

Assessing the Geometric Realizations of Hermitian Manifolds

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CSUSB REU 2020

August 14, 2020

Abstract

Algebraic curvature tensors are a useful way to study the geometric properties of surfaces in higher dimensions. While it's been shown in [3] that every algebraic curvature is geometrically realizable at a point on a pseudo-Riemannian manifold, the requirements for geometric realization on its complex analogue, the pseudo-Hermitian Manifold, are much stricter. Our aim is to gain a better algebraic understanding of the curvature of complex surfaces by recognizing these requirements, and to determine when a canonical algebraic curvature tensor is geometrically realizable on a Hermitian manifold. We will summarize what is already known in order to classify the remaining bilinear forms that allow for geometric realization.

1 Preliminaries

As a means to motivate our study of complex curvature, we must first introduce a few terms in Riemannian geometry.

Definition 1.1. Let $u, v \in V$. An *inner product* on V is a function that takes the ordered pair (u, v) and returns a scalar $\langle u, v \rangle \in \mathbb{R}$, and has the following properties:

1. Symmetry: $\langle u, v \rangle = \langle v, u \rangle$,
2. Additivity: $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$ for all $u, v, w \in V$,
3. Homogeneity: $\langle \lambda u, v \rangle = \lambda \langle u, v \rangle$ for all $\lambda \in \mathbb{R}$ and $u, v \in V$,
4. Non-degenerate: For all nonzero $v \in V$, there is a $w \in V$ so that $\langle v, w \rangle \neq 0$.

Definition 1.2. If we have $\langle v, v \rangle \geq 0$ for all $v \in V$, and $\langle v, v \rangle = 0$ if and only if $v = 0$, then we say the inner product is *positive definite*.

Definition 1.3. Let V be a real-valued n -dimensional vector space and let V^* be the corresponding dual space of V , where $V^* := \{\varphi : V \rightarrow \mathbb{R} \mid \varphi \text{ is a linear transformation}\}$.

An *algebraic curvature tensor* is $R \in \otimes^4 V^*$ so that for all $x, y, z, w \in V$, the following conditions are satisfied:

1. $R(x, y, z, w) = -R(y, x, z, w)$,
2. $R(x, y, z, w) = R(z, w, x, y)$,
3. $R(x, y, z, w) + R(z, x, y, w) + R(y, z, x, w) = 0$,

the last of which is known as the Bianchi Identity. We use $\mathcal{A}(V)$ to denote the set of all algebraic curvature tensors on V , which is itself a vector space.

Definition 1.4. Let $(V, \langle \cdot, \cdot \rangle)$ be a non-degenerate inner product space and $A : V \rightarrow V$ be a linear transformation. Then for all $x, y \in V$, we define $A^* : V \rightarrow V$ as the *adjoint* of A with the equation

$$(Ax, y) = (x, A^*y)$$

If $A^* = A$ then we refer to A as *symmetric* or *selfadjoint*, and if $A^* = -A$ then we refer to A as *antisymmetric* or *skew-adjoint*.

Definition 1.5. Let $S^2(V^*)$ be the set of all symmetric bilinear forms on V^* and $\Lambda^2(V^*)$ be the set of all antisymmetric bilinear forms on V^* .

If $\phi \in S^2$ and $\tau \in \Lambda^2$, then a *canonical algebraic curvature tensor* is of the form

1. $R_\phi^S(x, y, z, w) = \phi(x, w)\phi(y, z) - \phi(x, z)\phi(y, w)$, or
2. $R_\tau^\Lambda(x, y, z, w) = \tau(x, w)\tau(y, z) - \tau(x, z)\tau(y, w) - 2\tau(x, y)\tau(z, w)$.

2 Complex Curvature Models

We now introduce some definitions from complex geometry that will be used throughout the rest of the paper.

Definition 2.1. We define a *model space* $M = (V, \langle \cdot, \cdot \rangle, R)$, where $V = \text{span}\{e_1, \dots, e_n\}$, $\langle \cdot, \cdot \rangle$ is a non-degenerate bilinear inner product, and R is an algebraic curvature tensor.

Definition 2.2. Let $(V, \langle \cdot, \cdot \rangle)$ be a real, non-degenerate inner product space of dimension $2n$. Note that we will assume $2n \geq 4$, since the 2-dimensional setting is trivial [1].

If $J : V \rightarrow V$ is a linear map that satisfies

$$J^2 = -I, \text{ and } J^* = -J$$

Then we say that J is an *isometry*.

