ALGEBRAIC REQUIREMENTS FOR K-CURVATURE HOMOGENEOUS SPACES

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Abstract. We study the algebraic conditions that every one-model realizable as a curvature homogeneous-1 space. We study dimension $n = 3$ with a Riemannian and Lorentzian metric. In the case of symmetric spaces, we present families curvature tensors that could not correspond to a curvature homogeneous-1 space.

Contents

1. Introduction

The study of pseudo-Riemanninan and Riemannian manifolds causes us to question the curvature of the manifold. Curature homogeneity is a condition in both Riemanninan and pseudo-Riemannian geometry. We say that a manifold (M, g) is curvature homogeneous if there is a linear isometry between any two $p, q \in M$ that preserves the curvature tensor. This property can be extended up so that a manifold is curvature homogeneous up to order k if for any two $p, q \in M$ there exists a linear isometry $\phi : T_pM \to T_qM$ such that $\phi^*(\nabla^i R(q)) = \nabla^i R(p)$ for all $i \leq h$. Singer[1] has shown that for a Riemanninan manifold there exists some number k such if M is curvature homogeneous up to order k, then M is curvature homogeneous for all k, a condition called *locally homogeneous*. This was generalized to pseudo-Riemannian case by Cartan and Sternberg[2]. Kowalski and Prüfer[3] have

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shown that there are algebraic conditions that must be satisfied for a Riemannian manifold to be curvature homogeneous (k=0). Specifically they used these conditions to develop curvature tensors in dimension $n = 4$ that could not belong to curvature homogeneous spaces by not satisfying their algebraic conditions. In this paper, we will build off the work of Kowalski and Prüfer to develop algebraic conditions that must be satisfied to be curvature homogeneous to the $k = 1$ level. We do this for both the Riemannian and Lorentzian case in dimension $n = 3$. We present curvature tensors and their covariant derivatives that could not belong to any curvature homogeneous space up to the $k = 1$ level.

2. Definitions

It is useful to proceed with some definitions that will ease future discussion on this topic.

Definition 2.1. [Algebraic Curvature Tensor] Let V be vector space. An algebraic curvature tensor (ACT) R is a type $(0, 4)$ tensor that satisfies the following properties: If x, y, z, v are elements in V:

- $R(x, y, z, v) = -R(y, x, z, v)$.
- $R(x, y, z, v) = R(z, v, x, y)$.
- Bianchi Identity: $R(x, y, z, v) + R(x, v, y, z) + R(x, z, v, y) = 0$.

Definition 2.2. [Curvature Operator] The curvature operator, $\mathcal{R}$, is a type (1,3) that maps $\mathcal{R}(x, y) : V \to V$. It is defined so that

$$R(x, y, z, w) = g\big(\mathcal{R}(x, y)z, w\big).$$

Higher orders of the curvature operator are defined so that

$$\nabla^i R(x, y, z, w; t_1, .., t_i) = g\big(\nabla^i \mathcal{R}(x, y; t_1, ..., t_i)z, w\big).$$

Remark 2.3. Note that the symmetries of the algebraic curvature tensor R are also encoded in $\mathcal{R}$. For example $\mathcal{R}$ is anti-symmetric under exchange of x and y, so that $\mathcal{R}(x, y) = -\mathcal{R}(y, x)$.

Definition 2.4. [Connection] Let M be a manifold and $\mathcal{V}$ be the space of smooth vector fields on M. A connection, defined on M is a map

$$\nabla : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$$

which follows the following properties:

- $\nabla(v_1 + v_2; u_1 + u_2) = \nabla(v_1; u_1) + \nabla(v_1; u_2) + \nabla(v_2; u_1) + \nabla(v_2; u_2)$
- $\nabla(v_1, fu_1) = f\nabla(v_1, u_1)$ where f a scalar value.
- $\nabla(gv_1, u_1) = g\nabla(v_1, u_1) + u_1(g)v_1$ where g a scalar function.

