k-Plane Scalar Curvature and Structure Groups of Model Spaces

Preston Kelley

California State University San Bernardino

Abstract

In this paper we find the possible values of k-plane scalar curvatures for model spaces with a canonical algebraic curvature tensor built from a positive definite symmetric, bilinear form. We use this result to find all possible structure groups of model spaces of this type.

1 Introduction

Let V be a real vector space of finite dimension N with $V^* = \mathrm{Hom}(V, \mathbb{R})$ its dual space. An algebraic curvature tensor (ACT) is a function $R \in \otimes^4 V^*$ such that if $x, y, z, w \in V$, then

$$\begin{aligned} R(x, y, z, w) = R(z, w, x, y) = -R(y, x, z, w) \\ \text{and } R(x, y, z, w) + R(y, z, x, w) + R(z, x, y, w) = 0. \end{aligned} \tag{1}$$

We denote the space of algebraic curvature tensors over V as $\mathcal{A}(V)$. Let $S^2(V^*)$ denote the space of symmetric, bilinear forms over V. Throughout this paper, it will be assumed that the variable $\varphi \in S^2(V^*)$. With this assumption, a canonical ACT $R_\varphi \in \mathcal{A}(V)$ is defined as

$$R(x, y, z, w) = \varphi(x, w)\varphi(y, z) - \varphi(x, z)\varphi(y, w), \text{ for all } x, y, z, w \in V.$$

If there is an inner product $<,>\in S^2(V^*)$, we can form a so-called model space, $\mathfrak{M} = (V, <,>, R)$. In this paper, $<,>$ is assumed to be positive definite.

Given a model space $\mathfrak{M} = (V, <,>, R_\varphi)$ where $\dim V = N$, it is helpful to establish some terminology (see Section 2 of [1] for a more detailed discussion). It is well-known that there is a unique, self-adjoint linear map, $T : V \to V$, such that if $x, y \in V$, then $\varphi(x, y) = \langle Tx, y\rangle$. Throughout this paper, φ will be described using T. In particular, we define $\ker(\varphi)$ to be $\ker(T)$ and $\mathrm{rank}(\varphi)$ to be $\mathrm{rank}(T)$.

We will also relate the eigenvalues and eigenvectors of $T : V \to V$ to φ, specifically under an orthonormal basis for V that diagonalizes φ, which exists by the Spectral Theorem. This allows us to define φ to be positive definite when T is positive definite. If $\beta = \{e_1, \cdots, e_N\}$ is an orthonormal basis for V that diagonalizes φ, the eigenvalues of

φ are $\varphi(e_i, e_i)$, and $e_1, \cdots, e_N$ are eigenvectors. More generally, if $\beta = \{e_1, \cdots, e_N\}$ is any orthonormal basis, which may not diagonalize φ, then e_i is an eigenvector if and only if $\varphi(e_i, e_j) = \lambda_i \delta_{i,j}$ for all j. If the eigenvalues of φ are $\lambda_1, \cdots, \lambda_m$, then we will denote the eigenspaces of V as $E_{\lambda_1}, \cdots, E_{\lambda_m}$.

In a model space $\mathfrak{M} = (V, <,>, R)$, we can consider a non-degenerate 2-plane, π =span$\{x, y\}$, where $x, y \in V$. The sectional curvature of π is

$$\kappa(\pi) = \frac{R(x, y, y, x)}{\langle x, x\rangle \langle y, y\rangle - \langle x, y\rangle^2}. \tag{2}$$

One of the most significant facts about $\kappa(\pi)$ is that it is independent the basis chosen for π. There is a need, however, for a generalization of sectional curvature to higher-dimensional subspaces. One such construction is k-plane scalar curvature, also called k-Ricci curvature [2, 3].

Definition 1.1. Let $\mathfrak{M} = (V, <,>, R)$ and $L \subseteq V$ be a k-plane with an orthonormal basis $\{f_1, \cdots, f_k\}$. The k-plane scalar curvature of L is

$$\mathscr{K}(L) = \sum_{j>i=1}^{k} \kappa(f_i, f_j), \tag{3}$$

where $\kappa(f_i, f_j)$ is the sectional curvature of the 2-plane span$\{f_i, f_j\}$ (2).

Note that even though $\mathscr{K}(L)$ is invariant with respect to the basis chosen for L, our definition requires that the basis for L is orthonormal.

The application of curvature presented in this paper is finding the structure groups of model spaces. Here we will define the structure group for a specific type of model space, although this definition can be generalized (see [4]).