Definition 2.3. If $J : V \rightarrow V$ is a linear map that satisfies

$$J^2 = I, \text{ and } J^* = -J,$$

Then we say that J is a *para-isometry* [3].

Definition 2.4. In both cases, we refer to J as an *almost complex structure*.

Definition 2.5. Let J be an almost complex structure. A 2-dimensional subspace $\pi \subset V$ is called a *complex line* if $J\pi \subset \pi$. For an isometric J , the inner product on a complex line must be either positive definite, negative definite, or zero everywhere, but not mixed. However, if J is a para-isometry then the inner product on a complex line must be either mixed or zero everywhere [3].

Definition 2.6. [2]. A *complex structure* on a manifold M as a tensor field \mathcal{J} so that at each point $p \in M$, \mathcal{J}_p is an almost complex structure on T_pM and there are local coordinates $(x_1, y_1, \dots, x_n, y_n)$ on a neighborhood of p so that

$$\mathcal{J}\partial_{x_j} = \partial_{y_j} \quad \text{and} \quad \mathcal{J}\partial_{y_j} = -\partial_{x_j}.$$

Definition 2.7. [2]. A *Hermitian manifold* \mathcal{M} is a triple $\mathcal{M} := (M, g, \mathcal{J})$ where M is a $2n$ real-dimensional manifold, g a Riemannian metric, and \mathcal{J} a complex structure on TM .

We say that a complex curvature model $(V, \langle \cdot, \cdot \rangle, A)$ is *geometrically realizable* by a *Hermitian manifold* if for some point p on a Hermitian manifold \mathcal{M} there exists an isometry from φ from V to TM satisfying

$$\varphi^* \mathcal{J}_p = J, \quad \varphi^* g_p = \langle \cdot, \cdot \rangle, \quad \text{and} \quad \varphi^* R_p = A$$

where R_p is the Riemann curvature tensor of (M, g) at p .

2.1 Gray Identity

It has been shown in [1] that a complex curvature model is geometrically realizable if and only if the given algebraic curvature tensor satisfies

$$\begin{aligned} R(x, y, z, w) + R(Jx, Jy, Jz, Jw) &= R(Jx, Jy, z, w) + R(x, y, Jz, Jw) \\ &\quad + R(Jx, y, Jz, w) + R(x, Jy, z, Jw) \\ &\quad + R(Jx, y, z, Jw) + R(x, Jy, Jz, w), \end{aligned}$$

which is known as the *Gray Identity*.

3 Previous Results

We will be borrowing heavily from Diroff, who in 2012 characterized when certain canonical algebraic curvature tensors satisfy the Gray Identity, focusing in particular on cases where the bilinear form is equipped with a positive definite inner product, and the almost complex structure J is a isometry.

Lemma 3.1. (Gilkey [3]). *Let $A : V \rightarrow V$ be a linear map. Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space endowed on an almost complex structure J . Let $A^* = \pm A$, put $R_A := R_A^S$ if $A^* = A$ and $R_A := R_A^\Lambda$ if $A^* = -A$. Then*

1. *If $JA = AJ$ then R_A satisfies the Gray Identity.*
2. *If $JA = -AJ$ and if $\text{Rank}(A) \leq 2$ then R_A satisfies the Gray Identity.*
3. *If $JA = -AJ$ and if $\text{Rank}(A) \geq 2$ then R_A violates the Gray Identity.*

Lemma 3.2. (Diroff [2]). *Let $A : V \rightarrow V$ be a linear map. Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space endowed on an almost complex structure J . Then $\text{Rank}(AJ - JA) \neq 1$.*

Proof. We argue by contradiction. Put $B := AJ - JA$ and suppose $\text{Rank}(B) = 1$. First see that B and J anti-commute

$$JB = JAJ + A = (JA - AJ)J = -(AJ - JA)J = -BJ.$$

Since $B^* = (AJ - JA)^* = -JA + AJ = B$, by the Spectral Theorem for self-adjoint (positive definite) operators we can find a vector $e_1 \in V$ so that $Be_1 = \lambda e_1$ with $\lambda \in \mathbb{R}$ and $\lambda \neq 0$. Now consider

$$BJe_1 = -JBe_1 = -J(\lambda e_1) = -\lambda Je_1.$$

Thus Je_1 is an eigenvector of B corresponding to the eigenvalue $-\lambda$. Since it was assumed that $\lambda \neq 0$, we can conclude that B has two distinct eigenvalues and thus $\text{Rank}(B) \geq 2$, which gives us our contradiction. \square

Lemma 3.2 is important because it highlights the fact that Diroff only needed to consider linear operators for B that are diagonal and induce a positive definite inner product. For $\text{Rank}(B) = 1$, there is only one to consider:

$$B = \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \lambda \neq 0.$$

Another important result from Lemma 3.2 is that when $B = AJ - JA$, we have that B and J anti-commute. In fact, while Diroff only considered J as an isometry, immediate computation will show that same is true when J is a para-isometry.

