On the geometric realization of irreducible manifolds by decomposable model spaces

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Abstract

This paper proves that not every manifold realizing a decomposable model space is decomposable as the orthogonal product of smaller dimensional manifolds by providing two examples of warped product curvature models where the associated 0-model is decomposable but the 1-model is not. Since every 1-model can be geometrically realized, it follows that any manifold realizing this model space is irreducible.

1 Introduction

Substantial research has been done exploring the effect restrictions on algebraic curvature data have on the geometric properties of the manifold. Past studies (see [?]) have investigated the relationship between curvature homogenous manifolds and model spaces in both the Riemannian [?] and pseudo-Riemannian [?] settings. In [?], it was shown when certain operators defined using the curvature tensor of a model space commute, there is a geometric restriction placed on the model space and the manifolds. Other explorations of model space considerations about curvature operators and their restrictions on the geometry can be found in [?].

This paper continues the theme of geometric properties arising from algebraic restrictions through the study of irreducible manifolds. Specifically, we wish to show that there exist manifolds that realize decomposable curvature models despite being irreducible. In Section 2, we present some basic results from differential geometry that will be used throughout the remainder of the paper. In Section 3, we discuss the model space considerations necessary for such a manifold to exist and present our main theorem. Lastly, in Section 4, we provide proof of this theorem in the form of two examples of decomposable curvature models that are geometrically realizable as irreducible manifolds before summarizing our results in Section 5.

2 Preliminaries

2.1 Manifolds and curvature

Let M be an arbitrary smooth manifold, and denote the tangent space at a point $p \in M$ as T_pM. For the duration of the paper, when a vector space V is mentioned, we will assume V is real and finite-dimensional.

Definition 2.1. *An* inner product *on V is the function $< \cdot, \cdot >: V \times V \to \mathbb{R}$ satisfying the following properties for all $x, y, z \in V$ and $c \in \mathbb{R}$:*

(i) Linearity: $< x + y, z > = < x, z > + < y, z >$ and $< cx, y > = c < x, y >$,

(ii) Symmetry: $< x, y > = < y, x >$,

(iii) Non-degeneracy: $\forall x \in V \setminus \{0\}$ there exists $y \in V$ such that $< x, y > \neq 0$.

Note that the inner product is bilinear, since symmetry implies linearity in both the first slot and the second.

On each tangent space T_pM, there exists a choice of inner product called the metric, g, on M. The ordered pair (M, g) is called a *pseudo-Riemannian manifold*. For the remainder of this section, we will denote the pseudo-Riemannian manifold (M, g) by the name of its smooth manifold, M.

Let $(x_1, x_2, \ldots, x_n)$ be coordinates on M at point p. The following result, known as *the basis theorem*, provides a link between coordinates and tangent vectors [**?**].

Theorem 2.1. *If $(x_1, x_2, \ldots, x_n)$ is a coordinate system in M at p, then its coordinate vectors $\{\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_n}\}$ form a basis for the tangent space T_pM.*

The ∂_{x_i} are known as coordinate vector fields, and we define $\partial_{x_i} f = \frac{\partial}{\partial_{x_i}} f = f_{/i}$. The vector fields in the tangent space can be differentiated in a certain direction using a *connection* on a manifold.

Definition 2.2. *Let X_1, X_2, Y be tangent vectors in T_pM. A* connection *on a smooth manifold M is a function $\nabla : X_1 \times X_2 \to Y$ such that:*

(i) $\nabla_Y X_1 + X_2 = \nabla_Y X_1 + \nabla_Y X_2$,

(ii) $\nabla_{X_1 + X_2} Y = \nabla_{X_1} Y + \nabla_{X_2} Y$,

(iii) $\nabla_{f X_1} Y = f \nabla_{X_1} Y$,

(iv) $\nabla_{X_1}(fY) = X(f)Y + f\nabla_X Y$.

The covariant derivatives of the coordinate vector fields can be computed using a unique connection ∇ known as the *Levi-Civita connection*.

Definition 2.3. *On a pseudo-Riemannian manifold M there is a unique connection ∇ such that*

(i) $[X, Y] = \nabla_X Y - \nabla_Y X$

(ii) $X\left(g(Y, Z)\right) = g\left(\nabla_X Y, Z\right) + g\left(Y, \nabla_X Z\right)$

for all $X, Y, Z \in M$. ∇ is called the Levi-Civita connection *of M, and is characterized by the Koszul formula*

$$2g\left(\nabla_X Y, Z\right) = X\left(g(Y, Z)\right) + Y\left(g(Z, X)\right) - Z\left(g(X, Y)\right) - g\left(X, [Y, Z]\right) + g\left(Y, [Z, X]\right) + g\left(Z, [X, Y]\right).$$

If $\{\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_n}\}$ are coordinate vector fields forming a basis for T_pM, then $\nabla_{\partial_{x_i}} \partial_{x_j}$ is a linear combination of the coordinate vectors fields and can be computed using the Christoffel symbols of the connection.