Definition 2.5. [Levi-Civita Connection] The Levi-Civita connection is the unique connection that is metric compatible and torsion free. Specifically, we have $\nabla g = 0$ and $\nabla_X Y - \nabla_Y X = [X, Y]$ where X and Y are any two vector fields and $[X, Y]$ is the Lie bracket of X and Y.

We will use the Levi-Civita connection, ∇, to covariantly differentiate our curvature tensor, R in the following manner:

$$\nabla R(x_1, x_2, x_3, x_4; u) = u(R(x_1, x_2, x_3, x_4)) - R(\nabla_u x_1, x_2, x_3, x_4)$$
$$-R(x_1, \nabla_u x_2, x_3, x_4) - R(x_1, x_2, \nabla_u x_3, x_4) - R(x_1, x_2, x_3, \nabla_u x_4).$$

Similarly we define $\nabla^i R$ for $i \geq 2$. More details on this higher order differentiation can be found in [4]. In addition, we will denote ∇ as the the Levi-Civita connection for the remainder of this paper.

Definition 2.6. [0-Model] A 0-Model is a model space given by a vector space V, a non-degenerate inner-product g, and an algebraic curvature tensor R. We will denote it as (V, g, R).

Definition 2.7. [1-Model] A 1-Model is a model space given by a vector space, V, a non-degenerate inner-product g, algebraic curvature tensor R, and algebraic covariant derivative ∇R. We will denote it as $(V, g, R, \nabla R)$.

Definition 2.8. Let S be a type (2,1) tensor, define

$$S_x \cdot T(x_1, ..., x_k) = -T(S_x x_1, x_2, ..., x_k) - ... - T(x_1, ..., S_x x_k),$$

where T is a tensor type (0,k).

3. Theorems Inherited from Kowalski and Prüfer

As previously stated, this paper is heavily indebted to the work of Kowalski and Prüfer. In particular, the following theorem and corollary were adapted from their paper:

Theorem 3.1. *For any curvature homogeneous-k manifold (M, g) there exits an open covering $\{V_\alpha\}_{\{\alpha \in J\}}$ such that in each V_α there is an orthonormal moving frame $\{E_1^\alpha, .., E_n^\alpha\}$ on which all components of R, $\nabla^i R$ are constant for $i \leq k$.*

Corollary 3.2. *In each V_α there exists a flat connection $\tilde{\nabla}^\alpha$ (with torsion) such that $(\tilde{\nabla}^\alpha)^i g = (\tilde{\nabla}^\alpha)^i R = 0$ for $i \leq k$. Here $\tilde{\nabla}^\alpha$ is defined as the (unique) connection for which the vector fields $E_1^\alpha, ..., E_n^\alpha$ define an absolute parallelism. More explicitly: $\tilde{\nabla}^\alpha_{E_i^\alpha} E_j^\alpha = 0$.*

The existence of an absolute parallelism on the connection $\tilde{\nabla}^\alpha$ creates an opening for us to develop conditions that connect the metric and algebraic curvature tensor — which is where we are headed.

4. Algebraic Condition to be satisfied for CH_1 space

Before we can develop the conditions that must be met to be included in a curvature homogeneous up to $k = 1$ space (CH_1), we state the commutation relation for the covariant derivative as written by Gilkey[4].

Proposition 4.1. *On a manifold (M, g), if $x_1, ..., x_4$ are vectors and ∇R is the covariant derivative of the algebraic curvature tensor R, then the following relationship must hold.*

$$\nabla^2 \mathcal{R}(x_1, x_2; x_3, x_4) - (\nabla^2 \mathcal{R})(x_1, x_2; x_4, x_3) =$$

$$\mathcal{R}(x_4, x_3)\mathcal{R}(x_1, x_2) - \mathcal{R}(x_1, x_2)\mathcal{R}(x_4, x_3) - \mathcal{R}(\mathcal{R}(x_4, x_3)x_1, x_2) - \mathcal{R}(x_1, \mathcal{R}(x_4, x_3)x_2)$$

This proposition allows us to establish relationships between the covariant derivative of algebraic curvature tensors. Now we can establish algebraic conditions that a one model must satisfy to belong to a CH_1 space.