The proof is included because it shows that while we may not be able to use the Spectral Theorem, we may still use the same eigenvalue argument that will allow us to eliminate non-positive definite cases for which B is diagonal, whether J is an isometry or para-isometry, since in both instances we have that $BJ = -JB$.

Lemma 3.3. (Diroff [2]). *Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space endowed with an almost complex structure J . Let $A : V \rightarrow V$ be a linear map. Then*

1. R_A^S satisfies the Gray Identity if and only if $R_{AJ-JA}^S = R_{A+JAJ}^S$.
2. R_A^Λ satisfies the Gray Identity if and only if $R_{AJ-JA}^\Lambda = R_{A+JAJ}^\Lambda$.

We must note that while Diroff assumed a positive definite inner product, a careful analysis reveals that the assumption was not used in the proof, and thus Lemma 3.3 still holds when the inner product is not positive definite. Again, this is another important result that we will use to determine the geometric realizability of canonical algebraic curvature tensors on Hermitian manifolds.

Theorem 3.4. (Diroff [2]). *Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space endowed with an almost complex structure J . Let A be a linear map on V .*

1. *If $A^* = -A$ then the complex curvature model $(V, \langle \cdot, \cdot \rangle, J, R_A^\Lambda)$ is geometrically realizable on a Hermitian manifold if and only if*

$$AJ = JA$$

2. *If $A^* = A$ then the complex curvature model $(V, \langle \cdot, \cdot \rangle, J, R_A^S)$ is geometrically realizable on a Hermitian manifold if and only if there exists a complex line π so that*

$$AJ|_{\pi^\perp} = JA|_{\pi^\perp}$$

i.e. A commutes with J on the orthogonal complement of some complex line.

In addition to the results directly stated in Theorem 3.4, there are three major take-aways:

- (1) If A is antisymmetric, then no matter the rank of $(AJ - JA)$, whenever we have $R_{AJ-JA}^\Lambda = R_{A+JAJ}^\Lambda$, it must also be the case that $AJ - JA = \pm(A + JAJ)$.
- (2) If A is symmetric, then $R_{AJ-JA}^S = R_{A+JAJ}^S$ implies that $AJ - JA = \pm(A + JAJ)$, but only when $\text{Rank}(AJ - JA) \geq 3$.
- (3) If $\text{Rank}(B) = 0$, then A and J commute everywhere, particularly on the orthogonal complement of a complex line.

Corollary 3.5. (Diroff [2]). *Let $A : V \rightarrow V$ be a linear map. Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space endowed on an almost complex structure J . If $A^* = -A$ and if $AJ = -JA$, then $\text{Rank}(A) \neq 2$.*

Lemma 3.6. *Suppose $J^2 = I$. If $AJ - JA = \pm(A + JAJ)$, then $A = 0$.*

Proof. Without loss of generality, assume $AJ - JA = A + JAJ$. Then

$$\begin{aligned}
AJ - JA &= A + JAJ \\
J(AJ - JA) &= J(A + JAJ) \\
JAJ - A &= JA + AJ \\
AJ - JA &= A + AJ \\
J(AJ - A) &= A(I + J) \\
J(AJ - A)(I + J) &= A(I + J)(I - J) \\
2J(AJ - A) &= A(I^2 - J^2) \\
AJ - A &= 0 \\
AJ &= A
\end{aligned} \tag{1}$$

But also

$$\begin{aligned}
AJ - JA &= A + JAJ \\
(AJ - JA)J &= (A + JAJ)J \\
A - JAJ &= AJ + JA \\
A - AJ &= JA + JAJ \\
A(I - J) &= J(A + AJ) \\
A(I - J)(I + J) &= J(A + AJ)(I + J) \\
A(I^2 - J^2) &= 2J(AJ + A) \\
0 &= AJ + A \\
-A &= AJ
\end{aligned} \tag{2}$$

By equations (1) and (2) it follows that $A = -A$, and thus $A = 0$. \square

Our previous results conclude that Dieroff categorized R_A^S and R_A^Λ for when $(V, \langle \cdot, \cdot \rangle)$ is a positive definite inner product space endowed on an almost complex structure J , when $\text{Rank}(AJ - JA) \geq 3$ and when $\text{Rank}(AJ - JA) < 3$. Since para-isometries require a balanced signature and cannot exist when the inner product is positive definite, we attribute the elimination of those cases to Dieroff as well. By Theorem 3.4 and Lemma 3.6, all that is left to consider are the cases in which A is symmetric, the inner product is not positive definite, and J is either an isometry or a para-isometry.