Definition 2.4. *Let $(x_1, x_2, \ldots, x_n)$ be coordinates on M. The* Christoffel symbols of the first kind *are the Γ_{ij}^k such that*

$$\nabla_{\partial_{x_i}} \partial_{x_j} = \sum_{k=1}^{n} \Gamma_{ij}^k \partial_{x_k} \tag{1}$$

and the Christoffel symbols of the second kind, *Γ_{ijk}, are given by*

$$\Gamma_{ijk} = g\left(\nabla_{\partial_{x_i}} \partial_{x_j}, \partial_{x_k}\right).$$

Using the Koszul formula, we can express the Christoffel symbols of the second kind as

$$\Gamma_{ijk} = \frac{1}{2}(g_{jk/i} + g_{ik/j} - g_{ij/k}). \tag{2}$$

If $g(\cdot, \cdot)$ is the metric on M and ∇ is the Levi-Civita connection on M, then we define the Riemannian curvature tensor R on the coordinate vector fields $\partial_{x_i}, \partial_{x_j}, \partial_{x_k}, \partial_{x_l}$ as

$$R(\partial_{x_i}, \partial_{x_j}, \partial_{x_k}, \partial_{x_l}) := g(\nabla_{\partial_{x_i}} \nabla_{\partial_{x_j}} \partial_{x_k} - \nabla_{\partial_{x_j}} \nabla_{\partial_{x_i}} \partial_{x_k}, \partial_{x_l}). \tag{3}$$

Using Equation **??**, we can similarly define the first covariant derivative of R as

$$\begin{aligned}\nabla R(\partial_{x_i},\partial_{x_j},\partial_{x_k},\partial_{x_l};\partial_{x_m}) :=& \nabla_{\partial_{x_m}} R(\partial_{x_i},\partial_{x_j},\partial_{x_k},\partial_{x_l}) - R(\nabla_{\partial_{x_m}}\partial_{x_i},\partial_{x_j},\partial_{x_k},\partial_{x_l})\\ &- R(\partial_{x_i},\nabla_{\partial_{x_m}}\partial_{x_j},\partial_{x_k},\partial_{x_l}) - R(\partial_{x_i},\partial_{x_j},\nabla_{\partial_{x_m}}\partial_{x_k},\partial_{x_l})\\ &- R(\partial_{x_i},\partial_{x_j},\partial_{x_k},\nabla_{\partial_{x_m}}\partial_{x_l}).\end{aligned} \tag{4}$$

In general, we define $\nabla^i R$ to be the i^{th} covariant derivative of R, with the convention that $\nabla^0 R = R$.

Definition 2.5. *An* algebraic curvature tensor *R is a multilinear function $R : \otimes^4 V^* \to \mathbb{R}$ satisfying the following properties for all $x, y, z, w \in V$:*

(i) $R(x, y, z, w) = -R(y, x, z, w)$,

(ii) $R(x, y, z, w) = R(z, w, x, y)$,

(iii) *(First Bianchi Identity)* $R(x, y, z, w) + R(z, x, y, w) + R(y, z, x, w) = 0$.

The set of all algebraic curvature tensors on V is denoted $\mathcal{A}(V)$. Similarly, we define the first algebraic covariant derivative curvature tensor *as the multilinear function $\nabla R : \otimes^5 V^* \to \mathbb{R}$ satisfying the the same properties as R in the first four slots in addition to the following:*

(iv) *(Second Bianchi Identity)* $\nabla R(x, y, z, w; t) + \nabla R(x, y, w, t; z) + \nabla R(x, y, t, z; w) = 0$

The set of all ∇R on V is denoted as $\mathcal{A}_1(V)$.

Note that in the above definition, R and ∇R satisfy the same algebraic properties as the Riemannian curvature tensor and the first covariant derivative of the Riemannian curvature tensor, respectively [**?**].

Definition 2.6. *The* kernel *of $R \in \mathcal{A}(V)$ is defined as*

$$\ker R := \{v \in V \mid R(v, x, y, z) = 0 \ \forall x, y, z \in V\}. \tag{5}$$

Similarly, the kernel of $\nabla R \in \mathcal{A}_1(V)$ is defined as

$$\ker(\nabla R) := \{v \in V \mid \nabla R(v, x, y, z; w) = 0 \ \forall x, y, z, w \in V\}. \tag{6}$$

We prove in Proposition **??** that the definition of the kernel is not biased in favor of the first entry of R or ∇R, although the final slot of ∇R is somewhat different.

Definition 2.7. *Given a symmetric bilinear form $\varphi \in S^2(V)$, where $S^2(V)$ is the set of all symmetric bilinear forms on V, we define the* canonical algebraic curvature tensor *R_φ as*

$$R_\varphi(x, y, z, w) = \varphi(x, w)\varphi(y, z) - \varphi(x, z)\varphi(y, w).$$

for $x, y, z, w \in V$.

Additionally, if c is any positive real number, then $cR_\varphi = R_{\sqrt{c}\varphi}$.

On an appropriate basis, any $\varphi \in S^2(V)$ can be expressed as a diagonal matrix with a unique number of negative (or, "timelike") entries, p, and positive (or, "spacelike") entries, q. The tuple (p, q) is called the *signature* of φ. A Riemannian manifold has a metric with positive definite signature $(0, q)$, whereas a pseudo-Riemannian manifold has a metric with a nondegenerate signature (p, q), with $p, q \geq 1$.

