 and H_2 as shown in Figure 3. To do this, identify a meridinal curve on H_1 with a meridinal curve on H_2. Notice that the meridinal disks bounded by the identified meridinal curves form a sphere in the resulting manifold. Gluing all meridinal curves in this manner yields a sphere for each point around the longitude of H_2, i.e. we obtain $S^2 \times S^1$.

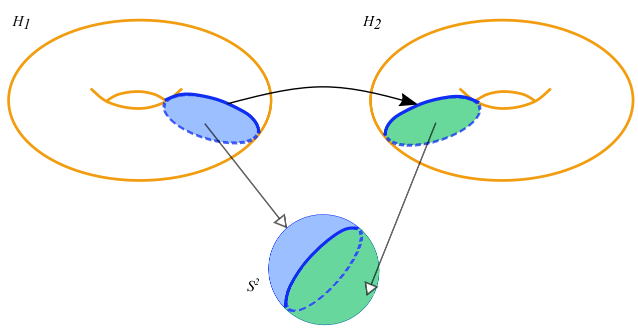

Figure 3: Construction of $S^2 \times S^1$ by gluing two solid tori (genus 1 handlebodies) from meridian to meridian. $S^2 \times S^1$ has Heegaard genus 1.

Other manifolds can also be obtained from two solid tori by gluing a meridinal curve of one torus to some other curve on the second torus. These manifolds are called *lens spaces*. We will describe lens spaces in more detail in Section 5, where we discuss link complements that involve lens spaces.

Lens spaces can be generalized even further. A torus is a genus 1 *handlebody*; so, rather than just manifolds (such as $S^2 \times S^1$) that can be constructed from two tori, we can look at manifolds that can be constructed from two genus g *handlebodies*.

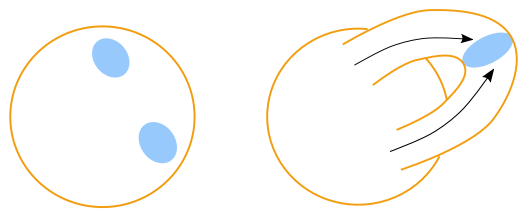

Figure 4: Constructing a handlebody.

Definition 1. Let B be a closed 3-ball, and let $\{D_1, ..., D_g, D'_1, ..., D'_g\}$ be a set of pairwise disjoint disks in ∂B. Let $\phi_i : D_i \to D'_i$ be a homeomorphism. Then a *handlebody* H is the genus g 3-manifold obtained after gluing along ϕ_1, gluing along ϕ_2, ..., gluing along ϕ_g.

Gluing two disks D_i, D'_i on ∂B, as shown in Figure 4, is equivalent to attaching a 1-handle $D^1 \times D^2$ to B by gluing $\partial D^1 \times D^2$ to ∂B. Thus, we will refer to the operation of gluing along ϕ_i as *attaching a 1-handle*. Further information about j-handle attachments can be found in [3].

Notice that a torus is obtained by gluing one 1-handle to a 3-ball. As shown in Figure 3, we can construct more complicated manifolds by gluing together two handlebodies in a particular way. In the construction of $S^2 \times S^1$, it was enough to identify the boundary of a single meridinal disk embedded in H_1 with a meridinal curve on the boundary of H_2. The identification of the rest of the torus is determined by the image of the meridinal curve. This is true for genus 1 handlebodies because one meridinal disk suffices as a *system of disks* for H_1:

Definition 2. A *system of disks* for a handlebody H is a set $\{D_1, ..., D_m\}$ of properly embedded, essential disks such that the complement of a regular neighborhood of $\bigcup D_i$ is a collection of 3-balls. Furthermore, a system of disks is *minimal* if the complement is connected.

Intuitively, one can see in Figure 3 that the meridinal disk in H_1 is a system of disks for H_1 because slicing along that disk yields a 3-ball.

So, to glue two handlebodies H_1, H_2 of genus g, it suffices to find a system of disks $\{D_1, ..., D_m\}$ for H_1, and then identify each ∂D_i with some curve l_i on ∂H_2. We call the pair $(H, \{l_i, ..., l_m\})$ a *Heegaard diagram* (see [2]).

Handlebodies are relatively simple manifolds to understand. A Heegaard diagram allows us to combine two handlebodies into a more complicated manifold. Conversely, we would like to be able to split up a given manifold into handlebodies:

Definition 3. A *Heegaard splitting* of a 3-manifold M is an ordered triple (Σ, H_1, H_2) where Σ is a closed surface embedded in M and H_1 and H_2 are handlebodies embedded in M such that $\partial H_1 = \Sigma = \partial H_2 = H_1 \cap H_2$ and $H_1 \cup H_2 = M$. The surface Σ is called a Heegaard surface. The *Heegaard genus* of M is the smallest possible genus of a Heegaard splitting of M.

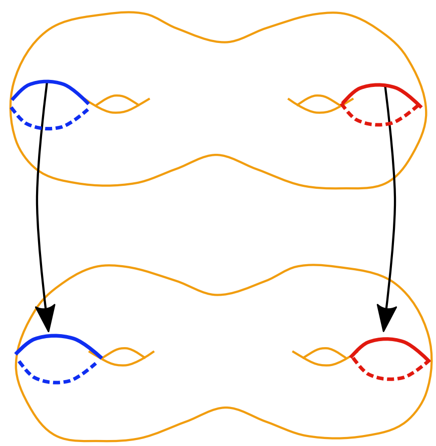

Figure 5: A genus 2 Heegaard splitting of $\#S^2 \times S^1$.