The cases will be studied and classified in the two sections that immediately follow. In Section 4 we will characterize the Jordan types for B in which the Gray Identity is not satisfied, considering J as an isometry and a para-isometry. In Section 5 we will characterize the Jordan types that do satisfy the Gray Identity. In both cases we will rely heavily on results from Mal'cev, found in [5], that will allow us to choose a convenient basis for which we can determine an inner product. For simplicity, all of the following cases are considered in dimension 4, but our results are easily extendable to higher dimensions.

4 Jordan Types for Which the Gray Identity is Not Satisfied

Let $(V, \langle \cdot, \cdot \rangle)$ be a non-degenerate inner product space endowed on an almost complex structure J . Suppose that $A^* = A$, $B = AJ - JA$, and that $\text{Rank}(B) \leq 2$. Then the following are the cases in which the Gray Identity is not satisfied:

4.1 $\text{Rank}(B) = 1$

$$B = \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \lambda \neq 0.$$

Thus λ is the only nonzero eigenvalue. We claim that there does not exist an almost complex structure J for a B of this rank and Jordan type which satisfies the Gray Identity.

Proof. By [5], we know that for a B of this Jordan type, there exists a basis such that

$$Be_1 = \lambda e_1.$$

Now consider

$$BJe_1 = -JBe_1 = -\lambda Je_1.$$

It follows that $-\lambda$ is also an eigenvalue for B , with corresponding eigenvector Je_1 . But this contradicts that λ is the only nonzero eigenvalue. Therefore since $BJ = -JB$ in both the isometric and para-isometric cases, no such J exists for a B of this particular Jordan type with $\text{Rank}(B) = 1$. \square

4.2 $\text{Rank}(B) = 2$

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We claim that there does not exist an almost complex structure J for a B of this rank and Jordan type which satisfies the Gray Identity.

Proof. By [5], we know that there exists a basis for a B of this Jordan type such that

$$Be_1 = 0; \quad Be_2 = e_1; \quad Be_3 = e_2; \quad Be_4 = 0,$$

and the nonzero entries of the inner product are

$$\langle e_1, e_3 \rangle = \langle e_2, e_2 \rangle = 1, \quad \text{and} \quad \langle e_i, e_i \rangle = \pm 1, \quad \text{for } i \geq 4.$$

We have that

$$Je_1 = \sum_{i=1}^4 a_i e_i; \quad Je_2 = \sum_{i=1}^4 b_i e_i; \quad Je_3 = \sum_{i=1}^4 c_i e_i; \quad Je_4 = \sum_{i=1}^4 d_i e_i.$$

Now consider

$$\begin{aligned} BJe_2 &= B\left(\sum_{i=1}^4 b_i e_i\right) \\ -JBe_2 &= B(b_2 e_2 + b_3 e_3) \\ -Je_1 &= b_2 e_1 + b_3 e_2 \end{aligned}$$

Thus, we have

$$Je_1 = -b_2 e_1 - b_3 e_2.$$

Similarly, we can show that

$$Je_2 = -c_2 e_1 - c_3 e_2.$$

Consider

$$\begin{aligned} BJe_4 &= B\left(\sum_{i=1}^4 d_i e_i\right) \\ -JBe_4 &= B(d_2 e_2 + d_3 e_3) \\ 0 &= d_2 e_1 + d_3 e_2. \end{aligned}$$

This implies that $d_2 = d_3 = 0$. Similarly, for any $2n \times 2n$ matrix B of this rank and Jordan type, we can show that on this basis

$$J = \begin{bmatrix} X & Y \end{bmatrix},$$

where

$$X = \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & 0 \\ a_3 & b_3 & c_3 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ a_{2n} & b_{2n} & c_{2n} & d_{2n} \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} f_1 & \dots & \dots & 2n_1 \\ 0 & \dots & \dots & 0 \\ 0 & \dots & \dots & 0 \\ \vdots & \dots & \dots & \vdots \\ f_{2n} & \dots & \dots & 2n_{2n} \end{bmatrix}.$$

To prove that J cannot exist as an isometry or a para-isometry, we want to show that the $(3, 3)$ of J^2 is 0, and therefore $J^2 \neq \pm I$. To this end, it is enough to show that $c_3 = 0$.