Definition 1.2. Let $\mathfrak{M} = (V, <,>, R)$, with $\dim V = N$. The structure group of $\mathfrak{M}$ is

$$\begin{gathered} G_{\mathfrak{M}} = \{A \in GL(N) \mid R(Ax, Ay, Az, Aw) = R(x, y, z, w), \\ \langle Ax, Ay\rangle = \langle x, y\rangle \, \forall x, y, z, w \in V\}. \end{gathered}$$

We may also consider the structure group for just one of $<,>$, R, or φ (if $R = R_\varphi$).

$$\begin{gathered} G_{<,>} = O(N) = \{A \in GL(N) \mid (\langle Ax, Ay\rangle) = \langle x, y\rangle \;\; \forall x, y \in V\} \\ G_R = \{A \in GL(N) \mid R(Ax, Ay, Az, Aw) = R(x, y, z, w) \, \forall x, y, z, w \in V\} \\ G_\varphi = \{A \in GL(N) \varphi(Ax, Ay) = \varphi(x, y) \, \forall x, y \in V\} \end{gathered}$$

We may write $R(Ax, Ay, Az, Aw)$ as $A^*R(x, y, z, w)$, $\langle Ax, Ay\rangle$ as $A^* \langle x, y\rangle$, and $\varphi(Ax, Ay)$ as $A^*\varphi(x, y)$. This is known as the precomposition of A on R, $<,>$, and φ, respectively. When $A^*R = R$ and $A^*(<,>) =<,>$, A is said to preserve R and $<,>$, respectively. If there is a subspace $W \subseteq V$ such that for all $A \in G_{\mathfrak{M}}$ we have $A : W \to W$, then W is said to be an *invariant subspace* under the action of $G_{\mathfrak{M}}$ (for convienence we will simply call W an invariant subspace without reference to the structure group whenever there is no possibility of confusion).

If $\mathfrak{M} = (V, <,>, R_\varphi)$ where φ is positive-definite and $\dim V = N \geq 3$, we can check whether a mapping $A : V \to V$ is in $G_{\mathfrak{M}}$ by considering the action of A on any basis for V, with the following method. Let $\beta = \{e_1, \cdots, e_N\}$ be a basis for V. It is well-known that the orthogonal group preserves the inner product over a vector space, so we begin by checking that $A \in O(N)$, so we know $<,>$ is preserved. Now, to check that R_φ is preserved, we only need to check that φ is preserved, since in the unbalanced-signature setting for φ, when $\dim V = N \geq 3$, we know $G_{R_\varphi} = G_\varphi$ ([4], Theorem 1.5). Now, we only need to check that $A^*\varphi(e_i, e_j) = \varphi(e_i, e_j)$ for all i, j.

If there is an invariant subspace $W \subseteq V$, we may want to consider $G_{\mathfrak{M}}$ restricted to W. We can denote this as $G_{\mathfrak{M}}|_W$, and define it as follows.

Definition 1.3. Let $\mathfrak{M} = (V, <,>, R)$ and let $W \subseteq V$ be an invariant subspace under the action of $G_{\mathfrak{M}}$. Then $G_{\mathfrak{M}}|_W$ is the subgroup of $G_{\mathfrak{M}}$ such that $A \in G_{\mathfrak{M}}|_W$ if $A \in G_{\mathfrak{M}}$ and $A(x) = x$ for all $x \in W^\perp$.

So if we have some $B \in G_{\mathfrak{M}}|_W$, we can write it as a matrix with the following notation:

$$B = \begin{bmatrix} I & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & I \end{bmatrix},$$

where the I's are the identity submatrices of appropriate size and b is the submatrix that acts on W. For example, if V has the basis $\{e_1, \cdots, e_6\}$, where $\{e_2, e_3, e_4\}$ is a basis for W, then the top-left I is 1, b is the 3x3 submatrix that acts on $\{e_2, e_3, e_4\}$, and the other I is the 2x2 identity matrix.

We now give a brief outline of the paper, with non technical statements of our theorems. The next section of this paper briefly describes some results that are necessary for our research. We follow in Section 3 with our findings relating to k-plane scalar curvature. Our first result in this section, Theorem 3.1, is that extremal k-planes have an orhtonormal basis of eigenvectors when φ is positive definite. The first corollary to this, Corollary 3.2, tells us the precise interval of k-plane scalar curvatures when φ is positive definite. We conclude this section with an example that demonstrates that Theorem 3.1 is not necessarily true when φ is not positive definite.

In Section 4, we apply these results to finding the structure groups of model spaces. Corollary 4.1 says that the eigenspaces of a model space where φ is positive definite are invariant. Theorem 4.3 proves, generally, that the structure group of a model space is the internal direct product of the structure groups restricted to invariant subspaces. Theorem 4.4 combines Corollary 4.1 and Theorem 4.3 to characterize all structure groups of model spaces where φ is positive definite and $\dim(V) \geq 3$. Finally, we end with Section 5, which includes some open questions, and then give acknowledgements and references in Section 6.