Lens spaces (including $S^2 \times S^1$) are thus manifolds of Heegaard genus 1, with a Heegaard splitting given by (T^2, H_1, H_2), where T^2 is a torus and H_1 and H_2 are each solid handlebodies of genus 1. We will be using Heegaard splittings as a tool for polyhedral decompositions of link complements.

In the remaining sections of this paper, we will construct links in the *connected sum* of manifolds. The connected sum of two 3-manifolds M_1 and M_2, denoted $M_1 \# M_2$, is the 3-manifold obtained by removing the interior of a 3-ball from each of M_1 and M_2, and then gluing the boundaries of each removed 3-ball to each other. When $M_1 = M_2$, we abbreviate the connected sum as $\#M_1$. Furthermore, we write $\#^k M$ to mean k copies of M joined together by repeatedly applying the connected sum operation. Thus $\#^2 M = \#M$ and trivially $\#^1 M = M$.

3 Topological Realizations of Fully Augmented Links

The Heegaard splitting of a manifold into two identical handlebodies motivates this next section. Since the complement of a fully augmented link in S^3 decomposes into two identical polyhedra $P^{\pm}$, it will be useful to construct handlebodies $H^{\pm}$ from $P^{\pm}$. To do this, we will use 1-handle attachments to the cell decomposition of $P^{\pm}$. Constructing these handlebodies will allow us to find distinct topological realizations of an FAL complement $S^3 \backslash L$. Of course, one such realization of a decomposition of $S^3 \backslash L$ into $P^{\pm}$ should be $S^3 \backslash L$ itself. We will begin by showing how to obtain this realization: starting with an FAL complement, we will build handlebodies from the cell decomposition, and then we will use the handlebodies to reconstruct the original FAL complement. This process is a new approach and an alternative to the process outlined in the introduction of this paper. Our leading example will be the Borromean Rings, shown in Figure 6a.

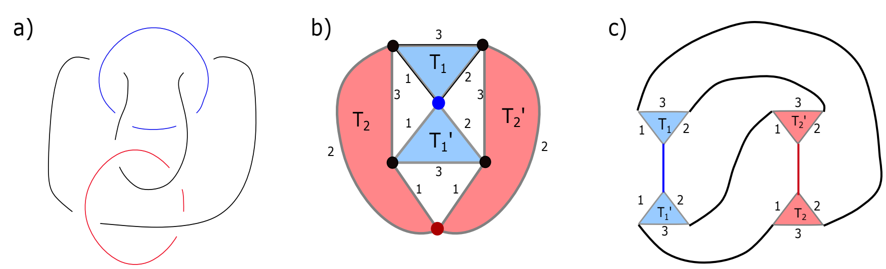

Figure 6: a) the Borromean rings; b) the cell decomposition of the complement of the Borromean rings; c) the cell decomposition after stretching vertices to arcs.

Begin with a cell decomposition of the complement of an FAL with $2g$ shaded triangles $T_1, ..., T_g, T_1', ..., T_g'$, along with an FAL gluing pattern (Figure 6b). Now, stretch each ideal vertex to an arc with endpoints on the boundaries of shaded triangles and unshaded regions on each side of the arc (Figure 6c). Since an arc comes from either a knot circle or a crossing circle, we will call an arc either a *knot arc* (black arcs in Figure 6c) or a *crossing* *arc* (blue and red arcs). Notice that identified shaded triangles are joined together by a crossing arc, and the gluing pattern identifies one endpoint of the crossing arc with the other.

We now describe how to form the handlebody H^+ from P^+ by gluing shaded triangles; applying the same operations to P^- will form H^-, which is a reflection of H^+. Note that P^+ begins as a 3-ball, and the cell decomposition is on the boundary of P^+. So, the set $\{T_1, ..., T_g, T'_1, ..., T'_g\}$ is a set of pairwise disjoint disks in ∂P^+, and we have a gluing map $\phi_i : T_i \to T'_i$ for each i. Now glue each T_i to T'_i, as shown in Figure 7b and 7c. By Definition 1, the resulting manifold is a genus g handlebody H^+. Intuitively, one can see that gluing each pair of identified triangles "adds a handle" to the manifold, which increases the genus of the manifold by 1. Furthermore, the boundary ∂H^+ of H^+ is a genus g surface.

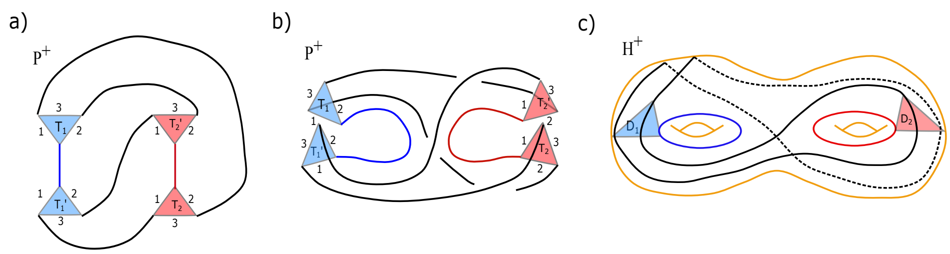

Figure 7: Gluing shaded triangles to form genus 2 handlebody.

We now describe properties of the genus g handlebody $H^\pm$. Let D_i denote the meridinal disk formed by gluing shaded triangles T_i and T'_i. Observe that the set of all D_i is a minimal system of disks for H, since slicing along all D_i yields a 3-ball. Furthermore, notice that the endpoints of the arcs on ∂T_i are glued to the endpoints of the arcs on $\partial T'_i$, resulting in three arcs that each "go around the handle" and puncture the boundary of D_i. In particular, since the endpoints of each crossing arc are identified, each crossing arc in the cell decomposition (Figure 7a) corresponds to a curve on H^+ (red/blue curve in Figure 7c) that goes once around a handle. We call this curve a *natural longitude*.