Consider the inner product

$$\begin{aligned} \langle Je_2, e_2 \rangle &= -\langle e_2, Je_2 \rangle \\ c_3 &= -c_3 \\ c_3 &= 0. \end{aligned}$$

Thus we have $J^2 \neq \pm I$, as required. \square

4.3 $\text{Rank}(B) = 2$

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \lambda \neq 0.$$

Thus λ is the only nonzero eigenvalue. We claim that there does not exist an almost complex structure J for a B of this rank and Jordan type which satisfies the Gray Identity.

Proof. By [5], we know that there exists a basis for a B of this Jordan type such that $Be_3 = \lambda e_3$. Now consider

$$BJe_3 = -JB e_3 = -\lambda J e_3.$$

It follows that $-\lambda$ is also an eigenvalue for B , with corresponding eigenvector Je_3 . But this contradicts that λ is the only nonzero eigenvalue. Thus, since $BJ = -JB$ in both the isometric and para-isometric cases, no such J exist for a B of this particular Jordan type with $\text{Rank}(B) = 2$. \square

4.4 $\text{Rank}(B) = 2$

$$B = \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \lambda \neq 0.$$

Thus λ is the only nonzero eigenvalue. We claim that there does not exist an almost complex structure J for a B of this rank and Jordan type which satisfies the Gray Identity.

Proof. By [5], we know that there exists a basis for a B of this Jordan type such that

$$Be_1 = \lambda e_1.$$

Now consider

$$BJe_1 = -JB e_1 = -\lambda J e_1.$$

It follows that $-\lambda$ is also an eigenvalue for B , with corresponding eigenvector Je_1 . But this contradicts that λ is the only nonzero eigenvalue. Thus, since $BJ = -JB$ in both

the isometric and para-isometric cases, no such J exist for a B of this particular Jordan type with $\text{Rank}(B) = 2$. □

4.5 $\text{Rank}(B) = 2$

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We claim that there does not exist an almost complex structure J for a B of this rank and Jordan type which satisfies the Gray Identity.

Proof. By [5], we know that without loss of generality, we can assume that for a B of this Jordan type, the nonzero entries of the inner product on this basis are:

$$\langle e_1, e_2 \rangle = \langle e_3, e_4 \rangle = 1.$$

We have that

$$\begin{aligned} & Be_1 = 0; \quad Be_2 = e_1; \quad Be_3 = 0; \quad Be_4 = e_3, \quad \text{and} \\ & Je_1 = \sum_{i=1}^4 a_i e_i; \quad Je_2 = \sum_{i=1}^4 b_i e_i; \quad Je_3 = \sum_{i=1}^4 c_i e_i; \quad Je_4 = \sum_{i=1}^4 d_i e_i. \end{aligned}$$

First consider

$$\begin{aligned} BJe_2 &= B\left(\sum_{i=1}^4 b_i e_i\right) \\ -JBe_2 &= b_2 Be_2 + b_4 Be_4 \\ -Je_1 &= b_2 e_1 + b_4 e_3. \end{aligned}$$

Thus, we have

$$Je_1 = -b_2 e_1 - b_4 e_3.$$

Similarly, we can show that

$$Je_3 = -d_2 e_1 - d_4 e_3.$$

Because J is antisymmetric, we have $\langle Je_2, e_3 \rangle = -\langle e_2, Je_3 \rangle$, and thus $b_4 = d_2$.

Suppose J is an isometry and consider

$$\begin{aligned}
J^2 e_1 &= J(-b_2 e_1 - b_4 e_3) \\
&= -b_2(-b_2 e_1 - b_4 e_3) - b_4(-b_4 e_1 - d_4 e_3) \\
-e_1 &= (b_2^2 + b_4^2)e_1 + (b_2 b_4 + b_4 d_4)e_3.
\end{aligned}$$

But this implies that $b_2^2 + b_4^2 = -1$. Thus, there does not exist an isometric J for a B of this Jordan type where $\text{rank}(B) = 2$.