2 Preliminaries

Consider $\mathfrak{M} = (V, <,>, R_\varphi)$. In [1], it is proven that a 2-plane with extremal sectional curvature has an orthonormal basis of eigenvectors. The method of proof originates from [5], which we will also utilize in 3.1. Calle and Dunn use this result to find the precise interval of sectional curvature values for a model space (Theorem 1.1), along with using the fact that the Grassmannian of 2-planes in V, which is the set of all 2-planes that pass through the origin in V, is compact and connected. This is also true for the Grassmannian of k-planes in V, denoted $Gr_k(V)$, which is the set of all k-planes that pass through the origin in V (see [7]).

One application of the results in [1] was found by [8], which relies on two important facts. If we are given $\mathfrak{M} = (V, <,>, R_\varphi)$, where $W \subseteq V$ is a subspace of V that is invariant under the action of $G_\mathfrak{M}$, then $W^\perp$, the orthogonal complement of W, is also invariant under the action of $G_\mathfrak{M}$ (a proof of this basic fact can be found in [8]). Since $A \in G_\mathfrak{M}$ preserves R and $<,>$, A also preserves sectional curvature. That means that if there is a unique, extremal 2-plane, π, then $A : \pi \to \pi$.

Using these facts, [8] finds a finite structure group by narrowing down the possible form of $A \in G_\mathfrak{M}$ to be diagonal, however, this is only useful if $\dim(V) = 3$ (Example 4.3 in [8]). Our goal in this paper is to adapt these methods to a finite-diminsional vector space. So we will note that since $A \in G_\mathfrak{M}$ preserves sectional curvature, A must also preserve k-plane scalar curvature. That means if we find a unique, extremal k-plane, $L \subseteq V$, then $A : L \to L$, meaning L is an invariant subspace. This also implies that $L^\perp$ is invariant.

Before considering more varied model spaces, it is worth stating some well-known facts.

Lemma 2.1. *In a model space $\mathfrak{M} = (V, <,>, R)$, the following conditions are equivalent:*

1. $G_\mathfrak{M} = G_{<,>}$
2. $R = \kappa R_{<,>}$*, for some* $\kappa \in \mathbb{R}$.
3. $\mathfrak{M}$ *has constant sectional curvature,* κ

A direct consequence of this fact is

Lemma 2.2. *If a model space $\mathfrak{M} = (V, <,>, R_\varphi)$, where $\dim(V) \geq 2$, has constant sectional curvature, then $G_\mathfrak{M} = O(N)$.*

3 k-Plane Scalar Curvature Results

In our first result, we use Klinger's method of rotating a 2-plane ([5]) and follow in Calle and Dunn's example ([1], Lemma 2.1), to show that when φ is positive definite, extremal k-planes have orthonormal bases of eigenvectors.

Theorem 3.1. *Let $\mathfrak{M} = (V, <,>, R_\varphi)$, where $\dim(V) = N$ and φ is positive definite. If $L \subseteq V$ is a k-plane whose k-plane scalar curvature is extremal, then there exists an orthonormal basis of eigenvectors for L.*

Proof. Restrict φ to L, which we will denote as $\varphi|_L$. Applying the Spectral Theorem to the positive definite $\varphi|_L$, find an orthonormal basis $\{f_1, \cdots, f_k\}$ for L such that $\varphi|_L(f_i, f_j) = \eta_i \delta_{i,j}$. Now extend this basis to create the orthonormal basis $\{f_1, \cdots, f_k, \cdots, f_N\}$ for our vector space V. We now have

$$\varphi = \left[\begin{array}{c|c} \eta & C \\ \hline C^T & \lambda \end{array} \right],$$

where $\eta_{ij} = \varphi(f_i, f_j)$ for $i, j \in \{1, \cdots, k\}$, C and C^T are the values of $\varphi(f_i, f_j)$ where exactly one of i or j is greater than k, and λ is the matrix of $\varphi(f_i, f_j)$ where both i and j are greater than k. So, to show that L has a basis of eigenvectors, we need to show that $C = 0$, which implies that $C^T = 0$. So we will consider the k-plane $L_\theta = \text{span}\{\cos\theta f_1 + \sin\theta f_\ell, f_2, \cdots, f_k\}$, where $\ell > k$.