2.2 Model spaces

In this section, we introduce the concept of model spaces and discuss their properties. A *model space* is a vector space V together with an inner product and an algebraic curvature tensor. For example, if φ is a symmetric bilinear form on V and $A \in \mathcal{A}(V)$, (V, φ, A) is a model space. A manifold with metric g and curvature R is said to realize this model if (T_pM, g_p, R_p) is *isomorphic* to (V, φ, A) at some point $p \in M$.

Definition 2.8. *Let $M = (V, \varphi, R)$ and $\bar{M} = (\bar{V}, \bar{\varphi}, \bar{R})$ be manifolds. Then M is* isomorphic *to $\bar{M}$ (denoted $M \cong \bar{M}$) if there exists an invertible function $T : V \to \bar{V}$ such that $T^*\bar{\varphi} = \varphi$ and $T^*\bar{R} = R$.*

For the purposes of the paper, it is convenient to distinguish two specific types of models.

Definition 2.9. *If V is a real, finite-dimensional vector space, $< \cdot, \cdot >$ an inner product on V, $R \in \mathcal{A}(V)$, and $\nabla R \in \mathcal{A}_1(V)$, then we define the* 0-model *$\mathcal{M}_0$ as*

$$\mathcal{M}_0 := (V, < \cdot, \cdot >, R)$$

and the 1-model *as*

$$\mathcal{M}_1 := (V, < \cdot, \cdot >, R, \nabla R).$$

Whenever $\dim(V) \geq 2$, V can be decomposed into the direct sum of two orthogonal vector spaces, V_1 and V_2, denoted $V = V_1 \oplus V_2$. If φ is an inner product on V such that $\varphi(v_1, v_2) = 0$ for any $v_1 \in V_1$ and $v_2 \in V_2$, then the vector space V with inner product φ can be decomposed as $(V, \varphi) = (V_1, \varphi_1) \oplus (V_2, \varphi_2)$. Similarly, if $R \in \mathcal{A}(V)$ is a curvature tensor on V, then R decomposes into the direct sum $R_1 \oplus R_2$ if $R(v_1, v_2, x, y) = 0$.

Definition 2.10. *A model space $\mathcal{M}$ is said to be* decomposable *if there exists a nontrivial orthogonal decomposition $V = V_1 \perp V_2$ that induces a direct sum decomposition $R = R_1 \oplus R_2$. If $\mathcal{M}$ is not decomposable, then $\mathcal{M}$ is said to be* indecomposable*.*

For example, if $R \in \mathcal{A}(V)$ and $\ker(R) \neq 0$, then there is a decomposition $(V, R) \cong (\bar{V}, \bar{R}) \oplus (\ker(R), 0)$, where $\bar{V}$ is a complimentary subspace. We will investigate this situation further in Section 3.

Similarly to Definition **??**, a manifold is said to be *reducible* if it can be expressed as the orthogonal cross product of smaller dimensional manifolds. If a manifold is not reducible, then we say it is *irreducible*. The cross product of manifolds appears in a common construction known as a *warped product*, however such a construction may or may not be initially reducible.

Definition 2.11. *Let M, N, P be manifolds. Then a* warped product *is the manifold $M \times N$ together with a smooth function $f : M \to \mathbb{R}$ and metric $g_{M\times N} = g_M \oplus f \cdot g_N$. If we define a second smooth function $h : M \times N \to \mathbb{R}$ and metric $g_{(M\times_f N)\times_h P} = (g_M \oplus f \cdot g_N) \oplus h \cdot g_P$, then the manifold $(M \times_f N) \times_h P$ together with f, h, and $g_{(M\times_f N)\times_h P}$ is called a multiply warped product.*

As a simple example, let $M = \mathbb{R}^2$, $N = \mathbb{R}$, and $f : \mathbb{R}^2 \to \mathbb{R}$ be the function $f(x, y)$. Then the manifold $\mathbb{R}^2 \times \mathbb{R}$ together with the metric $ds^2_{\mathbb{R}^2} + f(x, y) \cdot ds^2_{\mathbb{R}}$ is a warped product of flat space onto flat space. The warped products we discuss in subsequent sections will all be warped products of flat space warped onto flat space.

The family of model spaces discussed in the remainder of this paper are known as *warped product curvature models*, and the term *curvature model* will be used synonymously with model space.

Definition 2.12. *A model space $(V, < \cdot, \cdot >, R)$ is a* warped product curvature model *if there exists an orthogonal decomposition $V_1 \perp V_2$ of V and symmetric bilinear form $H \in S^2(V_1)$ such that if $x_i \in V_1$ and $y_i \in V_2$, the only nonzero curvature entries up to the usual symmetries are the following:*

$$R(y_i, x_j, x_k, y_l) = H(x_j, x_k) < y_i, y_l >, \qquad R(y_i, y_j, y_k, y_l) = cR_{<\cdot,\cdot>}(y_i, y_j, y_k, y_l).$$

3 Determination of $\mathcal{M}_k$ for decomposable R

In this section, we discuss the model space considerations that are sufficient for the geometric realization of a decomposable curvature model by an irreducible manifold. Although there are potentially a multitude of ways in which an algebraic curvature tensor R can decompose, the simplest decomposition is to split off the kernel of R. Then R decomposes into the direct sum of the nonzero curvature entries with 0. Furthermore, if R_φ is a canonical algebraic curvature tensor, then this decomposition is the only way in which R_φ can decompose. The following lemma formalizing this statement is adopted from [**?**].