Next, to obtain a link complement in S^3, we utilize 2-handle attachments to $H^\pm$. A 2-handle is a solid cylinder $D^2 \times D^1$ that can be attached to a manifold by gluing $\partial D^2 \times [0, 1]$ to an annulus on the manifold (see [3]). Glue a 2-handle to each natural longitude on H^+. To do this, glue $\partial D^2 \times [0, 1]$ to an annular neighborhood of the natural longitude, as shown in Figure

8a. Furthermore, when we attach a 2-handle to each natural longitude, we also drill out a core of the 2-handle in order to preserve the crossing circle in the link complement (in the rest of this paper, when we say 2-handle, we always mean a 2-handle with a drilled out core). Gluing a 2-handle to a natural longitude on H^+ yields a handlebody with genus one less than H^+. Furthermore, by an Euler characteristic argument, gluing a 2-handle to any nonseparating curve on H^+ reduces the genus of ∂H^+ by 1. Thus, after gluing 2-handles to all natural longitudes, the genus of H^+ is now 0, so H^+ is homeomorphic to a 3-ball with g drilled out arcs, as shown in Figure 8b. We apply the same 2-handle attachments to H^- so that H^- is now also homeomorphic to a 3-ball with g drilled out arcs. Curves on the surface of the 3-ball will become knot circles of the link, and drilled arcs will be halves of crossing circles.

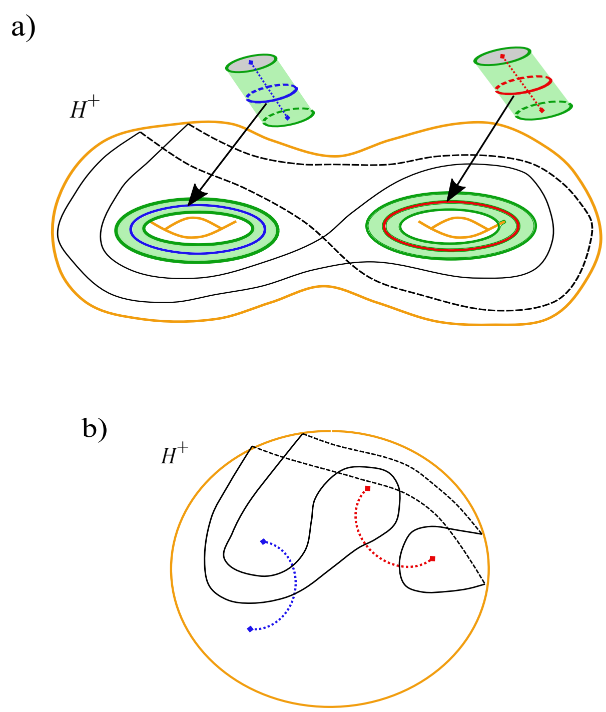

Figure 8: Gluing 2-handles (with drilled out cores) to natural longitudes on ∂H^+ to obtain a solid 3-ball.

It now remains to glue the unshaded faces of P^+ to their corresponding unshaded faces in P^-. Since all shaded triangles have been glued, the unshaded faces of $P^{\pm}$ correspond exactly to the boundary surface of $H^{\pm}$. Thus, as shown in Figure 9a, glue ∂H^+ to ∂H^- using the reflection map as the identification map. This results in endpoints of drilled arcs being glued, forming full crossing circles in S^3, as shown in Figure 9b. Furthermore, knot circles glue to their reflections. In this example, one can see that the resulting link in S^3, consisting of two crossing circles and one knot circle, is isotopic to the Borromean Rings, as desired.

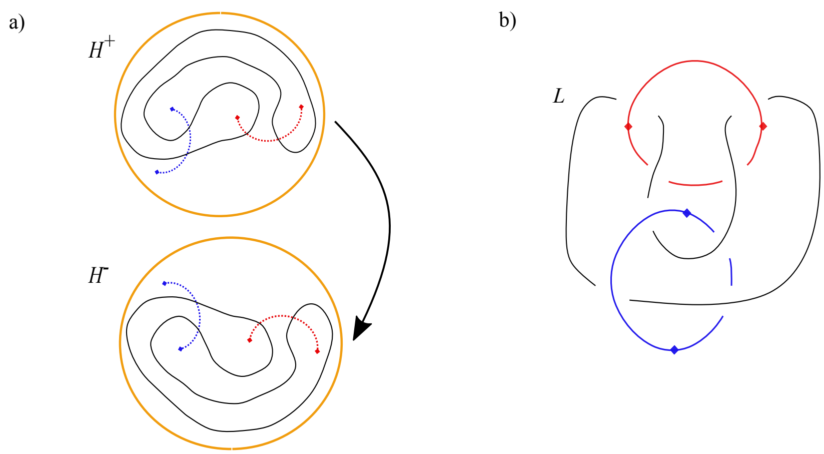

Figure 9: Gluing ∂H^+ to ∂H^- by reflection to obtain the complement of the Borromean Rings in S^3.

The above topological realization of the gluing pattern on the polyhedra $P^{\pm}$ associated with the Borromean rings is not unique. We obtained the Borromean rings after gluing H^+ to H^-, which occurred after gluing 2-handles to all natural longitudes on $H^{\pm}$. By changing the number of 2-handle attachments, we can find two other topological realizations:

Lemma 3.1. *The complement of the Borromean rings in S^3 is homeomorphic to each of the following:*
i) the complement of a link in $\#^2 S^2 \times S^1$
ii) the complement of a link in $S^2 \times S^1$

Proof. Begin with the cell decomposition of the complement of the Borromean rings. As above, stretch vertices to arcs, and then glue identified shaded triangles to obtain two genus 2 handlebodies $H^{\pm}$ each with a genus 2 surface as its boundary.