Proposition 4.2. *Let (M, g) be a CH$_1$ space. Then, in a neighborhood U_p of each point $p \in M$, there exists a type $(1,2)$ tensor field, S, such that the following conditions must be met:*

$$S_z \cdot g = 0, \tag{4.3}$$

for all $z \in T_mM$, $m \in U_p$,

$$\mathcal{S}_{X,Y,Z}(S_X \cdot R)(Y, Z, U, V) = 0 \tag{4.4}$$

for every $X \in T_mM, m \in U_p$ where $\mathcal{S}$ denotes the cyclic sum, and

$$\begin{aligned}(S_{x_4} \cdot \nabla R)(x_1, x_2, z, w; x_3) - (S_{x_3} \cdot \nabla R)(x_1, x_2, z, w; x_4) = \\ R(x_4, x_3, \mathcal{R}(x_1, x_2)z, w) - R(x_1, x_2, \mathcal{R}(x_4, x_3)z, w) \\ -R(\mathcal{R}(x_4, x_3)x_1, x_2, z, w) - R(x_1, \mathcal{R}(x_4, x_3)x_2, z, w)\end{aligned} \tag{4.5}$$

for all $x_1, x_2, x_3, x_4, z, w \in T_mM, m \in U_p$.

Proof. Let $\{E_1, ..., E_n\}$ be the orthonormal basis referenced in Corollary (3.2), and let $\tilde{\nabla}$ be a flat connection on which $\{E_1, ..., E_n\}$ form an absolute parallelism. It is sufficient to complete the proof on this basis. Define $S_x = \nabla_x - \tilde{\nabla}_x$. Then proof of Equation (4.3) and (4.4) follow from the paper written by Kowalski and Prüfer[3] because a space (M, g) is CH$_0$ if it is CH$_1$. The proof of Equation (4.5) follows quickly. If we can show that for a CH$_1$ space,

$$S_{E_q} \cdot \nabla R(E_i, E_j, E_k, E_\ell, ; E_m) = \nabla^2 R(E_i, E_j, E_k, E_\ell; E_q, E_m) \tag{4.6}$$

for $i, j, k, \ell, m, q \leq n$ then the proof of Equation (4.5) will follow from Proposition (4.1). To prove Equation (4.6) consider that

$$\nabla^2 R(E_i, E_j, E_k, E_\ell; E_m, E_q) =$$

$$E_q(\nabla R(E_i, E_j, E_k, E_\ell; E_m)) - \nabla R(\nabla_{E_q} E_i, E_j, E_k, E_\ell; E_m) - ... - \nabla R(E_i, E_j, E_k, E_\ell; \nabla_{E_q} E_m).$$

The assumption that our model space is CH$_1$ informs us that the entries of $\nabla R(E_i, E_j, E_k, E_\ell; E_m)$ are constant. The partial differentiation of these terms will therefore vanish, eliminating the first term: $E_q(\nabla R(E_i, E_j, E_k, E_\ell; E_m))$. Our definition of $\tilde{\nabla}$ as a flat connection on which we have an absolute parallelism ensures that

$$\tilde{\nabla} \cdot \nabla R(E_i, E_j, E_k, E_\ell; E_m, E_q) = 0,$$

completing our proof. □

5. Developing Unknowns

Let (M, g) be a CH$_1$ space, and let $\{E_1, ..., E_n\}$ be the orthonormal moving frame that makes the curvature entries of R, ∇R and g to be constant in a neighborhood U of $p \in M$. Throughout this paper, we will refer to the notation

$$R(E_i, E_j, E_k, E_l) = R_{ijkl}.$$

Recall that our tensor S_X was defined to be

$$S_X = \nabla_X - \tilde{\nabla}_X.$$

Our flat connection $\tilde{\nabla}$ was chosen as the unique connection for which $\{E_1, ..., E_n\}$ defined an absolute parallelism in any coordinate chart. This means that $\tilde{\nabla}_{E_i} E_j = 0$ for all $i, j < n$. From this we can see that on the basis $\{E_1, ..., E_n\}$,

$$\nabla_{E_i} E_j = S_{E_i} E_j \tag{5.1}$$

for $i, j \leq n$. To proceed, we introduce the same notation used by Kowalski and Prüfer to write S_{E_i} as a linear combination of the E's. We define

$$S_{E_i}E_j = \sum_{p=1}^{n} S_{ij}^{p} E_p \tag{5.2}$$

This notation is justified because the E_p form a basis for each tangent space.