At $\theta = 0$, we have the extremal k-plane curvature given by $\mathscr{K}(L) = \sum_{j>i=1}^{k} \eta_i \eta_j$. In general, we can break up the sum into two parts,

$$\mathscr{K}(L_\theta) = \sum_{p=2}^{k} \kappa(\cos\theta f_1 + \sin\theta f_\ell, f_p) + \sum_{j>i=2}^{k} \eta_i \eta_j,$$

where only the first sum is dependent on θ. So we can use the fact that $\{f_1, \cdots, f_N\}$ is an orthonormal basis, the properties of R_φ, and the double angle formula to get

$$\sum_{p=2}^{k} \kappa(\cos\theta f_1 + \sin\theta f_\ell, f_p) = \sum_{p=2}^{k} [\cos^2\theta R_\varphi(f_1, f_p, f_p, f_1) + \\ \sin(2\theta) R_\varphi(f_1, f_p, f_p, f_\ell) + \sin^2\theta R_\varphi(f_\ell, f_p, f_p, f_\ell)]$$

Now we use the fact that $\theta = 0$ is a critical point of the function $\mathscr{K}(L_\theta)$. So by taking the derivative, we get

$$\begin{aligned} 0 = \frac{d}{d\theta}[\mathscr{K}(L_\theta)] \, |_{\theta=0} &= \sum_{p=2}^{k} 2R_\varphi(f_1, f_p, f_p, f_\ell) \\ &= \sum_{p=2}^{k} \varphi(f_1, f_\ell)\varphi(f_p, f_p) \\ &= \varphi(f_1, f_\ell) \sum_{p=2}^{k} \eta_p \end{aligned}$$

By the assumption that φ is positive definite,

$$\eta_p = \varphi(f_p, f_p) > 0 \implies \sum_{p=2}^{k} \eta_p > 0 \implies \varphi(f_1, f_\ell) = 0.$$

Similarly, we can show that $\varphi(f_i, f_\ell) = 0$ for $2 \le i \le k$ by repeating the calculations for the k-planes $L_{\theta_i} = \text{span}\{f_1, \cdots, \cos\theta f_i + \sin\theta f_\ell, \cdots, f_k\}$, where $2 \le i \le k$ and $\ell > k$. So since φ is symmetric, $C = C^T = 0$. $\square$

Continuing to generalize Calle and Dunn's work ([1], Theorem 1.1), we find the interval of k-plane scalar curvatures. Here, we will use the fact that the Grassmannian of k-planes on V, $Gr_k(V)$, is compact and connected [7].

Corollary 3.2. *Let $\mathfrak{M} = (V, <,>, R_\varphi)$, where $\dim(V) = N$ and φ is positive definite, and let $\lambda_1, \cdots, \lambda_n$ be the eigenvalues of φ ordered from smallest to largest, and repeated according to multiplicity. Let*

$$m = \sum_{j>i=1}^{k} \lambda_i \lambda_j, \text{ and } M = \sum_{j>i=N-k+1}^{k} \lambda_i \lambda_j.$$

Then the set of k-plane scalar curvatures of $\mathfrak{M}$ is precisely the interval $[m, M]$.

Proof. Since $G_k(V)$ is compact, extremal k-planes exist. So if L is an extremal k-plane, then by Theorem 3.1, let $\{f_1, \cdots, f_k\}$ be an orthonormal basis of eigenvectors for L. Then we have

$$\mathscr{K}(L) = \sum_{j>i=1}^{k} \kappa(f_i, f_j) = \sum_{j>i=1}^{k} \varphi(f_i, f_i)\varphi(f_j, f_j).$$

Since $f_1, \cdots, f_k$ are eigenvectors, $\varphi(f_i, f_i)$ is an eigenvalue for all $1 \le i \le k$. Thus, m is the smallest such quantity, and M is the largest. So since $L \mapsto \mathscr{K}$ is continuous, we now just need to note that $G_k(V)$ is connected, meaning that the interval of k-plane scalar curvatures is connected. □

Remark 3.3. Now, we want to apply Theorem 3.1 and Corollary 3.2 to finding the structure groups of model spaces. As discussed in Section 2, if we find a unique, extremal k-plane L, then L is an invariant subspace of V under the action of $G_{\mathfrak{M}}$. So here we will outline the process of finding unique, extremal k-planes with an example, using Corollary 3.2.