Now, to realize a complement of a link in $\#^2 S^2 \times S^1$, glue H^+ to H^- along their boundaries, as shown in Figure 10a.

Otherwise, to realize a complement of a link in $S^2 \times S^1$, first glue a 2-handle to one of the natural longitudes on $H^{\pm}$. This yields two genus 1 handlebodies $H^{\pm}$, each with a genus 1 surface as its boundary. Then, gluing ∂H^+ to ∂H^- by reflection yields $S^2 \times S^1$, as shown in Figure 10b.

$\square$

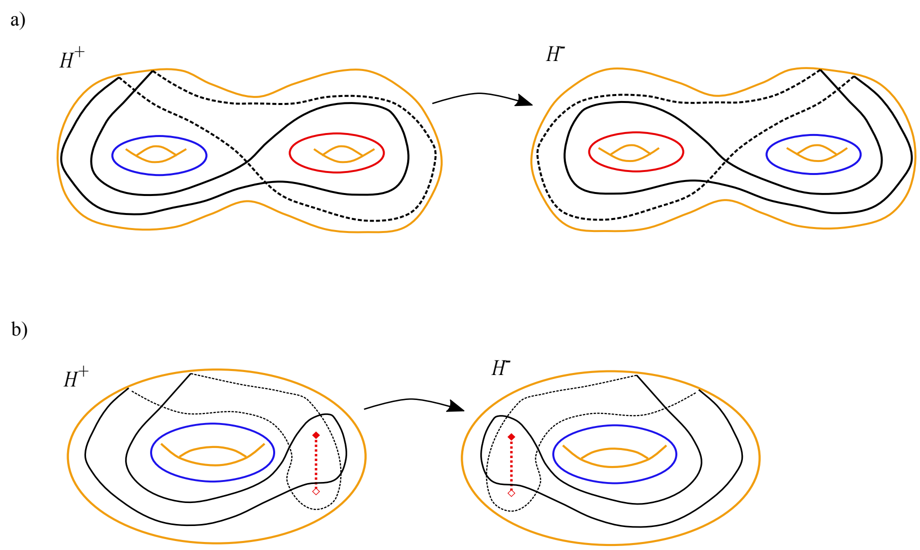

Figure 10: a) constructing a link complement in $\# S^2 \times S^1$; b) constructing a link complement in $S^2 \times S^1$.

In fact, Lemma 3.1 generalizes to the following result about complements of any FALs:

Theorem 3.1. *Let M be the complement of an FAL in S^3, and let $2g$ be the number of shaded triangles in the polyhedral decomposition of M. Then M is homeomorphic to the complement of a link in $\#^k S^2 \times S^1$ for each $1 \leq k \leq g$.*

Proof. Let M be the complement of any FAL in S^3, and let $2g$ be the number of shaded triangles on the polyhedra $P^{\pm}$ associated with M. Stretch vertices to arcs and glue identified shaded triangles to obtain genus g handlebodies $H^{\pm}$. Since the gluing pattern on $P^{\pm}$ yields an FAL, all identified pairs of triangles have a crossing arc that yields a natural longitude on the genus g handlebodies $H^{\pm}$, i.e. every handle of $H^{\pm}$ has a natural longitude. So, for any $1 \leq k \leq g$, glue a 2-handle to any $g-k$ of the natural longitudes. The result is a genus k handlebody $H^{\pm}$. Gluing two genus k handlebodies by the reflection on their boundary yields the connected sum $\#^k S^2 \times S^1$. Therefore, gluing ∂H^+ to ∂H^- results in a link in $\#^k S^2 \times S^1$. □

4 Topological Realizations of Polyhedral Partners

In the previous section, we described how to find topological realizations of FAL complements. We will now show that our approach of 1-handle and 2-handle attachments can be used for polyhedra with less restrictive gluings. FAL complements can be generalized to *polyhedral partners*:

Definition 4. (Meyer, Millichap, and Trapp [4]) Let $\mathcal{M}_{\mathcal{F}} = S^3 \setminus \mathcal{F}$ for some hyperbolic FAL $\mathcal{F}$, and let $P^{\pm}$ be the two associated totally geodesic ideal polyhedra. We say that M is a *polyhedral partner* of $\mathcal{M}_{\mathcal{F}}$ if M can be constructed from $P^{\pm}$ as follows:

(i) Corresponding unshaded faces of $P^{\pm}$ are identified in the same manner as $\mathcal{M}_{\mathcal{F}}$, and

(ii) If $\phi : G \to H$ identifies shaded faces G and H, then their corresponding faces are identified by conjugating ϕ with the reflection between $P^{\pm}$.

Polyhedral partners have been studied in [1] and [4]. In this paper, we will often omit the relation of polyhedral partner to a particular FAL complement. When we discuss a polyhedral partner M, one should be aware that M is implicitly the polyhedral partner of some $\mathcal{M}_{\mathcal{F}}$, since M can be decomposed and then re-glued in a different manner to obtain some FAL complement. However, in our context, the FAL complement that decomposes into the same polyhedra as our polyhedral partner M is of no particular importance to our topological realizations, and thus we will speak simply of a polyhedral partner M. Furthermore, we will only consider flat polyhedral partners. A polyhedral partner M, with associated polyhedra $P^{\pm}$, is *flat* if no shaded triangle in P^+ is identified with a shaded triangle in P^-. This corresponds to the notion of flat versus twisted crossing circles in FAL complements, as described in the introduction.

