# Using the Crushtaceans of Fully Augmented Links to Investigate Cheeger Constants

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**Abstract**

We investigate potential uses of graph theoretical tools in determining a measure of connectivity in links of a particular construction. Considering only fully augmented links in specific families, we find some results regarding how we can use graph theoretical tools to bound a measure of connectivity.

## 1 Fully Augmented Links: Forming FALs, circle packings, and Crushtaceans

In this paper, we will explore the connection between fully augmented links (FALs) and their crushtaceans. More specifically, we will explore how the Cheeger Constant of the FAL's crushtacean and the Cheeger Constant of the FAL itself are related.

Starting with a link diagram, we will now define how to form the Crushtacean. We identify the sections of the link diagram that are twist regions, or two strands that alternate passing over and under one another without interacting with other strands. We place an unknotted component around each of these twist regions. We then count the crossings in each twist region. If there are an even number of crossings, we remove the crossings and replace them with two parallel strands passing through the unknotted component. If there are an odd number of crossings, we replace the them with a single crossing where the overstrand is the same as it is in the outer two crossings. Our new link diagram is an FAL, call it $F$. We also label two distinct types of components in this diagram.

Definition 1.1. A *crossing circle* in an FAL is any of the unknotted components that we added around the twist regions.

Definition 1.2. A *knot circle* in an FAL is any of the components in our FAL that is not a crossing circle.

From an FAL, we can create a circle packing. We first color each of the crossing circles so that we can identify them later. We then identify three line segments on each side of the crossing circle. That is, we have a segment from the edge of the crossing circle to the point where a knot circle intersects the crossing disk, another from that point to the other point where a knot circle the crossing disk, and another from this point to the edge of the crossing disk. This is shown in Figure 1.

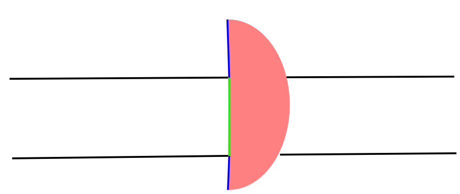


Figure 1: The Top Half of a Crossing Disk: Note that the line segments along the left side of the crossing disk alternate in color. This differentiates the three different line segments that we will use below to create our triangles.

From here, we will make two semicircles where the crossing disk is, with the colored line segments on each side. Then we can shrink the arcs of the semicircles, making two triangles that are incident at a single vertex. This creates a bowtie shape, as shown in Figure 2.

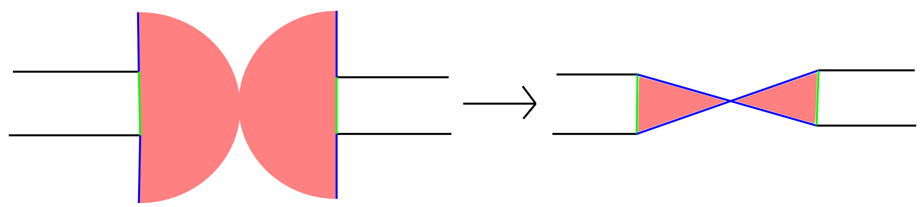


Figure 2: The next two steps of creating a circle packing from a link diagram. Note that we still keep the triangles shaded as we will use that later.

We will shrink the black components, those from our knot circles, until our shaded triangles are all connected to other shaded triangles on each of their 3 vertices. Each of our unshaded regions becomes a circle, including the ambient unshaded region. Each circle should be tangent with other circles that originated from regions that share a vertex. We also ensure that we mark which regions are shaded from the original crossing circle. In particular, we note which points of tangency would be the middle vertex of the bowtie shape created from each crossing disk. This new diagram is the circle packing of the FAL. The circle packing is the footprint of an associated polyhedron in $\mathbb{H}^3$ where the points of tangency between two circles is an ideal vertex in the polyhedron. This polyhedron is often called the manifold of our FAL. For a more in-depth treatment of these associated polyhedra, please refer to [1].