Example 3.4. Consider a model space $\mathfrak{M} = (V, <,>, R_\varphi)$, where V has an orthonormal basis $\beta = \{e_1, \cdots, e_5\}$ such that

$$\varphi = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}.$$

We can find extremal k-planes by looking at the k-planes formed by eigenvectors, then check if they are unique by looking at whether any other k-plane spanned by eigenvectors can have the same k-plane scalar curvature. So, beginning by looking at 3-planes, we find that both span$\{e_1, e_2, e_3\}$ and span$\{e_3, e_4, e_5\}$ are unique and extremal, since any other 3-planes constructed from eigenvectors will have k-plane scalar curvature strictly greater than 11 and strictly less than 40. Continuing by looking at 4-planes, we find that there is not a unique minimal 4-planes, since span$\{e_1, e_2, e_3, e_4\}$ and span$\{e_1, e_2, e_3, e_5\}$ both have a k-plane scalar curvature of 35. The maximal 4-plane, span$\{e_2, e_3, e_4, e_5\}$, is unique, however. □

Since Theorem 3.1 and Corollary 3.2 rely on the fact that φ is positive definite, it would be useful to know whether there are counterexamples when φ is not positive definite. We will highlight one such counterexample to 3.1 where φ has an unbalanced signature. In this counterexample, we find a minimal 3-plane, L, that is not spanned by eigenvectors.

Example 3.5. Consider $\mathfrak{M} = (V, <, >, R_\varphi)$, where V has an orthonormal basis $\{e_1, \cdots, e_5\}$ such that

$$\varphi = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

If we assume that extremal k-planes are spanned by eigenvectors, we find that the minimal 3-plane curvature is -1. But we can consider a 3-plane, $L = \text{span}\{e_1, \frac{e_2+e_3}{\sqrt{2}}, e_4\}$, which has k-plane curvature -1 as well. If we assume that this plane is spanned by eigenvectors, we arrive at a contradiction. So let f_1, f_2, and f_3 be eigenvectors such that

$$f_1 = a_1 e_1 + a_2 e_2,$$
$$f_2 = b_1 e_1 + b_2 e_2,$$
$$f_3 = c_1 e_3 + c_2 e_4 + c_3 e_5,$$

which has to be the case for $\varphi(f_i, f_i)$ to equal 1 or -1. Now L =span$\{f_1, f_2, f_3\}$, so we can write

$$f_1 = \alpha_1 e_1 + \alpha_2 \left(\frac{e_2 + e_3}{\sqrt{2}} \right) + \alpha_3 e_4,$$
$$f_2 = \beta_1 e_1 + \beta_2 \left(\frac{e_2 + e_3}{\sqrt{2}} \right) + \beta_3 e_4,$$
$$f_3 = \gamma_1 e_1 + \gamma_2 \left(\frac{e_2 + e_3}{\sqrt{2}} \right) + \gamma_3 e_4.$$

We can reach our contradiction by noting that our original equations for f are only satisfied when $f_1 = e_1$, $f_2 = e_1$, and $f_3 = e_4$, which means span$\{f_1, f_2, f_3\} \neq L$. Similarly, we reach a contradiction if we set $f_2 = b_1 e_3 + b_2 e_4 + b_3 e_5$. $\square$

4 Structure Group Results

Now, the next step in finding the structure groups of model spaces is to use Corollary 3.3 to find all possible invariant subspaces under the action of $G_{\mathfrak{M}}$.

Corollary 4.1. *Let $\mathfrak{M} = (V, <, >, R_\varphi)$, and let the distinct eigenvalues of φ be, ordered from smallest to largest, $\lambda_1, \cdots, \lambda_m$. If φ is positive definite and $\dim(V) \geq 3$, then the eigenspaces of V, denoted $E_{\lambda_1}, \cdots, E_{\lambda_m}$, are invariant under the action of $G_{\mathfrak{M}}$.*

Proof. If $\mathfrak{M}$ has constant sectional curvature, there is only one eigenspace and we are done. So assume that V has more than one eigenspace. Let $\dim(E_{\lambda_i}) = n_i$. Now, we can proceed by constructing unique, extremal k-planes. Each k-plane is invariant, meaning that its orthogonal complement will also be invariant, as discussed in Section 2.

If there are two eigenspaces, then $n_i \geq 2$ for at least one of $i = 1$ or $i = 2$, since $\dim(V) \geq 3$. So if $n_1 \geq 2$, then the n_1-plane E_{λ_1} is minimal and unique, thus an invariant subspace under the action of $G_{\mathfrak{M}}$. This implies that $E_{\lambda_1}^{\perp} = E_{\lambda_2}$ is also invariant. Similarly, if $n_2 \geq 2$, then the n_2-plane E_{λ_2} is maximal and unique, which implies that E_{λ_2} and E_{λ_1} are invariant under $G_{\mathfrak{M}}$.