Lemma 3.1. *Let $\varphi \in S^2(V)$. Assume* $\text{Rank}\{\varphi\} \geq 2$. *If there is a decomposition $V = V_1 \oplus V_2$ with $R_\varphi = R_1 \oplus R_2$, then either $V_1 \subset \ker \varphi$ or $V_2 \subset \ker \varphi$.*

It was shown in [?] that although the definition for $\ker(R)$ in Equation (**??**) appears to favor the first entry of R, the symmetries in Definition **??** prove that the definition for $\ker(R)$ holds regardless of which entry of R we consider. The proposition below verifies the analogous result for ∇R and is relevant to the discussions in the remainder of this section.

Proposition 3.1. *Let $\nabla R \in \mathcal{A}_1(V)$, and let $\ker(\nabla R)$ be defined as in Equation (**??**). Then*

$$\begin{aligned}\ker(\nabla R) &= \{v \in V \mid \nabla R(v, x, y, z; w) = 0 \ \forall x, y, z, w \in V\}\\ &= \{v \in V \mid \nabla R(x, v, y, z; w) = 0 \ \forall x, y, z, w \in V\}\\ &= \{v \in V \mid \nabla R(x, y, v, z; w) = 0 \ \forall x, y, z, w \in V\}\\ &= \{v \in V \mid \nabla R(x, y, z, v; w) = 0 \ \forall x, y, z, w \in V\}\\ &\subseteq \{v \in V \mid \nabla R(x, y, z, w; v) = 0 \ \forall x, y, z, w \in V\}.\end{aligned}$$

Proof. Let $v \in V$ be given and $x, y, z, w \in V$ be arbitrary. Using the symmetries of ∇R we have

$$\nabla R(v, x, y, z; w) = -\nabla R(x, v, y, z; w) = \nabla R(y, z, v, x; w) = -\nabla R(y, z, x, v; w).$$

By the second Bianchi Identity,

$$\nabla R(x, y, z, w; v) + \nabla R(x, y, w, v; z) + \nabla R(x, y, v, z; w) = 0.$$

But $\nabla R(x, y, w, v; z) = \nabla R(x, y, v, z; w) = 0$. Thus, $\nabla R(x, y, z, w; v) = 0$. □

It then follows that if a manifold M realizing the 0-model $\mathcal{M}_0 = (V, <\cdot,\cdot>, R_\varphi)$ decomposes as the orthogonal cross product of smaller manifolds, $M = M_1 \perp M_2$, then $\mathcal{M}_0 = \mathcal{M}_{0_1} \oplus \mathcal{M}_{0_2}$. Without loss of generality, we can assume this holds true for all decomposable algebraic curvature tensors R. Using the same reasoning, we can extend this concept to include the 1-model $\mathcal{M}_1$ and conclude that if the inner product decomposed and both R and ∇R decomposed in the same manner (for example, R and ∇R have the same kernel and decompose into a direct sum of their nonzero entries and 0), then the 1-model $\mathcal{M}_1 = (V, <\cdot,\cdot>, R, \nabla R)$ would decompose.

Furthermore, if $<\cdot,\cdot>$, R, and ∇R are decomposable, but ∇R has a different decomposition than R (for example, R has a kernel but ∇R does not) then the associated 1-model cannot decompose. Similarly, if ∇R is indecomposable, then $\mathcal{M}_1$ is indecomposable regardless of whether or not R and $<\cdot,\cdot>$ decompose.

We now state the main theorem of this paper, which we will prove in Section 4.

Theorem 3.1. *There exist manifolds that are irreducible yet realize decomposable model spaces.*

In [?], Gilkey proves that any 1-model is geometrically realizable at point on a pseudo-Riemannian manifold. Thus if $\mathcal{M}_1$ is indecomposable, any manifold M that realizes this model is irreducible. Lemma **??** below is again adopted from Gilkey's work in [?].

Lemma 3.2. *Let $\mathcal{M}_1$ be a 1-model. There exists a point P of a pseudo-Riemannian manifold M so that $\mathcal{M}_1$ is isomorphic to $\mathcal{M}_1(M, P)$.*

4 Two Examples

As previously discussed, we are interested in manifolds that are irreducible despite geometrically realizing a decomposable model space. We present two such model spaces in this section as proof of Theorem **??**. In Section 4.1, we will discuss a warped product curvature model of flat space warped onto flat space in which the algebraic curvature tensor is an R_φ. This example is of arbitrary signature (p, q) and is thus not restricted to either Riemannian or pseudo-Riemannian manifolds.

In Section 4.2, we will discuss a doubly warped product curvature model, also of flat space onto flat space, in which the algebraic curvature tensor is not an R_φ but has a non-trivial kernel of dimension 2. This example is restricted to pseudo-Riemannian manifolds, as it cannot be positive-definite.