We will now place a vertex in each interstice between the circles. We connect these vertices through the points of tangency between circles. We paint the edges that pass through the points of tangency that were the middle vertices of a bowtie. The final graph that we have is known as the Painted Crushtacean of the FAL $F$.

For an example of going from an FAL to a Crushtacean, see Figure 3.

# 2 Pretty Edge Cuts

In the Crushtacean, some edge cuts represent non-trivial cuts on an FAL using an $n$ punctured sphere together with some number of crossing disks. Since this type of cut, cutting along a surface that divides the volume of the polyhedron in two sections, is valuable for determining the Cheeger constant of the polyhedra corresponding to an FAL, we will give this type of edge cut a name.

Definition 2.1. Consider some trivalent connected graph $G = \{V, E\}$ with connected subgraphs $G_1 = \{V_1, E_1\}$ and $G_2 = \{V_2, E_2\}$ such that $V_1$ and $V_2$ partition $V$ and $|V_1|, |V_2| \geq 2$. Then an edge cut consisting of all edges connecting some $v_1 \in V_1$ to some $v_2 \in V_2$ is a *pretty edge cut.*

When considering the Crushtacean of a given FAL, we often use painted Crushtaceans since they include information on which edges of the Crushtacean correspond to crossing disks versus knot circles. This information also affects the Cheeger constant of the corresponding polyhedron for some FAL.

Definition 2.2. A *perfect matching* is some set of unordered pairs of vertices that matches each vertex to exactly one other vertex. Moreover, matched vertices must be connected via an edge.

Definition 2.3. A *painting* of a graph $G$ is some set of edges that includes all edges that connect matched vertices in a particular perfect matching on $G$. We will call such edges *painted edges.*

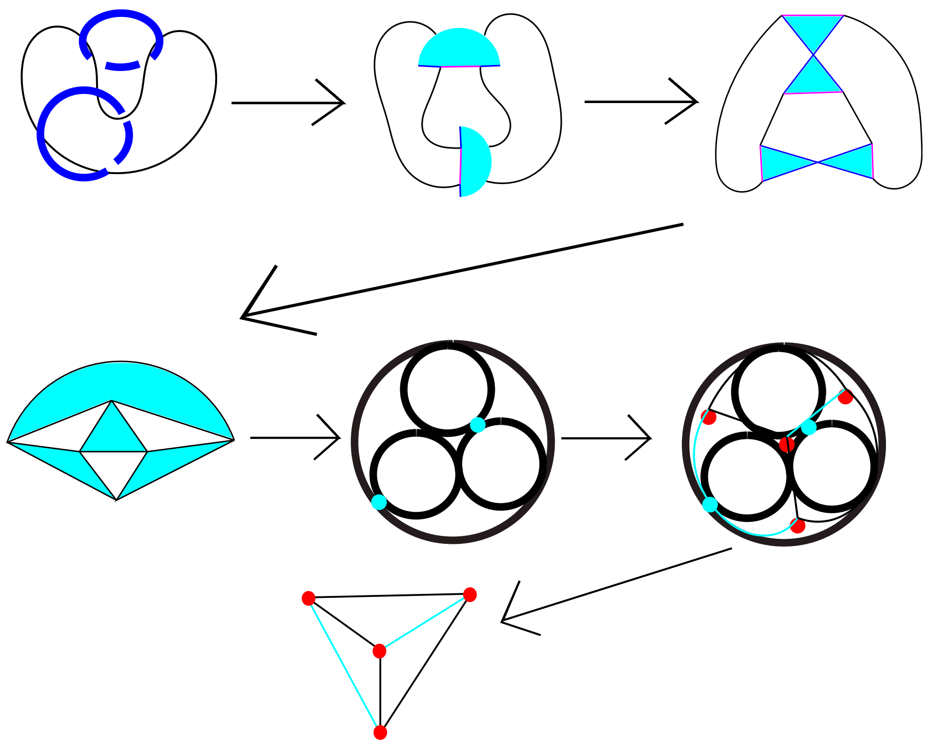


Figure 3: The FAL of the Borromean Rings to its Crushtacean

Note that, in most cases, a perfect matching or a painting of a graph is not the only perfect matching or painting on that graph. We also know that, based on the way we construct Crushtaceans, a painted Crushtacean falls into the category of a painted graph, hence its name.