With a Riemannian metric, we can apply the notation of Equation (5.2) to Equation (4.3). We see that:

$$\begin{aligned} 0 = S_{E_i} \cdot g(E_j, E_k) &= -g(S_{E_i}E_j, E_k) - g(E_j, S_{E_i}E_k). \\ &= -g\Big(\sum_{p=1}^{n} S_{ij}^{p} E_p, E_k\Big) - g\Big(E_j, \sum_{p=1}^{n} E_p\Big) \end{aligned}$$

Because g is bilinear, the only terms that will survive the orthonormal g are $p = k$ in the first term, and $p = j$ in the second term. This implies that

$$0 = S_{ij}^{k} + S_{ik}^{j} \tag{5.3}$$

for $i, j, k \leq n$. Because of Equation (5.3), we can reduce our unknowns, the S_{ij}^{k}'s, from n^3 to $n\binom{n}{2}$.

If we instead chose to work with a Lorentzian Metric, we would simply pick up a negative sign in some cases. Let E_1 be the time-like vector of our signature $(1, 2)$ metric so that

$$g(E_i, E_j) = \begin{cases} \delta_{i,j} & \text{if } j, k \neq 11 \\ -\delta_{i,j} & \text{if } i = 1 \text{ or } j = 1 \end{cases}$$

Then, the Lorentzian equivalent for Equation (5.3) becomes

$$0 = \begin{cases} S_{i,j}^{k} + S_{ik}^{j} & \text{if } j, k \neq 1 \\ S_{i,j}^{k} + S_{ik}^{j} & \text{if } j = k = 1 \\ S_{i,j}^{k} - S_{i,k}^{j} & \text{if } j = 1 \text{ and } k \neq 1 \end{cases}. \tag{5.4}$$

for $i, j, k \leq n$. Equation (5.3) allows us to reduce the number of unknown S_{ij}^{k}'s from n^3 to $n\binom{n}{2}$ Namely, we have independent S_{ij}^{k}'s for $1 \leq i \leq n, 1 \leq j < k \leq n$.

6. Developing System of Equations

We can now apply the notation defined in Equation (5.2) to the relationship between S, ∇R and R in Equation (4.5). With a similar reasoning used in the definition of the S_{ij}^{k}'s, we can define

$$\mathcal{R}(E_i, E_j)E_k = H_{ijk}^{p} e_p.$$

With a Riemannian metric, we see that $H_{ijk}^{l} = R_{ijkl}$. To do this, notice that $g((\mathcal{R}(E_i, E_j)E_k, E_l)) = R(E_i, E_j, E_k, E_l)$. Simultaneously: $g((\mathcal{R}(E_i, E_j)E_k, E_l) = H_{ijk}^{l}$ because g is Riemannian and the $E_i's$ are orthonormal.

With a Lorentzian metric, with a similar proof we see that:

$$H_{ijk}^{l} = \begin{cases} R_{ijkl} & \text{if } l \neq 1 \\ -R_{ijkl} & \text{if } l = 1 \end{cases}.$$

In both the Riemanninanan and Lorentzian case, by employing the multi-linearity of R and properties of the covariant derivative outlined in Definition (2.5), we see that:

$$S^p_{qi}\nabla R_{pjkl;m} - S^p_{qj}\nabla R_{ipkl;m} - S^p_{ql}\nabla R_{ijpl;m} - S^p_{ql}\nabla R_{ijkp;m} - S^p_{qm}\nabla R_{ijkl;p} = \tag{6.1}$$

$$H^p_{ijk}R_{qmpl} - H^p_{qmk}R_{ijpl} - H^p_{qmi}R_{pjkl} - H^p_{aqk}R_{ipkl}$$

where p is a summation index. Notice that Equation (6.1) is the same with the exchange of the pairs (i,j), (k,l) and (m,q). Thus, we have a system of 27 equations for dimension 3. Namely, corresponding to $1 \le i < j \le n$, $1 \le k < l \le n$, $1 \le m < q \le n$. The equations for $n = 3$ with a Riemannian metric are written in full in Appendix A. The equations for $n = 3$ with a Lorentzian metric are included in Appendix B.