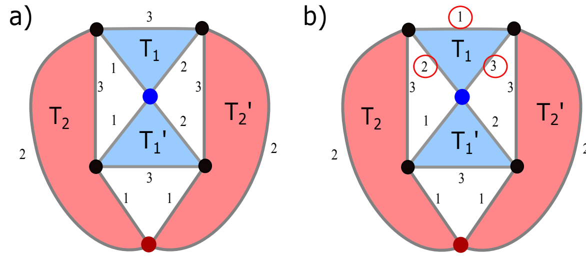

Figure 11: a) cell decomposition of an FAL complement; b) cell decomposition of a polyhedral partner, which differs from a) only by the edge numbering of T_1.

We will now discuss topological realizations of flat polyhedral partners. Let M be a flat polyhedral partner, and begin with the cell decomposition of the associated $P^{\pm}$, as shown in Figure 11b. First, as with FAL complements, stretch vertices into arcs and then glue 1-handles to identified shaded triangles. This transforms P^{+} from a 3-ball to a genus g handlebody H^{+} with a genus g surface as its boundary. Applying the same operations to P^{-}, we have a second genus g handlebody H^{-} with genus g surface as its boundary.

Distinct topological realizations of the glued $P^{\pm}$ can now be achieved by different choices of the number and/or order of 2-handle attachments to $H^{\pm}$. In particular, as exemplified in part (i) of Lemma 3.1, it is not required that any 2-handles be glued to the handlebodies $H^{\pm}$: after 1-handles are glued, the boundary of H^{+} consists of unshaded regions, which are identified with their reflections in H^{-}, and thus we can immediately glue genus g handlebodies H^{+} and H^{-} via the identification of their boundaries.

So, part (i) of Lemma 3.1 generalizes to the following result about any flat polyhedral partner:

Theorem 4.1. *Any flat polyhedral partner with $2g$ shaded regions on each of the associated polyhedra $P^{\pm}$ is the complement of some link in $\#^{g} S^2 \times S^1$.*

Proof. Let M be a flat polyhedral partner, and suppose that the associated polyhedra $P^{\pm}$ each have $2g$ shaded triangles. Attach 1-handles to obtain two genus g handlebodies, $H^{\pm}$, each with a genus g surface as boundary. Now, without gluing any 2-handles, glue ∂H^{+} to ∂H^{-}. Since the gluing map is the reflection, the resulting manifold containing the link is $\#^{g} S^2 \times S^1$. $\square$

We now want to consider gluing 2-handles to curves on $\partial H^{\pm}$ in order to achieve different topological realizations. Recall that in Section 3, we glued 2-handles to the natural longitudes, and that doing so yields a handlebody with genus reduced by 1 for each 2-handle that was glued. Natural longitudes are formed from crossing arcs in the cell decomposition, and thus in the case of FAL complements, there is a natural longitude around every handle in the handlebody obtained after gluing 1-handles. For polyhedral partners, since the gluing pattern does not require there to be crossing arcs, not every handle will necessarily have a natural longitude. Knowing the number of handles with natural longitudes will be helpful for analyzing possible 2-handle attachments.

We thus introduce the following definition. In a polyhedral decomposition with a specified gluing pattern, we say that two identified shaded triangles T_i and T_i' are *augmented* if, after stretching vertices to arcs, there is an arc e_i with one endpoint on ∂T_i and one endpoint on $\partial T_i'$, and the gluing pattern identifies the endpoints of that arc with each other. In Figure 11b, the red triangles are augmented, and the blue triangles are not augmented. Notice that the complement of a fully augmented link has a polyhedral decomposition where every pair of identified shaded triangles is augmented (Figure 11a).

Augmented triangles guarantee a natural longitude on the handlebody, allowing for the simplest type of 2-handle attachment. We will see in Section 5 that when we glue 2-handles to curves other than natural longitudes, the resulting manifold is not necessarily a handlebody.

Definition 5. Let M be a polyhedral partner. If k is the number of pairs of augmented shaded triangles, we say that M is a k-*augmented* polyhedral partner. Furthermore, if M is homeomorphic to the complement of some link L, then we call L a k-augmented link.

A fully augmented link with g crossing circles is thus a g-augmented link, and its complement is a g-augmented polyhedral partner. So, this definition generalizes FAL complements to k-augmented link complements. Recall that Theorem 3.1 showed that FAL complements can be realized as the connected sum of any $1 \leq k \leq g$ copies of $S^2 \times S^1$. The following theorem generalizes this result to k-augmented link complements:

Theorem 4.2. *Let M be a k-augmented flat polyhedral partner with $2g$ shaded triangles on the associated $P^{\pm}$. Then M is homeomorphic to the complement of some link in $\#^j S^2 \times S^1$ for each $g-k \leq j \leq g$.*

Proof. Consider the polyhedra $P^{\pm}$ with $2g$ shaded triangles associated with M. Since M is k-augmented, each genus g handlebody obtained from attaching 1-handles to $P^{\pm}$ has a natural longitude around exactly k of the handles. So, for any $g-k \leq j \leq g$, glue a 2-handle to $g-j$ of the natural longitudes, reducing the genus of $H^{\pm}$ to j. Now, glue ∂H^{+} to ∂H^{-} by the reflection to obtain a link complement in $\#^{j} S^{2} \times S^{1}$. $\square$

5 Almost Augmented Links

We will now focus on *almost augmented links*, which are links whose complements are $(g-1)$-augmented flat polyhedral partners, also referred to as *almost augmented polyhedral partners*. The goal of this section is to find topological realizations of these manifolds. The main result of this section is Theorem 5.1, which shows that any almost augmented polyhedral partner is homeomorphic to a link complement in either S^3 or the connected sum of two lens spaces.