THEOREM 2.4. *Consider the trivalent graph $G$ with vertices $V$, edges $E$, and some painting. If we take some $k$-pretty edge cut on $G$, the parity of the painted edges in the $k$-edge cut is the same as the parity of $k$.*

*Proof.* Recall that the sum of degrees of vertices in any graph must be even. From here, we can note that since all vertices in a trivalent graph have degree three, the sum of degrees of vertices in a trivalent graph is $3|V|$. For $3|V|$ to be even, $|V|$ must be even.

Let's consider one of the two connected subgraphs of $G$ obtained after removing the edges in our $k$-edge cut, call it $G_1$ and its set of vertices $V_1$. Add $k$-many additional vertices to $G_1$, connecting each new vertex to exactly one vertex in $G_1$ until all vertices from $G_1$ have degree three. From here, arbitrarily number the $k$ vertices not in $G_1$ 1 through $k$. Then, connect sequential vertices with an edge, and connect vertex 1 with vertex $k$. We clearly have a new trivalent graph composed of our new vertices and edges together with $G_1$. The number of vertices in this new trivalent graph must be even, so $|V_1| + k$ is even. For this to be the case, $|V_1|$ and $k$ must have the same parity. Now, let's consider two cases.

First, if $k$ is odd, then $G_1$ has an odd number of vertices. The perfect matching on $G$ must connect pairs of vertices, so the total number of vertices in $V_1$ matched with another vertex in $V_1$ must be even. Then there must be an odd number of vertices in $V_1$ that are paired with vertices outside of $V_1$. The edges connecting such vertices would be painted. Thus, there are an odd number of painted edges in our $k$-edge cut.

Next, assume that $k$ is even. Then $G_1$ must have an even number of vertices. The number of vertices in $V_1$ matched to another vertex in $V_1$ is even, so we have an even number of vertices matched to vertices outside of $V_1$. Then the edges connecting these vertices to their match must be both painted and involved in the $k$-edge cut. Thus, there are an even number of painted edges in our $k$-edge cut. ☐

# 3 Triangle Vertex Expansion

In order to determine how relationships between Crushtaceans affected the Cheeger Constant of the graph, we first will define an operation on graphs along with an inverse-like operation and then define how to determine the Cheeger Constant of a graph.

Definition 3.1. We will define an operation on trivalent graphs called *triangle vertex expansion.* To perform a triangle vertex expansion, we will select a vertex in our trivalent graph. We will place three vertices such that each one is connected to a different vertex that our selected vertex was incident with. We will then connect each of our new vertices to each other new vertex. Finally, we will remove our selected vertex and the three edges incident with our selected vertex. Note that in our trivalent graph, this operation will produce a new graph that is also trivalent. This is demonstrated in Figure 4.

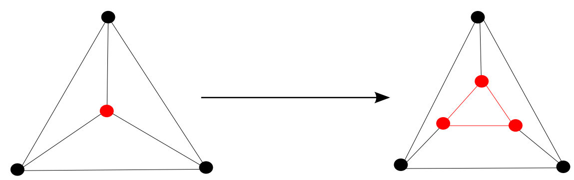


Figure 4: **Triangle Vertex Expansion:** When we perform a triangle vertex expansion, the red vertex is replaced with the red vertices and edges on the right.

This operation also has something that can act as an inverse in most cases.

Definition 3.2. We will define an operation called a *triangle vertex contraction.* To perform this operation, we will find a 3 cycle bounding a face in a trivalent graph. Then, since the graph is trivalent, each vertex in the 3 cycle is connected to each other vertex in the 3 cycle as well as one vertex not in the 3 cycle. We will then remove the 3 cycle and replace it with a single vertex connected to each of the vertices that one of the vertices in the 3 cycle was connected to.