4.1 Warped product curvature model with 1-dimensional kernel

For the duration of this section, let $M = \mathbb{R}^{n-1} \times (0, \infty)$ with coordinates $(x_1, x_2, \ldots, x_n)$ and arbitrary signature (p, q). Let g be a metric on M with nonzero entries given by

$$\begin{aligned} g(\partial_{x_1}, \partial_{x_1}) &= \epsilon_1 x_n^2, \\ g(\partial_{x_2}, \partial_{x_2}) &= \epsilon_2 x_n^2, \\ &\vdots \\ g(\partial_{x_{n-1}}, \partial_{x_{n-1}}) &= \epsilon_{n-1} x_n^2, \\ g(\partial_{x_n}, \partial_{x_n}) &= \epsilon_n, \end{aligned}$$

where $\epsilon_i = \pm 1$.

Lemma 4.1. *Let (M, g) be as defined above. Then*

(i) The nonzero curvature entries up to symmetry are

$$R(\partial_{x_i}, \partial_{x_j}, \partial_{x_j}, \partial_{x_i}) = -(\epsilon_i \epsilon_j) x_n^2$$

where $i \neq j$ and $i, j \in \{1, \ldots, n-1\}$.

(ii) $\nabla R(\partial_{x_i}, \partial_{x_j}, \partial_{x_j}, \partial_{x_i}; \partial_{x_n}) = -2(\epsilon_i \epsilon_j) x_n \neq 0$

Proof. By Equation (**??**), the only nonzero Christoffel symbols are Γ_{iin} and Γ_{ini}. Equation (**??**) then implies

$$\nabla_{\partial_{x_i}} \partial_{x_i} = -\epsilon_i x_n \partial_{x_n}$$

and

$$\nabla_{\partial_{x_i}} \partial_{x_n} = \frac{\epsilon_i}{x_n} \partial_{x_i}$$

are the only nonzero values of $\nabla_{\partial_{x_i}} \partial_{x_j}$. From Equation (**??**), we have

$$R(\partial_{x_i}, \partial_{x_j}, \partial_{x_k}, \partial_{x_l}) = g(\nabla_{\partial_{x_i}} \nabla_{\partial_{x_j}} \partial_{x_k} - \nabla_{\partial_{x_j}} \nabla_{\partial_{x_i}} \partial_{x_k}, \partial_{x_l}).$$

By inspection, we see that if $i = j$,

$$\nabla_{\partial_{x_i}} \nabla_{\partial_{x_j}} \partial_{x_k} - \nabla_{\partial_{x_j}} \nabla_{\partial_{x_i}} \partial_{x_k} = 0$$

which would imply $R(\partial_{x_i}, \partial_{x_i}, \partial_{x_k}, \partial_{x_l}) = 0$. Thus for nonzero curvature entries, we have $i \neq j$. Furthermore, $\nabla_{\partial_{x_i}} \nabla_{\partial_{x_j}} \partial_{x_k} - \nabla_{\partial_{x_j}} \nabla_{\partial_{x_i}} \partial_{x_k} \neq 0$ implies either $\nabla_{\partial_{x_i}} \nabla_{\partial_{x_j}} \partial_{x_k} \neq 0$ or $\nabla_{\partial_{x_j}} \nabla_{\partial_{x_i}} \partial_{x_k} \neq 0$.

Case I. Suppose $\nabla_{\partial_{x_i}} \nabla_{\partial_{x_j}} \partial_{x_k} \neq 0$. Then $j = k$ by inspection.

Case II. Suppose $\nabla_{\partial_{x_j}} \nabla_{\partial_{x_i}} \partial_{x_k} \neq 0$. Then $i = k$ by inspection.

Since $R(\partial_{x_i}, \partial_{x_j}, \partial_{x_k}, \partial_{x_l}) = -R(\partial_{x_j}, \partial_{x_i}, \partial_{x_k}, \partial_{x_l})$, we can assume $j = k$. Hence

$$\begin{aligned} \nabla_{\partial_{x_i}} \nabla_{\partial_{x_j}} \partial_{x_k} - \nabla_{\partial_{x_j}} \nabla_{\partial_{x_i}} \partial_{x_k} &= \nabla_{\partial_{x_i}} \nabla_{\partial_{x_j}} \partial_{x_j} - \nabla_{\partial_{x_j}} \nabla_{\partial_{x_i}} \partial_{x_j} \\ &= \nabla_{\partial_{x_i}} (-\epsilon_j x_n \partial_{x_n}) - \nabla_{\partial_{x_j}} \cdot 0 \\ &= \partial_{x_i}(-\epsilon_j x_n) \partial_{x_n} + (-\epsilon_j x_n) \nabla_{\partial_{x_i}} \partial_{x_n} \\ &= 0 - \epsilon_j x_n \left(\frac{\epsilon_i}{x_n} \partial_{x_i} \right) \\ &= -(\epsilon_i \epsilon_j) \partial_{x_i}. \end{aligned}$$

Thus

$$\begin{aligned}R(\partial_{x_i},\partial_{x_j},\partial_{x_k},\partial_{x_l}) &= R(\partial_{x_i},\partial_{x_j},\partial_{x_j},\partial_{x_l})\\ &= g(-\epsilon_i\epsilon_j\partial_{x_i},\partial_{x_l})\\ &= -(\epsilon_i\epsilon_j)g(\partial_{x_i},\partial_{x_l})\end{aligned}$$

is nonzero only when $i=l$. Therefore, the only nonzero curvature entries up to the usual symmetries are $R\left(\partial_{x_i},\partial_{x_j},\partial_{x_j},\partial_{x_i}\right) = -(\epsilon_i\epsilon_j)x_n^2$. This completes the proof for (i).