We first have the following result as an immediate corollary of Theorem 4.2:

Corollary 5.1. *Let M be an almost augmented polyhedral partner with $2g$ shaded triangles on each $P^{\pm}$. Then M is homeomorphic to the complement of some link in $\#^{k} S^{2} \times S^{1}$ for each $1 \leq k \leq g$.*

As discussed in Sections 3 and 4, we can obtain different realizations of polyhedral partners through different choices of 2-handle attachments to curves on the boundaries of the handlebodies $H^{\pm}$. Thus far, we have only glued 2-handles to natural longitudes. Notice that a natural longitude is a *nonseparating* curve: the surface $\partial H^{\pm}$ remains connected after cutting along the curve. We can in fact glue 2-handles to any nonseparating curves on the surface of the handlebody. In Corollary 5.1, the $\#^{k} S^{2} \times S^{1}$ manifolds were obtained by attaching 2-handles to natural longitudes only. We will now find other realizations of almost augmented polyhedral partners by also attaching 2-handles to nonseparating curves that are not natural longitudes.

We will focus on the case in which 2-handles have first been glued to all $g-1$ natural longitudes, resulting in $H^{\pm}$ each as a solid torus. In Corollary 5.1, we then glued the boundaries of these solid tori to each other, obtaining $S^2 \times S^1$, and any remaining curves on $\partial H^{\pm}$ became knot circles in the resulting link. See Figure 10b. Now, we would instead like to attach 2-handles to the remaining curves on the tori prior to gluing ∂H^{+} to ∂H^{-}. To understand the manifolds that result from these 2-handle attachments, we first investigate the types of curves that can remain on the genus 1 handlebody surfaces. To describe a curve γ on a torus, we write $\gamma = (p, q)$ for some relatively prime $p, q \in \mathbb{Z}$, where p is the *longitude number* and q is the *meridian number*.

We now explain how to use the cell decomposition of almost augmented polyhedral partner to determine the (p, q) values of each remaining curve on the torus. We first discuss how to calculate (p, q) of a curve on a torus using signed intersections, and then we translate the method from the torus to the cell decomposition.

As shown in Figure 12a, assign an orientation to the curve γ. Let m be the meridinal curve on the torus, and assign an orientation to m as well. Then for each point in $\gamma \cap m$, assign ± 1 using the "right hand rule" as follows. If a counter-clockwise rotation of the positive end of m aligns with the positive end of γ, assign $+1$; otherwise, assign -1. Then the longitude number p is equal to the sum of the ± 1 values for all intersection points. The meridinal number q is calculated similarly by looking at the intersection point of γ with an oriented longitudinal curve l (see Figure 13a).

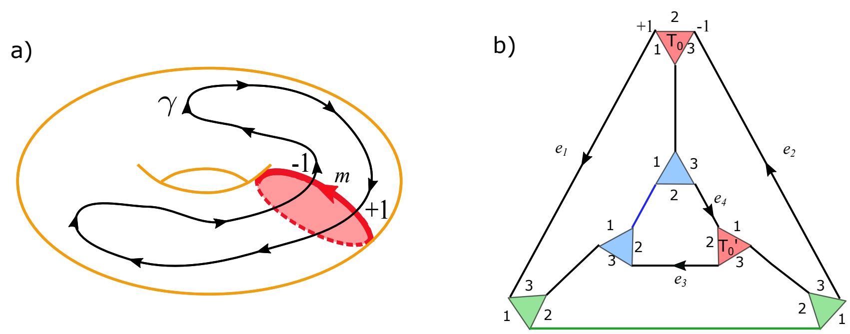

Figure 12: (a) determining the longitude number of γ; (b) determining the longitude number of the curve formed by arcs $(e_1, ..., e_4)$.

We can now translate signed meridinal and longitudinal intersections from the torus to the cell decomposition; this will allow us to determine the curves prior to constructing the handlebody. First, we assign an orientation to all knot arcs, as shown in Figure 12b. To do this, choose an arc e_1 on the cell decomposition, and give e_1 some orientation, i.e. an assignment of an initial vertex and terminal vertex. Since the terminal vertex of e_1 glues to the initial vertex of some other arc e_2, we can continue to orient the arcs according the gluing pattern. When the terminal vertex of some e_k glues back to the initial vertex of e_1, this completes the curve. If there exists some arc $e_i \notin (e_1, ..., e_k)$, repeat this process beginning with e_i until all arcs are oriented.

Next, we want to use an oriented meridinal curve to calculate the longitude number p of the curves formed by the arcs. Let $(e_1, ..., e_k)$ be the arcs in the cell decomposition that will glue to form a curve on the torus. Let T_0 be one triangle of the non-augmented pair of identified triangles. Since glued triangles form a system of disks in the handlebody, we know that ∂T_0 bounds an embedded disk in the torus and thus is a meridinal curve. So, we can use ∂T_0 to determine the longitude number of a curve on the torus by looking at the intersection points between that curve and ∂T_0. Let n be the number of arcs in $(e_1, ..., e_k)$ with an endpoint on ∂T_0. Note that $0 \leq n \leq 3$. If all arc endpoints on ∂T_0 are terminal endpoints, or if all arc endpoints on ∂T_0 are initial endpoints, then $p = n$. Otherwise, there must be an arc with a terminal endpoint on ∂T_0 and an arc with an initial endpoint on ∂T_0, and hence $p = n - 2$. This corresponds to intersections with opposite signs canceling each other out.