7. Applications

The system of equations developed and presented in Appendices A and B include both ∇R and R. In the special case in which $\nabla R = 0$, we are left with relationships among the independent entries of R that must be satisfied by the 1-model to belong to a CH_1 manifold. By definition when $\nabla R = 0$ corresponds to a symmetric space. A 1-model whose covariant derivative vanishes at each entry, and whose algebraic curvature tensor fails to meet the conditions established by the equations outlined in the previous sections cannot be the curvature tensor of a symmetric space,.

7.1. **Symmetric Spaces 3D Riemanninan.** The right hand side of the equations in Appendix A correspond to the conditions set on R when the entries of ∇R vanish. Klinger[5] showed that for a Riemannian metric there is a constant transformation that takes us to a basis in which maximal entries of the algebraic curvature tensor vanish. We can set

$$R_{1213} = R_{1223} = R_{1323} = 0.$$

This leaves us in $\dim M = 3$ with three independent curvature entries: $R_{1221} = Z$, $R_{1331} = Y$, and $R_{2332} = V$. Kowalski and Prüfer call this basis a 'Chern' basis. With this Chern basis in hand, the necessary relationships between the entries of the curvature tensor become simple. The equations become:

$$Z(V - Y) = 0, \tag{7.1}$$

$$Y(Z - V) = 0, \tag{7.2}$$

$$V(Y - Z) = 0. \tag{7.3}$$

From here it is evident that for Z, Y, and V to satisfy the these equations they must fall into one of two cases:

(1) $Z = Y = V$, or
(2) at most one Z, Y or V is nonzero.

Any 1-model with the covariant derivative $\nabla R = 0$ and with R not included in these two cases in this Chern basis cannot be realized as symmetric space.

7.2. **Symmetric Spaces 3D Lorentzian.** We can take a similar approach to the system of equations from Appendix B. Unfortunately, in dimension $n = 3$, there is no Chern basis that will cause certain entries of the curvature tensor to vanish. We proceed with our six independent entries of R outlined in Appendix B:

$$R_{1221} = Z, \qquad R_{1331} = Y,$$
$$R_{1231} = X, \qquad R_{1232} = W,$$
$$R_{2332} = V, \qquad R_{1332} = U.$$

By setting the 15 independent entries of ∇R to zero, we have a system of equations that must be satisfied by our curvature operator, R, in order for our 1-model to be realized as a curvature homogeneous up to $k = 1$ space. These relations are as follows:

$$WY - UX = 0, \tag{7.4}$$
$$UW - VX = 0, \tag{7.5}$$
$$UZ - WX = 0, \tag{7.6}$$
$$VZ - W^2 - X^2 + YZ = 0, \tag{7.7}$$
$$U^2 - VY + X^2 - YZ = 0. \tag{7.8}$$

While these five equations with six unknowns have a less straightforward solution than that of the dimension $n = 3$ Riemannian case, with the aid of a computing software we see there are five possible scenarios that would satisfy these equations:

Entry	Case 1	Case 2	Case 3	Case 4	Case 5
$U =$	U	0	U	0	0
$V =$	$\frac{WU}{X}$	0	V	V	Z
$W =$	W	0	0	W	0
$X =$	X	X	0	0	0
$Y =$	$\frac{XU}{W}$	Y	0	0	$-Z$
$Z =$	$\frac{WX}{U}$	$\frac{X^2}{Y}$	$\frac{W^2}{V}$	$\frac{W^2}{V}$	Z

Note that case 5 corresponds to constant sectional curvature. Any 1-model in which the entries of ∇R vanish and the entries of R do not satisfy one these five cases cannot be realized as a Lorentzian symmetric space.