Definition 3.3. If we have a perfect matching on a trivalent graph and perform triangle vertex expansion, the *obvious matching* on this new graph is the one where we match all vertices from the parent graph as they were in the parent graph. Then the vertex that was matched with the expanded vertex will be matched with the new vertex from the triangle expansion that it is connected to via an edge and the other two vertices from the triangle vertex expansion are matched with each other.

Due to this obvious perfect matching, when we perform triangle vertex expansions, we can see that there will be a painting such that if the parent graph has a $k$ edge cut with $n$ painted edges, we can find an analogous $k$ edge cut with $n$ painted edges after triangle vertex expansion.

Definition 3.4. The *Cheeger Constant of some graph* $G = \{V, E\}$ is the minimal

$$\frac{k}{\min\{Vol(A), Vol(B)\}}$$

where $A$ and $B$ are two partitioning subsets of $V$, an edge cut of $k$ edges separates these two subsets, and $Vol(U)$ where $U$ is some set of vertices is the sum of the degrees of all vertices in $U$.

Now that we have this operation, our next goal was to find how the Cheeger constants of a graph before and after triangle vertex expansion were related.

Theorem 3.5. *The graph resulting from a triangle vertex expansion of some trivalent graph $G$ can have a Cheeger constant no greater than the Cheeger constant of $G$.*

*Proof.* Let $G = \{V, E\}$ be a trivalent graph and choose some vertex $v \in V$ to perform triangle vertex expansion. We will call this new graph $G' = \{V', E'\}$ and the three new vertices in this graph $v_1, v_2$, and $v_3$. We know that the Cheeger constant of $G$ is determined by some $k$-edge cut on $G$ that divides the vertices of $G$ into two partitioning subgraphs of $V$, call them $V_1$ and $V_2$. Because $G$ is trivalent, the volume of any set of vertices in $G$ is three times the number of vertices in the set. Then we find that the Cheeger constant of $G$ can be written as

$$\frac{k}{3 \cdot \min\{|V_1|, |V_2|\}}$$

Consider an analogous $k$-edge cut in $G'$ where we cut the edges in $G'$ corresponding to the cut edges in $G$. If one of the cut edges in $G$ were connected to $v$ and some $u \in V$ we instead cut the edge connecting $u$ to one of our $v_i$s. If this occurs more than once, repeat this. Then we have partitioning subsets $V_1'$ and $V_2'$ of $V'$. One of these subsets will have 3 vertices where there was only 1 in the corresponding subset of $V$, so its cardinality will increase by two. Then we get that there is some $k$-edge cut that divides $G'$ into $V_1'$ and $V_2'$ such that $3 \cdot \min\{|V_1'|, V_2'|\} \geq 3 \cdot \min\{|V_1|, |V_2|\}$. Then we find that

$$\frac{k}{3 \cdot \min\{|V_1|, |V_2|\}} \geq \frac{k}{3 \cdot \min\{|V_1'|, V_2'|\}}$$

Thus, the Cheeger constant of $G$ provides an upper bound for the Cheeger constant of $G'$. □

NOTE 3.6. By the argument used in the above theorem, we can see that for a $k$-cut in some trivalent graph $G$, there is an analogous $k$-cut in any graph that results from a triangle vertex expansion on $G$.

NOTATION 3.7. Define the elements of a set of graphs $B_4$ in the following manner: Starting with $K_4$, find a 4-cut that divides the graph into two subgraphs with two vertices each. Then, choose one vertex from each of these subgraphs and perform triangle vertex expansion on the selected vertices. Continue choosing one vertex on each side of the 4-cut and performing triangle vertex expansions.