Now suppose J is a para-isometry. It can be checked that in order to have

$$J^2 = I, \quad AJ - JA = B, \quad \text{and} \quad BJ = -JB,$$

it must be the case that

$$b_2 = \pm d_4 = \pm 1, \quad \text{and} \quad b_3 = b_4 = 0.$$

This yields the following possibilities for J :

$$J_x = \pm \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad \text{and} \quad J_y = \pm \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

For J_x we must have

$$A = \begin{bmatrix} a & -\frac{1}{2} & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & e & -\frac{1}{2} \\ 0 & 0 & 0 & e \end{bmatrix},$$

and for $-J_x$ we must have

$$A = \begin{bmatrix} a & \frac{1}{2} & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & e & -\frac{1}{2} \\ 0 & 0 & 0 & e \end{bmatrix}.$$

For J_y we must have

$$A = \begin{bmatrix} a & -\frac{1}{2} & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & e & \frac{1}{2} \\ 0 & 0 & 0 & e \end{bmatrix},$$

and for $-J_y$ we must have

$$A = \begin{bmatrix} a & \frac{1}{2} & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & e & -\frac{1}{2} \\ 0 & 0 & 0 & e \end{bmatrix}.$$

Now we know by Theorem 3.4 that in order to satisfy the Gray Identity, we need $R_{AJ-JA} = R_{A+JAJ}$, for either $J = J_x$ or $J = J_y$. Recall that $B = AJ - JA$. We can calculate that

$$R_{AJ-JA}(e_2, e_4, e_4, e_2) = 1.$$

Immediate computation will show that when $J = J_x$ or $J = J_y$, in both cases we have

$$R_{A+JAJ}(e_2, e_4, e_4, e_2) = 0.$$

Thus, $R_{AJ-JA} \neq R_{A+JAJ}$, which violates the Gray Identity, and the proof is complete. \square

4.6 Rank(B) = 2

$$B = \begin{bmatrix} x & y & 0 & 0 \\ -y & x & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad y \neq 0.$$

We claim that this case is not possible for an isometric J , and the only possibility for a para-isometric J is when $x = 0$, but the solution is trivial.

Proof. By [5], we know that there exists a basis for a B of this Jordan type such that

$$Be_1 = xe_1 - ye_2; \quad Be_2 = ye_1 + xe_2; \quad Be_3 = 0; \quad Be_4 = 0,$$

and the nonzero entries of the inner product on this basis are:

$$\langle e_1, e_2 \rangle = 1, \text{ and } \langle e_i, e_i \rangle = \pm 1, \text{ for } i \geq 3.$$

We have that

$$Je_1 = \sum_{i=1}^4 a_i e_i; \quad Je_2 = \sum_{i=1}^4 b_i e_i; \quad Je_3 = \sum_{i=1}^4 c_i e_i; \quad Je_4 = \sum_{i=1}^4 d_i e_i.$$

Consider

$$\begin{aligned}
BJe_1 &= B\left(\sum_{i=1}^4 a_i e_i\right) \\
-JBe_1 &= a_1 Be_1 + a_2 Be_2 \\
-J(xe_1 - ye_2) &= a_1(xe_1 - ye_2) + a_2(ye_1 + xe_2) \\
(-xa_1 + yb_1)e_1 + (-xa_2 + yb_2)e_2 &= (xa_1 + ya_2)e_1 + (xa_2 - ya_1)e_2.
\end{aligned}$$

Thus we have the following equations:

$$\begin{aligned}
-xa_1 + yb_1 &= xa_1 + ya_2 \\
y(b_1 - a_2) &= 2xa_1,
\end{aligned} \tag{3}$$

and

$$\begin{aligned}
-xa_2 + yb_2 &= xa_2 - ya_1 \\
y(b_2 + a_1) &= 2xa_2.
\end{aligned}$$

Next, let us consider

$$\begin{aligned}
BJe_3 &= B\left(\sum_{i=1}^4 c_i e_i\right) \\
-JBe_3 &= c_1 Be_1 + c_2 Be_2 \\
&= c_1(xe_1 - ye_2) + c_2(ye_1 + xe_2) \\
0 &= (xc_1 + yc_2)e_1 + (xc_2 - yc_1)e_2.
\end{aligned}$$

Which yields the equations

$$\begin{aligned}
xc_1 + yc_2 &= 0 \\
xc_1 &= -yc_2,
\end{aligned}$$

and

$$\begin{aligned}
xc_2 - yc_1 &= 0 \\
xc_2 &= yc_1.
\end{aligned}$$

Since $x \neq 0$, we have $c_1 = -\frac{y}{x}c_2$, and $c_1 = \frac{x}{y}c_2$, which implies that $c_2 = c_1 = 0$. Similarly, we can show that $d_1 = d_2 = 0$. Furthermore, for any $2n \times 2n$ matrix B of this rank and Jordan type, we can show that on this basis

$$J = \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}, \text{ where } X = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}, \text{ and } Y \text{ is some } (2n-2) \times (2n-2) \text{ matrix.}$$

To prove that J cannot exist as an isometry or a para-isometry for $x \neq 0$, we want to show that the $(1, 1)$ entry of J^2 is 0, and therefore $J^2 \neq \pm I$. To this end, it is enough to show that $a_1 = a_2 = 0$.