8. Conclusions

We have established a set of algebraic conditions that must be satisfied for a 1-model to correspond to a curvature homogeneous up to $h = 1$ space. In the case of symmetric space, We present solutions to the entries of the curvature tensor whose 1-model could not correspond to a CH_1 space. Future studies could expand to higher orders curvature homogeneous spaces. It would also be meaningful to find a 1-model that satisfies the algebraic conditions specific to a CH_1 space outlined in this paper, but do not meet the conditions for a CH_0 space given by Kowalski and Prüfer[3]. There are also applications of this method to homothety curvature homogeneous spaces.

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Appendix A. Dimension 3 Riemannian Equations

In this appendix we present the system of equations for which the S_{ij}^k's must have a solution to in order for a 1-Model to have potential in being geometrically realized as a CH_1 space with a Riemanninan metric in 3 dimensions. Because these equations are quite long, the 6 independent curavture entries to the algebraic curvature tensor were renamed in the following manner:

$$R_{1221} = Z \qquad R_{1331} = Y$$

$$R_{1231} = X \qquad R_{1232} = W$$

$$R_{2332} = V \qquad R_{1332} = U.$$

The 15 independent entries of ∇R were renamed as well:

$$\nabla R_{12211} = A \qquad \nabla R_{12311} = B \qquad \nabla R_{12321} = C$$

$$\nabla R_{13321} = D \qquad \nabla R_{12212} = E \qquad \nabla R_{13311} = F$$

$$\nabla R_{12312} = G \qquad \nabla R_{12322} = H \qquad \nabla R_{23322} = I$$

$$\nabla R_{13322} = J \qquad \nabla R_{13313} = K \qquad \nabla R_{12313} = L$$

$$\nabla R_{12323} = N \qquad \nabla R_{23323} = P \qquad \nabla R_{13323} = Q.$$

Additionally, each S_{ij}^k has been renamed as A_{ij}^k The system of equations in matrix form is:

Nine rows of this matrix are obvious duplicates of each other, reducing our system of 27 equations to 18. It can be shown using Mathematica that this matrix has maximal rank.