For example, in Figure 12b, the arc e_1 has its initial endpoint on ∂T_0, and e_2 has its terminal endpoint on ∂T_0. Hence the longitude number of the curve formed by $(e_1, ..., e_4)$ is $2 - 2 = 0$. Similarly, in Figure 12a, the curve γ intersects m twice, each time with a different orientation, and hence the longitude number of γ is 0.

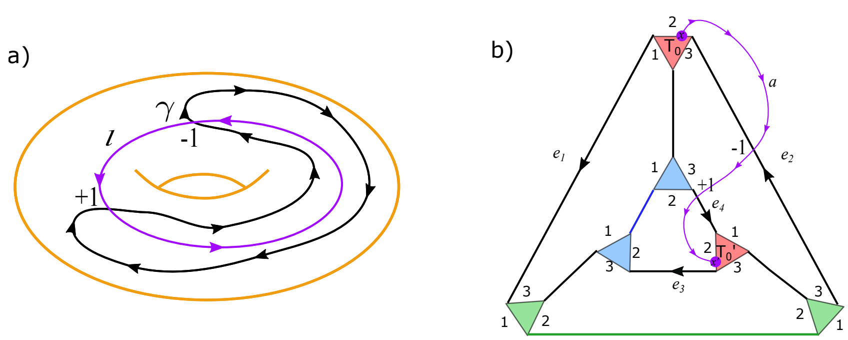

Figure 13: (a) determining the meridian number of γ; (b) determining the meridian number of the curve formed by arcs $(e_1, ..., e_4)$.

To determine the meridian number q of each curve formed by $(e_1, ..., e_k)$, choose any point x on ∂T_0 and its identified point x' on $\partial T_0'$, as shown in Figure 13b. Let a be any arc, with some orientation, from x to x' that does not intersect ∂T_0 or $\partial T_0'$ at any other points. Notice that a becomes a longitudinal curve when T_0 and T_0' are glued. Thus, for each point of intersection between a and any $e_i \in (e_1, ..., e_k)$, use the previously described "right hand rule" to assign ± 1. Then let q be the sum of all ± 1 values from intersection points. The meridian number is $q \pmod p$, where p is the number of longitudes as determined above. Different choices of the arc a lead to values of q that are equivalent mod p.

Let M be an almost augmented flat polyhedral partner. Stretch vertices on the cell decomposition of M to arcs, glue 1-handles to all identified shaded triangles, and glue 2-handles to all natural longitudes. This yields two genus 1 handlebodies $H^{\pm}$ each with a genus 1 surface as boundary. Let $T^2 = \partial H^+ = \partial H^-$. We now want to characterize the curves on T^2.

Lemma 5.1. *Up to homeomorphism of M, each remaining curve on T^2 is one of $\{(0,0), (1,0), (3,1), (3,2)\}$. Furthermore, at least one curve is not $(0,0)$.*

Proof. First we discuss curves that are the same up to homeomorphism of M. Given a curve $\gamma = (p, q)$ on T^2, a Dehn twist on a meridian of T^2 changes γ to the curve $(p, q + kp)$ for some $k \in \{-1, 1\}$. Such a twist extends to a homeomorphism of H^+ via the operation of slicing along the meridinal disk, doing one full twist, and gluing back together. The manifolds obtained by gluing H^+ to H^- with or without the Dehn twist are homeomorphic. Thus applying Dehn twists can change a curve on T^2 without changing M itself. In particular, this gives us that for any integer q, a $(1, q)$ curve can be obtained from a $(1, 0)$ curve by Dehn twists. Furthermore, for any integer $q \equiv 1$ or $2 \pmod 3$, the $(3, q)$ curve can be obtained from a $(3, 1)$ or a $(3, 2)$ curve by Dehn twists.

We will show that any curve $(p, q) \notin \{(0,0), (1,0), (3,1), (3,2)\}$ cannot exist on T^2. Let D_0 be the disk resulting from the pair of identified triangles T_0, T_0' that are not augmented. First, no curve on T^2 can have a longitude number greater than 3. This follows from the fact that ∂D_0 is a meridian of T^2 that is intersected three times by curves. So a curve can have a longitude number of at most 3. Furthermore, we cannot have a $(3, 3)$ simple closed curve on a torus, as p and q must be relatively prime.

We also claim that no curve on T^2 can be of the form $(2, q)$ for any q. If a curve has longitude number exactly 2, then there is no other disjoint curve on T^2 with longitude number exactly 1, which contradicts the fact that ∂D_0 is punctured exactly three times. To verify this, observe that the signed intersection of the torus knots $(2, q)$ and $(1, q')$ is given by $(2 * l + q * m)(1 * l + q' * m) = q - 2q'$, where $m * l = 1$ is the signed intersection of the meridian m and a longitude l. Then, $q - 2q' \neq 0$ since q must be odd, and thus $(2, q)$ and $(1, q')$ are not disjoint.

Therefore, if (p, q) is on T^2, then $(p, q) \in \{(0, 0), (1, 0), (3, 1), (3, 2)\}$. To prove the final part of the lemma, suppose that T^2 contains only $(0, 0)$ curves. Each $(0, 0)$ curve must puncture ∂D either 0 or 2 times. Clearly, no number of $(0, 0)$ curves will lead to ∂D to be punctured exactly 3 times in total. Therefore, there must be some curve on T^2 that is not $(0, 0)$.