If there are three or more eigenspaces, we begin by creating the unique, minimal $(n_1 + n_2)$-plane, $E_{\lambda_1} \oplus E_{\lambda_2}$. Now we have two invariant subspaces of V, $E_{\lambda_1} \oplus E_{\lambda_2}$ and its orthogonal complement $E_{\lambda_3} \oplus \cdots \oplus E_{\lambda_m}$. Next, we construct the unique, minimal $(n_1 + n_2 + n_3)$-plane, $E_{\lambda_1} \oplus E_{\lambda_2} \oplus E_{\lambda_3}$. Now V can be partitioned into three invariant subspaces, $E_{\lambda_1} \oplus E_{\lambda_2}$, E_{λ_3}, and $E_{\lambda_4} \oplus \cdots \oplus E_{\lambda_m}$. Continuing in this process, we will find that $E_{\lambda_3}, \cdots, E_{\lambda_m}$ are all invariant after constructing all possible unique, minimal k-planes using $E_{\lambda_1}, \cdots, E_{\lambda_{m-1}}$, where $k \geq n_1 + n_2$. To conclude, we can construct the maximal $(n_2 + \cdots + n_m)$-plane, $E_{\lambda_2} \oplus \cdots \oplus E_{\lambda_m}$, which shows that the subspace E_{λ_1} is invariant, meaning E_{λ_2} must also be invariant. ☐

Since this method of proving that the eigenspaces are invariant may seem vague, we will give an example.

Example 4.2. Let $\mathfrak{M} = (V, <,>, R_{\varphi})$, where $\dim(V) = 6$ and φ is positive definite. Suppose $\beta = \{e_1, \cdots, e_N\}$ is an orthonormal basis for V that diagonalizes φ, and under this basis,

$$\varphi = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}.$$

Using Corollary 4.1, we construct unique, extremal k-planes.

1. Construct the 3-plane span$\{e_1, e_2, e_3\}$. Then the eigenspaces that we know are invariant are $E_1 \oplus E_2$, and the orthogonal complement of our 3-plane, $E_3 \oplus E_4$.

2. Construct the 4-plane span$\{e_1, e_2, e_3, e_4\}$, which demonstrates that $E_1 \oplus E_2$, E_3, and E_4 are invariant.

3. Finally, construct the 5-plane span$\{e_2, e_3, e_4, e_5, e_6\}$, which shows that E_1 is invariant. As a direct consequence, E_2 is also invariant.

This means if $A \in G_{\mathfrak{M}}$, the form of A is reduced as follows,

$$A = \begin{bmatrix} a_1 & b_1 & c_1 & 0 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & 0 & 0 \\ a_3 & b_3 & c_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & d_1 & e_1 & f_1 \\ 0 & 0 & 0 & d_2 & e_2 & f_2 \\ 0 & 0 & 0 & d_3 & e_3 & f_3 \end{bmatrix} \xrightarrow{1.} \begin{bmatrix} a_1 & b_1 & c_1 & 0 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & 0 & 0 \\ a_3 & b_3 & c_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & d_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & e_2 & f_2 \\ 0 & 0 & 0 & 0 & e_3 & f_3 \end{bmatrix} \xrightarrow{2.} \begin{bmatrix} a_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & b_2 & c_2 & 0 & 0 & 0 \\ 0 & b_3 & c_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & d_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & e_2 & f_2 \\ 0 & 0 & 0 & 0 & e_3 & f_3 \end{bmatrix} \xrightarrow{3.}.$$

□

Since we now know the eigenspaces are invariant, we will now give a general result that gives us $G_{\mathfrak{M}}$ relative to invariant subspaces.

Theorem 4.3. *Let $\mathfrak{M} = (V, <,>, R)$, $\dim(V) = N$, and let $V_1, \cdots, V_m$ be the distinct subspaces of V that are invariant under the action of $G_{\mathfrak{M}}$, with $V = \bigoplus_{i=1}^m V_i$. Then $G_{\mathfrak{M}} = G_{\mathfrak{M}}|_{V_1} \times \cdots \times G_{\mathfrak{M}}|_{V_m}$, the group-theoretic internal direct product of the structure groups of $\mathfrak{M}$ restricted to its invariant subspaces.*

Proof. Denote $\dim(V_i)$ as n_i. Let $\beta = \{e_1, \cdots, e_N\}$ be an ordered basis for V such that $\beta_i = \{e_{(n_1+\cdots+n_{i-1}+1)}, \cdots, e_{(n_1+\cdots+n_i)}\}$ is a basis for V_i. Now, we can write any $A \in G_{\mathfrak{M}}$ as a product of B_i's, where each $B_i \in G_{\mathfrak{M}}|_{V_i}$, since each V_i is invariant. So then we just need to show that the remaining group-theoretic properties hold, which are that each $G_{\mathfrak{M}}|_{V_i}$ is normal in $G_{\mathfrak{M}}$ and $\left[G_{\mathfrak{M}}|_{V_1} \cdots G_{\mathfrak{M}}|_{V_{m-1}}\right] \cap G_{\mathfrak{M}}|_{V_m} = I$. So let

$$B_i = \begin{bmatrix} I & 0 & 0 \\ 0 & b_i & 0 \\ 0 & 0 & I \end{bmatrix} \in G_{\mathfrak{M}}|_{V_i}.$$