Consider the inner product

$$\begin{aligned}\langle Je_1, e_1 \rangle &= -\langle e_1, Je_1 \rangle \\ a_2 &= -a_2 \\ a_2 &= 0.\end{aligned}$$

Next, consider

$$\begin{aligned}\langle Je_2, e_2 \rangle &= -\langle e_2, Je_2 \rangle \\ b_1 &= -b_1 \\ b_1 &= 0.\end{aligned}$$

By Equation (3), it follows that $a_1 = 0$. Therefore, when $x \neq 0$, J cannot exist as an isometry or a para-isometry.

If $x = 0$ and J is a para-isometry, then this case only works trivially. Consider the inner products

$$\begin{aligned}\langle Je_3e_4 \rangle &= -\langle e_3, Je_4 \rangle \\ -c_4 &= -d_3 \\ c_4 &= d_3.\end{aligned}$$

and

$$\begin{aligned}\langle Je_3e_3 \rangle &= -\langle e_3, Je_3 \rangle \\ -c_3 &= c_3 \\ c_3 &= 0.\end{aligned}$$

Similarly we can find that $d_4 = 0$. This yields the only possibilities for J :

$$J = \pm \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad J = \pm \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}.$$

Immediate computation will show that for A we must have

$$A = \pm \begin{bmatrix} \frac{y}{2} & \frac{y}{2} & 0 & 0 \\ \frac{y}{2} & \frac{y}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{or} \quad A = \pm \begin{bmatrix} -\frac{y}{2} & \frac{y}{2} & 0 & 0 \\ \frac{y}{2} & -\frac{y}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since we know from [4] that if $A^* = A$ and $\text{Rank}(A) = 1$, then $R_A = 0$. Thus, $R_{AJ-JA} = R_{A+JAJ}$, but since $R_A = 0$ the solution is trivial. \square

5 Jordan Types for Which the Gray Identity is Satisfied

Let $(V, \langle \cdot, \cdot \rangle)$ be a non-degenerate inner product space endowed on an almost complex structure J . Suppose that $A^* = A$, $B = AJ - JA$, and that $\text{Rank}(B) \leq 2$. Then the following are the cases in which the Gray Identity is satisfied:

5.1 $\text{Rank}(B) = 2$

$$B = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \lambda_1, \lambda_2 \neq 0.$$

Thus λ_1, λ_2 are the only two nonzero eigenvalues. This case is possible whether J is an isometry or a para-isometry, but only if $\lambda_1 = -\lambda_2$.

Proof. Suppose that $\lambda_1 \neq -\lambda_2$. By [5], we know that for a B of this Jordan type, there exists a basis such that

$$Be_1 = \lambda_1 e_1, \quad Be_2 = \lambda_2 e_2, \quad Be_3 = 0, \quad \text{and} \quad Be_4 = 0.$$

Then

$$BJe_1 = -JBe_1 = -\lambda_1 Je_1, \text{ and}$$

$$BJe_2 = -JBe_2 = -\lambda_2 Je_1.$$

It follows that $-\lambda_1, -\lambda_2$ are also nonzero eigenvalues for B , with corresponding eigenvectors Je_1 and Je_2 , respectively. But this contradicts that λ_1, λ_2 are the only two nonzero eigenvalues. Since $BJ = -JB$ in both the isometric and para-isometric cases, no such J exist for a B of this particular Jordan type with $\text{Rank}(B) = 2$, unless $\lambda_1 = -\lambda_2$.

Now suppose $\lambda_1 = -\lambda_2$, and let $\lambda_1 = p \neq 0$. Then we have

$$B = \begin{bmatrix} p & 0 & 0 & 0 \\ 0 & -p & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Let us first consider the case where J is an isometry. Notice that $\text{span}\{e_1, e_2\}$ forms a complex line, and when J is an isometry there are no mixed signatures. Consider

$$J = \pm \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad A = \pm \begin{bmatrix} x & \frac{p}{2} & 0 & 0 \\ \frac{p}{2} & x & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

As required, we have $J^2 = -I$, $AJ - JA = B$, and $BJ = -JB$. Since $\text{Rank}(A) = 2$, it is easy to show that $R_{AJ-JA} = R_{A+JAJ}$, and thus satisfies the Gray Identity.