□

Now, since Lemma 5.1 tells us exactly what (p, q) curves can result on T^2, we know exactly what 2-handle attachments are possible to the solid tori. Before stating our final result, we provide the following definition:

Definition 6. A *lens space*, denoted $L(p, q)$, is the 3-manifold obtained from two solid tori by identifying a meridinal curve of one torus to a (p, q) curve of the other torus.

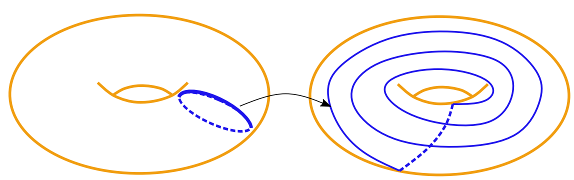

Figure 14: Construction of the lens space $L(3, 1)$.

Other definitions of lens spaces can be found in [6].

Almost augmented polyhedral partners can be topologically realized as follows:

Theorem 5.1. *Any almost augmented polyhedral partner is homeomorphic to the complement of a link in S^3 or in the connected sum $\#L(3, 1)$, or both.*

Proof. Let M be an almost augmented polyhedral partner. From the cell decomposition, stretch vertices to arcs, then glue 1-handles to all identified shaded triangles, and then glue 2-handles to all natural longitudes. By Lemma 5.1, each remaining curve on $\partial H^{\pm}$ must be $(0, 0)$, $(1, 0)$, $(3, 1)$, or $(3, 2)$.

If there is a $(1,0)$ curve on $\partial H^{\pm}$, then gluing a 2-handle to that curve reduces the genus of $\partial H^{\pm}$ and of $H^{\pm}$ both to zero. Thus, gluing H^+ to H^- after attaching this 2-handle yields a link in S^3.

If there is no $(1,0)$ curve on $\partial H^{\pm}$, then Lemma 5.1 shows that $\partial H^{\pm}$ must contain either a $(3,1)$ or a $(3,2)$ curve, to which we glue a 2-handle. As shown in Figure 15, gluing a 2-handle to a curve $\gamma = (p,q)$ then attaching a 3-ball is equivalent to forming a lens space $L(p,q)$. Thus the manifold obtained by attaching a 2-handle to H^+ along γ is the result of removing a 3-ball from $L(p,q)$. To see this, compare Figure 14 to Figure 15.

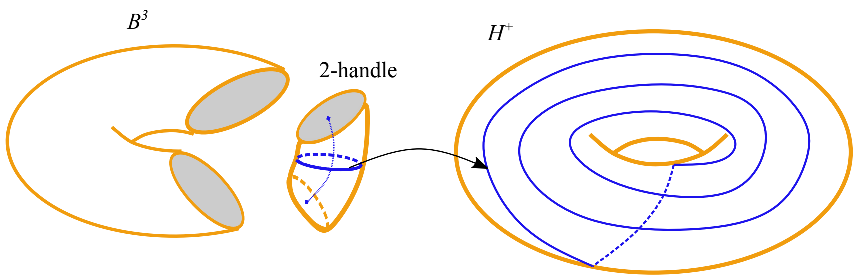

Figure 15: Gluing a 2-handle to a $(3,1)$ curve on a torus

Since gluing a 2-handle to γ reduces the genus of ∂H^+ by 1, the boundary of H^+ is S^2. However, note that H^+ is no longer a handlebody.

We now glue H^+ to H^-, where each is a copy of $L(p,q)\backslash B^3$, by identifying the boundaries ∂H^+ and ∂H^-. Since ∂H^+ and ∂H^- are each S^2, the operation of gluing $L(p,q)\backslash B^3$ to $L(p,q)\backslash B^3$ by identifying a sphere S^2 in each manifold is exactly the operation of the connected sum of $L(p,q)$ with $L(p,q)$. Hence, we obtain a link complement in $\#L(p,q)$, where (p,q) is the curve γ to which we glued a 2-handle. Note that the manifolds $\#L(3,2)$ and $\#L(3,1)$ are homeomorphic, since $\pm qq' \equiv 1 \pmod{p}$ suffices to show that $L(p,q)$ and $L(p,q')$ are homeomorphic [6].

$\square$

6 Future Work

- Analyze topological realizations of twisted polyhedral partners.
- Which 2-handle attachments lead to links in S^3?

- Compare the results of attaching 2-handles to nonseparating curves versus separating curves.
- Characterize 2-handle attachments that yield nested links.
- Generalize the results of almost augment links to k-augmented links. What types of curves can exist on the genus $g-k$ surfaces, and what are the resulting manifolds after further 2-handle attachments?

7 Acknowledgements

Thank you to my mentor, Dr. Rolland Trapp, for the invaluable guidance, insight, and creative ideas throughout this this project. I would also like to thank Dr. Corey Dunn for coordinating this REU program. This work was funded by California State University, San Bernardino as well as the NSF (grant no. 1758020).

References

[1] J. Harnois, H. Olson, and R. Trapp, *Hyperbolic Tangle Surgeries and Nested Links*, Algebraic and Geometric Topology **18**.

[2] J. Johnson, *Notes on Heegaard Splittings.*

[3] B. Martelli, *An Introduction to Geometric Topology*, 2016.

[4] J. Meyer, C. Millichap, and R Trapp, *Arithmeticity and hidden symmetries of fully augmented pretzel link complements*, New York Journal of Mathematics **26** (2020).

[5] J. Purcell, *An Introduction to Fully Augmented Links*, Contemporary Mathematics **541** (2011).

[6] D. Rolfsen, *Knots and Links*, 1976.

[7] N. Saveliev, *Lectures on the topology of 3-manifolds*, 1999.