THEOREM 3.8. *For any graph $G = \{V, E\} \in B_4$, the upper bound of the Cheeger Constant of $G$ is*

$$\frac{4}{\frac{3}{2}|V|.}$$

*Proof.* By performing a triangle vertex expansion on each side of the 4-cut in $G$, we ensure that we have an equal number of vertices on each side of the 4-cut. This, together with the trivalency of the graph, gives us that the volume of each side of the edge-cut is $\dfrac{3}{2}|V|$. Then, we know that since we are looking for the minimum fraction consisting of the number of edges removed over the minimum volume of the sides of the edge cut, the Cheeger Constant of this graph cannot be more than

$$\frac{4}{\frac{3}{2}|V|.}$$

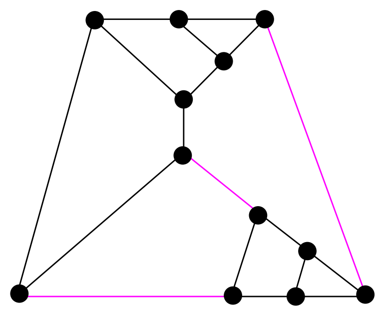
□

NOTE 3.9. The bound given in Theorem 3.8 is not always the Cheeger Constant of a graph in this family of graphs. Take, for example, the graph in Figure 5.

The upper bound given by Theorem 3.8 is $\frac{4}{18} \approx 0.2222$. However, the 3-edge cut colored pink yields a Cheeger Constant of $\frac{3}{15} = 0.2$. Thus, the 4-edge cut described in Theorem 3.8 does not necessarily realize the Cheeger Constant, and we cannot say that the Cheeger Constant of such a graph is strictly equal to $\dfrac{4}{\frac{3}{2}|V|}$.

We must now consider how a triangle vertex expansion affects the types of edge cuts in a graph. In particular, we want to see if a triangle vertex expansion can create a new 1 or 2 edge cut, or disconnect the graph entirely.

Figure 5: Graph of $K_4$ after a particular set of 4 triangular expansions.



Conjecture 3.10. For some 3-edge connected trivalent graph $G$ together with some triangle vertex expansion on $G$ that produces a graph $G'$, we find that $G'$ is also 3-edge connected.

*Proof.* In a 3-edge connected graph, we can remove any pair of edges without disconnecting the graph. Moreover, any trivalent graph is at most 3-edge connected, so showing that we can remove any pair of edges without disconnecting the graph is sufficient to say that a trivalent graph is 3-edge connected.

First, let's take the case where both edges removed are not in the 3-cycle between the three new vertices. Then, if the path that existed in between two vertices in $G$ did not go through the expanded vertex, we are done. Otherwise, we can follow the path in $G$ until we reach one of our new vertices. This vertex is connected to each other new vertex, one of which, call it $v'$, is connected to the next vertex from the path in $G$. We will take the edge to $v'$ before continuing along the original path from $G$. The new vertices also are connected to the rest of the graph. If one of the vertices is new, we can take the path for the expanded vertex after going around the three cycle until it is connected to a vertex in the path from $G$ and then following the path from $G$. If both vertices are new, then they are connected by a single edge by construction.

Next let's take the case where one of the removed edges is in the three cycle. The new vertices are still connected to one another, either directly by an edge or with an intermediary visit to the other new vertex. Then we will consider a path from $G$ that is possible when the edge from $G$ is removed. If the path includes the expanded vertex, we can do the same thing as we did in the previous case since we established that the new vertices are still connected to one another.

The final case is the one where both edges removed are in the 3-cycle. In this case, we will have two of our new vertices connected to one another via an edge and the third new vertex, call it $u$ that is not incident with the other two. In $G$, we can consider the path that we would take if we removed the edge connecting the expanded vertex to the only vertex that $u$ is still incident with after the edge cut. That is, looking at Figure 6, we consider the path we would take if we removed the pink edge.

Then, to find a path where no more than one vertex is one of the new vertices, we can take the original path from $G$, and if we arrive at the new section, we can go to the new vertex connected to the old vertex. Then we can go from there to a new vertex that connected to the vertex that followed the expanded vertex in $G$. If both vertices are in the new vertices and are directly connected by an edge, we are done. Otherwise, we know that each of the new vertices is connected to one of the vertices in $G$. Then we can take the path described above to get to the other vertex.