We now wish to prove (ii) by computing $\nabla R(\partial_{x_i},\partial_{x_j},\partial_{x_j},\partial_{x_i};\partial_{x_n})$. Using the usual symmetries of R and the proof of (i), we can simplify equation (**??**) to get

$$\begin{aligned}\nabla R(\partial_{x_i},\partial_{x_j},\partial_{x_j},\partial_{x_i};\partial_{x_n}) &= \nabla_{\partial_{x_n}} R(\partial_{x_i},\partial_{x_j},\partial_{x_j},\partial_{x_i})\\ &= \nabla_{\partial_{x_n}}\left(-(\epsilon_i\epsilon_j)x_n^2\right)\\ &= -2(\epsilon_i\epsilon_j)x_n\\ &\neq 0.\end{aligned}$$

□

Corollary 4.1.

(i) $\operatorname{span}\{\partial_{x_n}\}\subseteq\ker(R)$

(ii) $\operatorname{span}\{\partial_{x_n}\}\not\subseteq\ker(\nabla R)$

Proof. This result follows directly from Lemma **??** and Equations (**??**) and (**??**). □

Lemma 4.2. *Given (M,g) defined above and change of basis*

$$X_i := \frac{1}{\sqrt{x_n}}\partial_{x_i}, \qquad X_n := \partial_{x_n},$$

the algebraic curvature tensor $R=R_\varphi$, where φ is defined as $\varphi(X_m,X_m)=\epsilon_m\ \forall m=1,\ldots,n-1$.

Proof. Let (M,g) be as defined above, and recall that the nonzero entries of g are given by

$$\begin{aligned}g(\partial_{x_1},\partial_{x_1}) &= \epsilon_1 x_n^2\\ g(\partial_{x_2},\partial_{x_2}) &= \epsilon_2 x_n^2\\ &\vdots\\ g(\partial_{x_{n-1}},\partial_{x_{n-1}}) &= \epsilon_{n-1} x_n^2\\ g(\partial_{x_n},\partial_{x_n}) &= \epsilon_n\end{aligned}$$

where $\epsilon_i=\pm1$. Fix a change of basis such that

$$X_i := \frac{1}{\sqrt{x_n}}\partial_{x_i}, \qquad X_n := \partial_{x_n}$$

Then the nonzero entries of R are

$$\begin{aligned}R(X_i,X_j,X_j,X_i) &= R\left(\frac{1}{\sqrt{x_n}}\partial_{x_i},\frac{1}{\sqrt{x_n}}\partial_{x_j},\frac{1}{\sqrt{x_n}}\partial_{x_j},\frac{1}{\sqrt{x_n}}\partial_{x_i}\right)\\ &= \frac{1}{x_n^2}R(\partial_{x_i},\partial_{x_j},\partial_{x_j},\partial_{x_i})\\ &= \frac{1}{x_n^2}\left(-(\epsilon_i\epsilon_j)x_n^2\right)\\ &= -\epsilon_i\epsilon_j\end{aligned}$$

for $i, j = 1, \ldots, n-1$ and $i \neq j$. Define $\varphi \in S^2(V)$ to be such that $\varphi(X_m, X_m) = \epsilon_m$ for all $m = 1, \ldots, n-1$ are the only nonzero values of φ. Then

$$R(X_i, X_j, X_k, X_l) = \varphi(X_i, X_l)\varphi(X_j, X_k) - \varphi(X_i, X_k)\varphi(X_j, X_l)$$

is nonzero when either $i = l$ and $j = k$ or $i = k$ and $j = l$. Without loss of generality, take $i = l$ and $j = k$. Then,

$$\begin{aligned} R(X_i, X_j, X_j, X_i) &= \varphi(X_i, X_i)\varphi(X_j, X_j) - \varphi(X_i, X_j)\varphi(X_j, X_i) \\ &= -\epsilon_i \epsilon_j \end{aligned}$$

is the only nonzero curvature entry. Therefore, $R = R_\varphi$. □

Lemma 4.3. *Let (M, g) be as given above, and let R and ∇R be as stated in Lemma* **??**. *Then the 0-model $\mathcal{M}_0 = (M, g, R)$ has the unique decomposition at every point in M as*

$$\mathcal{M}_0 = (\ker(R)^\perp, g_1, R|_{\ker(R)^\perp}) \oplus (\ker(R), g_2, 0)$$

where $g_1(\partial_{x_i}, \partial_{x_i}) = \epsilon_i$ for $1 \leq i < n$ and $g_2(\partial_{x_n}, \partial_{x_n}) = \epsilon_n$, but the associated 1-model $\mathcal{M}_1 = (M, g, R, \nabla R)$ does not decompose.