$$\begin{bmatrix}
A & 2H & C-3G & E & -2C-(C-G) & 2B & 0 & 0 & 0 \\
B-H & J & -D+E-2L & C+G & -D+L & -A+F & 0 & 0 & 0 \\
C+G & -E+I & -J-N & -B+H & A+N-(-J-N) & D & 0 & 0 & 0 \\
B-H & J & -D+E-2L & C+G & -D+L & -A+F & 0 & 0 & 0 \\
F-2J & 0 & 2G-K & 2D-(-D-L) & K & -2B & 0 & 0 & 0 \\
D-I-(-D-L) & -G & H-Q & -F+J-(J+N) & B+Q & -C & 0 & 0 & 0 \\
C+G & -E+I & -J-N & -B+H & A+N-(-J-N) & D & 0 & 0 & 0 \\
D-I-(-D-L) & -G & H-Q & -F+J-(J+N) & B+Q & -C & 0 & 0 & 0 \\
2J-(J+N) & -2H & -P & -2D+I & 2C+P & 0 & 0 & 0 & 0 \\
0 & A+2N & E-2L & 0 & 0 & 0 & E & -2C-(C-G) & 2B \\
-N & B+Q & G-K-(C-G) & 0 & 0 & 0 & C+G & -D+L & -A+F \\
L & C+P-(-C+G) & H-Q & 0 & 0 & 0 & -B+H & A+N-(-J-N) & D \\
-N & B+Q & G-K-(C-G) & 0 & 0 & 0 & C+G & -D+L & -A+F \\
-2Q & F & 2L-(-D-L) & 0 & 0 & 0 & 2D-(-D-L) & K & -2B \\
K-P & D-L & J+N & 0 & 0 & 0 & -F+J-(J+N) & B+Q & -C \\
L & C+P-(-C+G) & H-Q & 0 & 0 & 0 & -B+H & A+N-(-J-N) & D \\
K-P & D-L & J+N & 0 & 0 & 0 & -F+J-(J+N) & B+Q & -C \\
2Q & -2N-(J+N) & I & 0 & 0 & 0 & -2D+I & 2C+P & 0 \\
0 & 0 & 0 & 0 & A+2N & E-2L & -A & -2H & 2G-(C-G) \\
0 & 0 & 0 & -N & B+Q & G-K-(C-G) & -B+H & -J & -E+L-(-D-L) \\
0 & 0 & 0 & L & C+P-(-C+G) & H-Q & -C-G & E-I & J+N \\
0 & 0 & 0 & -N & B+Q & G-K-(C-G) & -B+H & -J & -E+L-(-D-L) \\
0 & 0 & 0 & -2Q & F & 2L-(-D-L) & -F+2J & 0 & -2G+K \\
0 & 0 & 0 & K-P & D-L & J+N & -2D+I-L & G & -H+Q \\
0 & 0 & 0 & L & C+P-(-C+G) & H-Q & -C-G & E-I & J+N \\
0 & 0 & 0 & K-P & D-L & J+N & -2D+I-L & G & -H+Q \\
0 & 0 & 0 & 2Q & -2N-(J+N) & I & -J+N & 2H & P
\end{bmatrix} \times \begin{bmatrix} A^2_{11} \\ A^3_{11} \\ A^3_{12} \\ A^2_{21} \\ A^3_{21} \\ A^3_{22} \\ A^2_{31} \\ A^3_{31} \\ A^3_{32} \end{bmatrix} = \begin{bmatrix}
0 \\ -UX+WY \\ UW-VX \\ -UX+WY \\ 2UZ-2WX \\ VZ-W^2+X^2-YZ \\ UW-VX \\ VZ-W^2+X^2-YZ \\ -2UZ+2WX \\ 2UX-2WY \\ -UZ+WX \\ U^2-VY-X^2+YZ \\ -UZ+WX \\ 0 \\ -UW+VX \\ U^2-VY-X^2+YZ \\ -UW+VX \\ -2UX+2WY \\ -2UW+2VX \\ -U^2+VY-VZ+W^2 \\ UZ-WX \\ -U^2+VY-VZ+W^2 \\ 2UW-2VX \\ UX-WY \\ UZ-WX \\ UX-WY \\ 0
\end{bmatrix}$$

Appendix B. Dimension 3 Lorentzian Equations

In this appendix we present the system of equations for which the A^k_{ij}'s must have a solution to in order for a 1-Model to have potential in being geometrically realized as a CH_1 space with a Lorentzian metric in 3 dimensions. Because these equations are quite long, the 6 independent curvature entries to the algebraic curvature tensor were renamed in the following manner:

$$\begin{aligned}
R_{1221} &= Z & \qquad R_{1331} &= Y \\
R_{1231} &= X & \qquad R_{1232} &= W \\
R_{2332} &= V & \qquad R_{1332} &= U.
\end{aligned}$$

The 15 independent entries of ∇R were renamed as well:

$$\begin{aligned}
\nabla R_{12211} &= A & \qquad \nabla R_{12311} &= B & \qquad \nabla R_{12321} &= C \\
\nabla R_{13321} &= D & \qquad \nabla R_{12212} &= E & \qquad \nabla R_{13311} &= F \\
\nabla R_{12312} &= G & \qquad \nabla R_{12322} &= H & \qquad \nabla R_{23322} &= I \\
\nabla R_{13322} &= J & \qquad \nabla R_{13313} &= K & \qquad \nabla R_{12313} &= L \\
\nabla R_{12323} &= N & \qquad \nabla R_{23323} &= P & \qquad \nabla R_{13323} &= Q.
\end{aligned}$$