In every case, removing 2 edges of $G'$ produces a connected graph. Thus, $G'$ is 3-edge connected. $\square$

Notation 3.11. Define the following graphs as members of some set of graphs $B_3$ : Starting with $K_4$, perform triangle vertex expansion on one vertex. Note that the choice of vertex does not impact the resulting graph since $K_4$ is a complete graph and thus any vertex we select will result in isomorphic graphs. Then, we can conceptualize this as two sections, one from the expansion and one from the original graph. Expand the same number of times in each section of this graph.

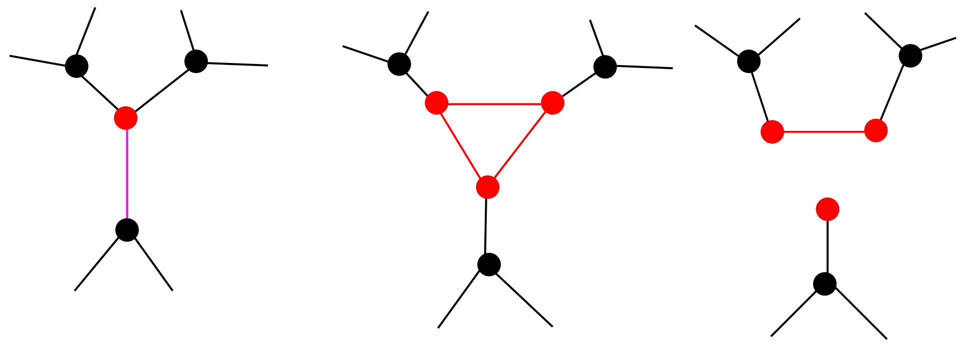


Figure 6: If we expand the red vertex in the right graph and then consider the two edge cut shown on the left, we consider paths in the graph on the left that do not require the vertical pink edge.

THEOREM 3.12. *The Cheeger constant of any $G = \{V, E\} \in B_3$ will be*

$$\frac{2}{|V|}.$$

*Proof.* Note that by the above theorem, we have that all graphs in this family are 3-edge connected. Then clearly a 3-cut will be the minimum needed to separate the graph. Then we can see, by the construction of the graph, that there is a 3-edge cut that separates the new section from the old section. Since we started out with 3 vertices on each side, we add the same number of vertices to each side, and each vertex is of degree 3, we are dividing the volume of the entire graph in half when we perform the 3-edge cut. Then we have maximized the denominator and minimized the numerator of the fraction that forms the Cheeger Constant candidate. That is to say, we have found an edge cut that must result in the Cheeger Constant. Then our Cheeger Constant is

$$\frac{3}{\frac{1}{2} \cdot 3 \cdot |V|} = \frac{2}{|V|}.$$

□

# 4 Tying Painted Crushtaceans Back to the Associated Polyhedra

Using the previous section on $k$-edge cuts in painted triangle vertex expansions of $K_4$, we can note a pattern and make a generalization. Each pretty edge cut corresponds to a surface in $\mathbb{H}^3$ that cuts the associated manifold. Moreover, we can find the area of this cutting surface. Each $k$-edge cut corresponds to some $k$ punctured sphere. For this $k$ pretty edge cut, we will call the number of painted edges $n$. This $n$ corresponds to the number of crossing disks that will be needed to complete the cutting of our manifold. Then, since we know that the area of a $k$ punctured sphere is $2(k-2)\pi$ and the area of a crossing disk is $2\pi$, we find that the area of the cutting surface is

$$n(2\pi) + 2(k-2)\pi.$$

This cutting surface is valuable due to the method by which we find the Cheeger Constant of a manifold.