Proof. From Lemma **??**, we have the only nonzero curvature entries are $R(\partial_{x_i}, \partial_{x_j}, \partial_{x_j}, \partial_{x_i})$ where $i, j < n$ and $i \neq j$. By Corollary **??**(i), $\operatorname{span}\{\partial_{x_n}\} \subseteq \ker(R)$. Since we showed in Lemma **??** that $R = R_\varphi$, we can use Lemma **??** to conclude (M, g, R) has the unique decomposition

$$(M, g, R) \cong (\bar{V}, g_1, \bar{R}) \oplus (\ker(R), g_2, 0).$$

By Corollary **??**(ii), $\operatorname{span}\{\partial_{x_n}\} \not\subseteq \ker(\nabla R)$. Therefore, the 1-model $\mathcal{M}_1 = (M, g, R, \nabla R)$ does not decompose. □

We now prove Theorem **??** using the above results.

Proof of Theorem **??**. Let M, g, R, and ∇R be defined as in Lemma **??**. By Lemma **??**, the 0-model $\mathcal{M}_0 = (M, g, R)$ decomposes as the direct sum

$$\mathcal{M}_0 = (\bar{V}, g_1, \bar{R}) \oplus (\ker(R), g_2, 0).$$

But by the same Lemma, the associated 1-model, $\mathcal{M}_1 = (M, g, R, \nabla R)$, does not decompose. Since $\mathcal{M}_1$ does not decompose, any manifold that realizes this 1-model cannot decompose, even though the 0-model $\mathcal{M}_0$ decomposes. □

4.2 Doubly-warped product curvature model with 2-dimensional kernel

For the duration of this section, let $M = \mathbb{R}^{2n} \times (0, \infty) \times (0, \infty)$ with coordinates $(x_1, x_2, \ldots, x_{2n}, a, b)$. Let g be a metric on M with nonzero entries given by:

$$g(\partial_{x_p}, \partial_{x_p}) = a^2, \quad g(\partial_{x_q}, \partial_{x_q}) = b^2, \quad g(\partial_a, \partial_b) = 1$$

where $1 \leq p \leq n$ and $n + 1 \leq q \leq 2n$.

Lemma 4.4. *Let (M, g) be as defined above. Then*

(i) The nonzero curvature entries up to symmetry are

$$R(\partial_{x_p}, \partial_{x_q}, \partial_{x_q}, \partial_{x_p}) = -ab. \tag{7}$$

(ii) $\nabla R(\partial_{x_p}, \partial_{x_q}, \partial_{x_q}, \partial_{x_p}; \partial_a) = -b$ *and* $\nabla R(\partial_{x_p}, \partial_{x_q}, \partial_{x_q}, \partial_{x_p}; \partial_b) = a$.

The proof of Lemma **??** follows the same procedure as the proof of Lemma **??**.

Corollary 4.2.

(i) $\operatorname{span}\{\partial_a, \partial_b\} = \ker(R)$

(ii) $\operatorname{span}\{\partial_a, \partial_b\} \not\subseteq \ker(\nabla R)$

Proof. This result follows directly from Lemma **??** and equations (**??**) and (**??**). $\square$

Lemma 4.5. *Let (M, g) be as defined above, let $p \in M$, and let R be the nonzero curvature entries described in Lemma* **??***. If $\partial_{x_1} \in V_1$, then the only decomposition of the model space (T_pM, g_p, R_p) is*

$$(T_pM, g_p, R_p) \cong (V_1, g_1, R) \oplus (V_2, g_2, 0).$$

That is, $V_2 \subseteq \ker(R)$.

Proof. Let $T_pM = V_1 + V_2$ such that $V_1 \perp V_2$. Then for any $v \in T_pM$, $v = v_1 + v_2$, where $v_1 \in V_1$ and $v_2 \in V_2$. Let $\partial_{x_1} \in V_1$. Since $v_2 = c_1\partial_{x_1} + c_2\partial_{x_2} + \cdots + c_{2n}\partial_{x_{2n}} + c_{2n+1}\partial_a + c_{2n+1}\partial_b$, bilinearity of the inner product implies

$$\begin{aligned} g(\partial_{x_1}, v_2) &= g(\partial_{x_1}, c_1\partial_{x_1} + c_2\partial_{x_2} + \cdots + c_{2n}\partial_{x_{2n}} + c_{2n+1}\partial_a + c_{2n+1}\partial_b) \\ &= c_1 g(\partial_{x_1}, \partial_{x_1}) + c_2 g(\partial_{x_1}, \partial_{x_2}) + \cdots c_{2n} g(\partial_{x_1}, \partial_{x_{2n}}) + c_{2n+1} g(\partial_{x_1}, \partial_a) + c_{2n+1} g(\partial_{x_1}, \partial_b) \\ &= c_1 g(\partial_{x_1}, \partial_{x_1}) + c_2 \cdot 0 + \cdots + c_{2n} \cdot 0 + c_{2n+1} \cdot 0 + c_{2n+2} \cdot 0 \\ &= c_1 g(\partial_{x_1}, \partial_{x_1}) \\ &= c_1 a^2, \end{aligned}$$

with $a \neq 0$. But $g(\partial_{x_1}, v_2) = 0$, so $c_1 = 0$. Similarly, the multilinearity of the curvature tensor implies