Then for any $A \in G_{\mathfrak{M}}$,

$$A = B_1 \cdots B_m = \begin{bmatrix} b_1 & & 0 \\ & \ddots & \\ 0 & & b_m \end{bmatrix} \text{ and } A^{-1} = \begin{bmatrix} b_1^{-1} & & 0 \\ & \ddots & \\ 0 & & b_m^{-1} \end{bmatrix}$$

for some $B_1, \cdots, B_m$. So if we consider $C \in G_{\mathfrak{M}}|_{V_i}$, then

$$\begin{aligned} ABA^{-1} &= \begin{bmatrix} b_1 & & 0 \\ & \ddots & \\ 0 & & b_m \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} b_1^{-1} & & 0 \\ & \ddots & \\ 0 & & b_m^{-1} \end{bmatrix} \\ &= \begin{bmatrix} I & 0 & 0 \\ 0 & b_i\, c\, b_i^{-1} & 0 \\ 0 & 0 & I \end{bmatrix}. \end{aligned}$$

Since $B_iCB_i^{-1}$ is a product of elements in $G_{\mathfrak{M}}|_{V_i}$, $B_iCB_i^{-1} \in G_{\mathfrak{M}}|_{V_i}$, so $G_{\mathfrak{M}}|_{V_i}$ is normal in $G_{\mathfrak{M}}$. Next, note that any A is a product of some B_i's, since each B_i in the product acts independently on some V_i. Finally, I is the only element common to every B_i, which concludes the proof. $\square$

Now, we combine the previous statements to find all structure groups of $\mathfrak{M} = (V, <, >, R_\varphi)$, where $\dim(V) \geq 3$ and φ is positive definite.

Theorem 4.4. *Let $\mathfrak{M} = (V, <, >, R_\varphi)$, and let the distinct eigenvalues of φ be $\lambda_1, \cdots, \lambda_m$. Denote the eigenspace of V with eigenvalue λ_i as E_{λ_i}, where $\dim(E_{\lambda_i})$ is n_i. If φ is positive definite and $N = \dim(V) \geq 3$, then the structure group of $\mathfrak{M}$ is isomorphic to $O(n_1) \oplus \cdots \oplus O(n_m)$, the group-theoretic external product of orthogonal groups of dimension n_i.*

Proof. Using Corollary 4.1 and Theorem 4.3, we know $G_{\mathfrak{M}} \cong G_{\mathfrak{M}}|_{E_{\lambda_1}} \times \cdots \times G_{\mathfrak{M}}|_{E_{\lambda_m}}$. So we just need to show that $G_{\mathfrak{M}}|_{E_{\lambda_i}} \cong O(n_i)$. Let $\{e_1, \cdots, e_N\}$ be an ordered, orthonormal basis for V that diagonalizes φ, such that $\lambda_i \leq \lambda_j$ for $i < j$. Then if $r = n_1 + \cdots + n_{i-1} + 1$ and $s = n_i - 1$, the basis for E_{λ_i} is $\{e_r, \cdots, e_{r+s}\}$. So let $B_i \in G_{\mathfrak{M}}|_{E_{\lambda_i}}$ such that

$$B_i = \begin{bmatrix} I & 0 & 0 \\ 0 & b_i & 0 \\ 0 & 0 & I \end{bmatrix}, \text{ where } b_i = \begin{bmatrix} a_{1,1} & \cdots & a_{1,s} \\ \vdots & \ddots & \vdots \\ a_{s,1} & \cdots & a_{s,s} \end{bmatrix} \in O(n_i).$$

Since $b_i \in O(n_i)$, b_i is self adjoint. So if we denote the adjoint of b_i as b_i^*, then $(b_ib_i^*)_{i,j} = \delta_{i,j}$. This also means that b_i preserves $<, > |_{E_{\lambda_i}}$, the restriction of the inner product to E_{λ_i}. So if $r \leq j \leq r+s$, $r \leq \ell \leq r+s$, and $j \neq \ell$,

$$\begin{aligned} B_i^*\varphi(e_j, e_j) &= \varphi(a_{1,j}e_r + \cdots + a_{s,j}e_{r+s}, a_{1,j}e_r + \cdots + a_{s,j}e_{r+s}) \\ &= (a_{1,j}^2 + \cdots + a_{s,j}^2)\lambda_i = (b_ib_i^*)_{j,j}\lambda_i = \lambda_i, \text{ and} \\ B_i^*\varphi(e_j, e_\ell) &= \varphi(a_{1,j}e_r + \cdots + a_{s,j}e_{r+s}, a_{1,\ell}e_r + \cdots + a_{s,\ell}e_{r+s}) \\ &= (a_{1,j}a_{1,\ell} + \cdots + a_{s,j}a_{s,\ell})\lambda_i = (b_ib_i^*)_{j,\ell}\lambda_i = 0 \end{aligned}$$