Additionally, each S^k_{ij} has been renamed as A^k_{ij} The system of equations in matrix form is:

$$\begin{bmatrix}
-A & 2H & C-3G & E & -2C-(C-G) & 2B & 0 & 0 & 0 \\
-B-H & J & -D+E-2L & C+G & -D+L & -A+F & 0 & 0 & 0 \\
-C-G & E+I & -J-N & B+H & -A+N-(-J-N) & D & 0 & 0 & 0 \\
-B-H & J & -D+E-2L & C+G & -D+L & -A+F & 0 & 0 & 0 \\
-F-2J & 0 & 2G-K & 2D-(-D-L) & K & -2B & 0 & 0 & 0 \\
-2D-I-L & G & H-Q & F+J-(J+N) & -B+Q & -C & 0 & 0 & 0 \\
-C-G & E+I & -J-N & B+H & -A+N-(-J-N) & D & 0 & 0 & 0 \\
-2D-I-L & G & H-Q & F+J-(J+N) & -B+Q & -C & 0 & 0 & 0 \\
-J+N & 2H & -P & 2D+I & -2C+P & 0 & 0 & 0 & 0 \\
0 & -A+2N & E-2L & 0 & 0 & 0 & E & -2C-(C-G) & 2B \\
-N & -B+Q & G-K-(C-G) & 0 & 0 & 0 & C+G & -D+L & -A+F \\
-L & -2C+G+P & H-Q & 0 & 0 & 0 & B+H & -A+N-(-J-N) & D \\
-N & -B+Q & G-K-(C-G) & 0 & 0 & 0 & C+G & -D+L & -A+F \\
-2Q & -F & 2L-(-D-L) & 0 & 0 & 0 & 2D-(-D-L) & K & -2B \\
-K-P & -D+L & J+N & 0 & 0 & 0 & F+J-(J+N) & -B+Q & -C \\
-L & -2C+G+P & H-Q & 0 & 0 & 0 & B+H & -A+N-(-J-N) & D \\
-K-P & -D+L & J+N & 0 & 0 & 0 & F+J-(J+N) & -B+Q & -C \\
-2Q & J+3N & I & 0 & 0 & 0 & 2D+I & -2C+P & 0 \\
0 & 0 & 0 & 0 & -A+2N & E-2L & A & -2H & 2G-(C-G) \\
0 & 0 & 0 & -N & -B+Q & G-K-(C-G) & B+H & -J & -E+L-(-D-L) \\
0 & 0 & 0 & -L & -2C+G+P & H-Q & C+G & -E-I & J+N \\
0 & 0 & 0 & -N & -B+Q & G-K-(C-G) & B+H & -J & -E+L-(-D-L) \\
0 & 0 & 0 & -2Q & -F & 2L-(-D-L) & F+2J & 0 & -2G+K \\
0 & 0 & 0 & -K-P & -D+L & J+N & D+I-(-D-L) & -G & -H+Q \\
0 & 0 & 0 & -L & -2C+G+P & H-Q & C+G & -E-I & J+N \\
0 & 0 & 0 & -K-P & -D+L & J+N & D+I-(-D-L) & -G & -H+Q \\
0 & 0 & 0 & -2Q & J+3N & I & 2J-(J+N) & -2H & P
\end{bmatrix}
\times
\begin{bmatrix}
A^2_{11} \\ A^3_{11} \\ A^3_{12} \\ A^2_{21} \\ A^3_{21} \\ A^3_{22} \\ A^2_{31} \\ A^3_{31} \\ A^3_{32}
\end{bmatrix}
=
\begin{bmatrix}
0 \\
-UX+WY \\
UW-VX \\
-UX+WY \\
2UZ-2WX \\
VZ-W^2-X^2+YZ \\
UW-VX \\
VZ-W^2-X^2+YZ \\
2UZ-2WX \\
2UX-2WY \\
-UZ+WX \\
U^2-VY+X^2-YZ \\
-UZ+WX \\
0 \\
-UW+VX \\
U^2-VY+X^2-YZ \\
-UW+VX \\
2UX-2WY \\
-2UW+2VX \\
-U^2+VY-VZ+W^2 \\
-UZ+WX \\
-U^2+VY-VZ+W^2 \\
2UW-2VX \\
-UX+WY \\
-UZ+WX \\
-UX+WY \\
0
\end{bmatrix}$$