$$\begin{aligned} R(\partial_{x_1}, \partial_{x_{2n}}, v_2, \partial_{x_1}) &= R(\partial_{x_1}, \partial_{x_{2n}}, c_2\partial_{x_2} + \cdots + c_{2n}\partial_{x_{2n}} + c_{2n+1}\partial_a + c_{2n+1}\partial_b, \partial_{x_1}) \\ &= c_2 R(\partial_{x_1}, \partial_{x_{2n}}, \partial_{x_2}, \partial_{x_1}) + \cdots + c_{2n+1} R(\partial_{x_1}, \partial_{x_{2n}}, \partial_a, \partial_{x_1}) + c_{2n+1} R(\partial_{x_1}, \partial_{x_{2n}}, \partial_b, \partial_{x_1}) \\ &= c_2 \cdot 0 + \cdots + c_{2n} R(\partial_{x_1}, \partial_{x_{2n}}, \partial_{x_{2n}}, \partial_{x_1}) + c_{2n+1} \cdot 0 + c_{2n+2} \cdot 0 \\ &= c_{2n} R(\partial_{x_1}, \partial_{x_{2n}}, \partial_{x_{2n}}, \partial_{x_1}) \\ &= -c_{2n} ab, \end{aligned}$$

where $a, b \neq 0$. But $R(\partial_{x_1}, \partial_{x_{2n}}, v_2, \partial_{x_1}) = 0$, so $c_{2n} = 0$. Repeating the above calculation for $R(x_1, x_q, v_2, x_1) = 0$, where $n+1 \leq q \leq 2n$, shows $c_q = 0$. Thus, $v_2 = c_2\partial_{x_2} + \cdots c_n\partial_{x_n} + c_{2n+1}\partial_a + c_{2n+1}\partial_b$, which implies $V_2 \subseteq \operatorname{span}\{\partial_{x_2}, \ldots, \partial_{x_n}, \partial_a, \partial_b\}$ and $V_1 \subseteq \operatorname{span}\{\partial_{x_1}, \ldots, \partial_{x_{2n}}\}$. However, this contradicts Equation (**??**), since

$$R(\partial_{x_1}, v_2, v_2, \partial_{x_1}) = (c_2^2 + c_3^2 + \cdots + c_n^2)(-ab) = 0.$$

This implies $c_2 = \cdots = c_n = 0$.

Hence $\partial_{x_2}, \ldots, \partial_{x_n} \notin V_2$ and $V_2 \subseteq \operatorname{span}\{\partial_a, \partial_b\}$. Then Lemma **??** implies $V_2 \subseteq \ker(R)$. $\square$

Since we had the additional assumption that ∂_{x_i} was restricted to one of the subspaces of T_pM, we were not able to demonstrate that (T_pM, g_p, R_p) must only decompose as in Lemma **??**. If this hypothesis could be removed, then one could provide an alternate proof of Theorem **??** in the following way.

Let M, g, R, and ∇R be defined as in Lemma **??**. If Lemma **??** could be proven without this additional assumption, then the 0-model $\mathcal{M}_0 = (M, g, R)$ must only decompose as the direct sum

$$(M, g, R) = (V_1, g_1, R|_{V_1}) \oplus (V_2, g_2, 0)$$

where $V_2 \subseteq \ker(R)$. But by Lemma **??**, $\ker(\nabla R) \neq \ker(R)$. Hence the 1-model $\mathcal{M}_1 = (M, g, R, \nabla R)$ cannot decompose. Therefore, any manifold that realizes this 1-model cannot decompose.

5 Conclusion

The warped product curvature models presented in this paper show that not every manifold realizing a decomposable model space is reducible. The first curvature model we discussed was an R_φ with a one-dimensional kernel, whereas the second curvature model was not an R_φ but had a nontrivial two-dimensional kernel. Although there was an additional restriction placed on the second curvature model effecting the decomposition of the 0-model, if this restriction could be removed, the second curvature model would provide an additional family of irreducible manifolds that geometrically realize decomposable model spaces.

The research discussed in this paper seems to raise more questions than it answers. A few of the more pressing ones are discussed in Section 5.1 below.

5.1 Open Questions

1. Perhaps the most obvious question is whether or not the additional hypothesis in Lemma **??** can be removed. If it is proved that this is possible, then the alternate proof of Theorem **??** can be completed.

2. Is the example discussed in Section 4.2 signature dependent? That is, is it not possible to produce a two-dimensional kernel with Riemannian manifolds using a warped product curvature model? It seems like the construction in Section 4.2 is flexible enough that it could be arranged to work in a Riemannian signature.

3. There are many different ways to make the kernel nontrivial. This paper only explores cases where the kernel is one- or two-dimensional. Is it possible to find a model with any size kernel? Similarly, there are other ways for an algebraic curvature tensor to decompose than by splitting off the kernel. Do models spaces with other decompositions exist? A twisted product curvature model seems like a possible candidate.

4. This paper raises a number of questions regarding warped products. Given a warped product curvature model, when must it decompose? In what ways can it decompose? When is a warped product an R_φ?

5. Roughly speaking, holonomy measures the extent to which parallel transport around a closed loop fails to preserve geometric data. The holonomy of a manifold is the holonomy group of the Levi-Civita connection on tangent bundle. To what extent does the algebraic data of a decomposable model space interact with the holonomy group of any manifold that geometrically realizes it? Is there a relationship between curvature invariant subspaces and the holonomy group of a manifold? A good starting point for exploring this would be [**?**].

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