So since b_i preserves $<, >$ and φ, restricted to E_{λ_i}, $B_i \in G_{\mathfrak{M}}|_{E_{\lambda_i}}$. This means $G_{\mathfrak{M}}|_{E_{\lambda_i}} \cong O(n_i)$, which concludes the proof. $\square$

Example 4.5. We will find the structure group from Example 4.2. In terms of eigenspaces, $G_{\mathfrak{M}}|_{E_1} \cong G_{\mathfrak{M}}|_{E_3} \cong O(1)$ and $G_{\mathfrak{M}}|_{E_2} \cong G_{\mathfrak{M}}|_{E_4} \cong O(2)$. This means $G_{\mathfrak{M}} \cong O(1) \oplus O(1) \oplus O(2) \oplus O(2)$. So if $A \in G_{\mathfrak{M}}$, then A is of the form

$$A = \begin{bmatrix} \pm 1 & 0 & 0 & 0 \\ 0 & a_1 & 0 & 0 \\ 0 & 0 & \pm 1 & 0 \\ 0 & 0 & 0 & a_2 \end{bmatrix},$$

where $a_1, a_2 \in O(2)$ are 2x2 submatrices.

5 Conclusion

We used previous methods for finding the bounds of sectional curvature and generalized these to find the bounds for k-plane scalar curvature in model spaces where φ is positive definite. Part of this was demonstrating that when φ is positive definite, extremal k-planes have an orthonormal bases of eigenvectors. This result is not true when φ is not positive definite. We also used previous methods for finding invariants in structure groups that contain a metric to characterize $G_{\mathfrak{M}}$ for $\mathfrak{M} = (V, <,>, R_\varphi)$ when $\dim(V) \geq 3$ and φ is positive definite. To do this, we first showed that the eigenspaces of model spaces of this type are invariant. We also showed generally that the the structure group of a model space is the internal direct product of the structure groups restricted to invariant subspaces.

There exist many routes for further research from our results. One potential route may be to study k-cvc(ε) in cases where φ is positive definite (see [2]). Another may be to look for similar results in different types of model spaces, such as those with a canonical algebraic curvature tensor built from an antisymmetric, bilinear form (see [6]). We will now present some specific open questions.

5.1 Open Questions

1. Are there restrictions to φ such that when φ is not positive definite, we can prove that there is an orthonormal basis of eigenvectors for extremal k-planes?
2. Can Corollary 1 be proved when φ is not positive definite?
3. Is there an alternate method to finding the structure group of $\mathfrak{M} = (V, <,>, R_\varphi)$, where φ is not positive definite and/or its signature is unbalanced?
4. What is the structure group for $\mathfrak{M} = (V, <,>, R_\tau)$, where $R_\tau = R_\varphi \pm R_\psi$?

6 Acknowledgements

I would like to thank Dr. Corey Dunn for his enthusiasm and guidance. I would also like to thank Dr. Rolland Trapp for his support. This project was funded by NSF grant DMS-1758020 and California State University, San Bernardino.

References

[1] Maxine Calle and Corey Dunn. Sharp sectional curvature bounds and a new proof of the spectral theorem. *arXiv preprint arXiv:1809.04200*, 2018.

[2] Maxine E Calle. k-plane constant curvature conditions. *Rose-Hulman Undergraduate Mathematics Journal*, 20(2):6, 2019.

[3] Bang-Yen Chen. Relations between ricci curvature and shape operator for submanifolds with arbitrary codimension. *Glasgow Mathematical Journal*, 41:33–41, 02 1999.

[4] Corey Dunn, Cole Franks, and Joseph Palmer. On the structure group of a decomposable model space. *Beiträge zur Algebra und Geometrie/Contributions to Algebra and Geometry*, 56(1):199–216, 2015.

[5] Ruedi Klinger. A basis that reduces to zero as many curvature components as possible. In *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, volume 61, pages 243–248. Springer, 1991.

[6] Elise McMahon. Linear dependence of canonical alegbraic curvature tensors of symmetric and anti-symmetric builds. *CSUSB REU*, 2014.

[7] John Milnor and James D Stasheff. *Characteristic Classes.(AM-76), Volume 76*, volume 76. Princeton university press, 1974.

[8] Montana Williams. A method to finding invariants using sectional curvature. *CSUSB REU*, 2019.