DEFINITION 4.1. Let $C$ be some surface in $\mathbb{H}^3$ that cuts a manifold $M$ into two pieces. Then the *Cheeger Constant of the manifold $M$* is the minimal

$$\frac{A(C)}{\min\{V(A), V(B)\}}$$

where $A(C)$ is the area of our cutting surface, $V(A)$ is the volume of one of the pieces, and $V(B)$ is the volume of the other piece.

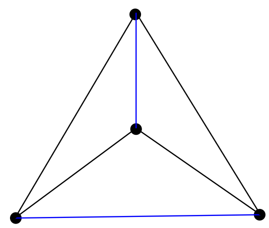


Figure 7: Painted Crushtacean of the Borromean Rings

Let's consider an example family of graphs discussed in Theorem 3.8. We can find the Crushtacean of the Borromean Rings as shown in Figure 7.

In Figure 7, we can perform a pretty edge cut consisting of 4 unpainted edges that gives us subgraphs with half of the volume of the graph each. Interestingly, we find that the corresponding cut in the corresponding manifold also divides the volume in half. In this case, the reason for this comes down to what we see in the circle packing of the Borromean Rings. The circle packing of the Borromean rings can be seen in Figure 3. The 4 edge cut corresponds to a 4 punctured sphere. When we form the manifold that corresponds to this circle packing, we can see that we are cutting along a surface across which our octahedron has reflective symmetry. This is because the surface passes through each vertex in the circle packing that corresponds to the edges we cut. The only surface to go through each of these vertices is the surface across which we have reflective symmetry. Since we have this reflective symmetry across our cutting surface, the cut in $\mathbb{H}^3$ that corresponds to our four edge cut divides the volume of our manifold in half. In [2] Purcell's use of central subdivision in the nerve corresponds directly to our use of triangle vertex expansion in the Crushtacean. Purcell finds that each central subdivision in the nerve results in two additional octahedra in the associated manifold. For more details, see section 3.3 of [2]. Since all regular ideal octahedra have the same volume, and since for the family of graphs described in Theorem 3.8 we perform triangle vertex expansions on either side of our edge cut, we have a family of graphs for which a 4 punctured sphere can divide the volume of the associated manifold in half. Note that this is assuming that we use the obvious matching when we perform triangle vertex expansions, since otherwise we may need to use a 4 punctured sphere together with some crossing disks to cut our manifold.

From here, we can determine an upper bound on the Cheeger Constant of the manifolds associated with FALs that correspond to a graph in the family from Theorem 3.8. Since we perform triangle vertex expansions in pairs, we will add 4 octahedra to our manifold each time we perform a pair of triangle vertex expansions. Moreover, since each regular ideal octahedron has the same volume, call it $v_8$, we find that the following is an upper bound on the Cheeger Constant of our manifold that results after $n$ pairs of triangle vertex expansions

$$\frac{4\pi}{\min\{V(A), V(B)\}}.$$

However, $V(A) = V(B)$ and are half the volume of the total manifold. Since the total manifold is comprised of $4n + 2$ octahedra, we can then find that our upper bound becomes

$$\frac{4\pi}{\frac{1}{2}(4n+2)v_8} = \frac{4\pi}{(2n+1)v_8}.$$

# 5 Future Work

We explored some elements of how the Cheeger Constant of a Crushtacean could relate to the Cheeger Constant of the FAL, but there are a few other questions to consider:

1. Is there an upper or lower bound on the Cheeger Constant of the FAL in terms of the Cheeger Constant of the FAL?
2. Given some Crushtacean, can we determine the type of surface that would produce the optimal cut that gives us the Cheeger Constant in the FAL?
3. We only considered FALs in depth that were triangle vertex expansions on $K_4$. How do they behave when we allow other types of graphs?
4. Removing the necessity to start with $K_4$, can we define families of Crushtaceans that have a certain formula for their Cheeger Constant or a bound on their Cheeger Constant?
5. Can we find other interesting operations on trivalent graphs that give us some relationship between the Cheeger Constant of the parent graph and the resulting graph?

# 6 Acknowledgements

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# References

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