Next we shall consider the case where J is a para-isometry, and show by contradiction that no such J exists. Notice that $\text{span}\{e_1, e_2\}$ forms a complex line, and when J is an isometry there are only mixed signatures. Computation of the inner products on J will show that

$$a_1 = b_2 = c_3 = d_4 = 0.$$

It then follows that for J and A we have the following possibilities:

$$J = \pm \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{or} \quad J = \pm \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix},$$

and

$$A = \pm \begin{bmatrix} x & \frac{p}{2} & 0 & 0 \\ -\frac{p}{2} & x & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then we have

$$A + JAJ = \pm \begin{bmatrix} 2x & 0 & 0 & 0 \\ 0 & 2x & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Now, suppose we have $R_{AJ-JA} = R_{A+JAJ}$, and consider

$$\begin{aligned} R_{AJ-JA}(e_1, e_2, e_2, e_1) &= R_{A+JAJ}(e_1, e_2, e_2, e_1) \\ p^2 &= -4x^2, \end{aligned}$$

which gives us our contradiction. □

5.2 $\text{Rank}(B) = 1$

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This case is only possible if J is a para-isometry.

Proof. First we will show that it is not possible in the isometry case. By [5], we know that without loss of generality, we can assume that for a B of this Jordan type, the nonzero entries of the inner product on this basis are:

$$\langle e_1, e_2 \rangle = \langle e_3, e_3 \rangle = 1, \text{ and } \langle e_4, e_4 \rangle = -1.$$

We have that

$$Be_1 = 0; \quad Be_2 = e_1; \quad Be_3 = 0; \quad Be_4 = 0, \text{ and}$$

$$Je_1 = \sum_{i=1}^4 a_i e_i; \quad Je_2 = \sum_{i=1}^4 b_i e_i; \quad Je_3 = \sum_{i=1}^4 c_i e_i; \quad Je_4 = \sum_{i=1}^4 d_i e_i.$$

Then

$$\begin{aligned} BJe_2 &= B(b_2 e_2) \\ -JBe_2 &= b_2 e_1. \end{aligned}$$

Thus, $Je_1 = -b_2 e_1$. Since we first assumed that J is an isometry, we have that

$$J^2 e_1 = -e_1 = b_2^2 e_1.$$

But this implies that $b_2^2 = -1$. Thus, there does not exist an isometric J for a B of this Jordan type where $\text{rank}(B) = 1$.

If J is a para-isometry, then this case works in at least dimension 4, if

$$J = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix},$$

and

$$A = \begin{bmatrix} 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & w \end{bmatrix},$$

where either $z = 0$ or $w = 0$, but not both.

As required, we have $J^2 = I$, $AJ - JA = B$, and $BJ = -JB$. Since we know from [4] that if $B^* = B$ and $\text{Rank}(B) = 1$, then $R_B = 0$. Thus it is easy to show that $R_{AJ-JA} = R_{A+JAJ}$, and since $\text{Rank}(A) = 2$ we have a non-trivial example of an A and a para-isometric J in which the Gray Identity is satisfied. □

6 Conclusions

In our assessment of geometric realizations of canonical algebraic curvature tensors on Hermitian manifolds, we found that even when we have an inner product that is not positive definite, Dirroff's Theorem still holds. In other words, when the Gray Identity is satisfied it still must be the case that A and J commute on the orthogonal complement of a complex line. The para-isometric case that was found to satisfy the Gray Identity is remarkable because we had $\text{Rank}(B) = 1$, but the fact that we had $\text{Rank}(A) = 2$ allowed us to find a solution that was non-trivial. Our findings illustrate the exceptional condition of matrix commutativity.

7 Open Question

What can be said of sums of canonical algebraic curvature tensors as they relate to the Gray Identity? Are there two or more canonical algebraic curvature tensors that do not individually satisfy the Gray Identity, but do as a sum?

8 Acknowledgments

I would like to thank Dr. Corey Dunn; this research would not have been possible without his expert guidance and insight. Thanks to Dr. Rolland Trapp for his wisdom, encouragement, and valuable prizes. This research was generously funded by California State University, San Bernardino, and the NSF grant DMS-1758020.

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