On the Classification of Belted Sum Decomposition of Nested Links

Brandon Castillo

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Abstract

One way to simplify complicated hyperbolic links is through belted sum decompositions. Belted sum decompositions separate hyperbolic link complements into simpler pieces which in turn create more tractable geometry. Belted sum decompositions of Fully Augmented Links (FALs) are understood. This project extends known results on belted sum decompositions of FALs to a more general class of links, called nested links. In particular, a classification of belted sum prime nested links is given by analyzing 3ps in their complements.

1 Introduction

There are many important concepts in Knot Theory but this paper will only look at Fully Augmented Links, better known as FALS, and Nested Links, which are a subclass of generalized FALS. A knot can be thought of as picking up a string with two ends, twisting and turning it, and then gluing the two end pieces together. A series of tangles and crossing will most likely occur and this is called a knot. A link is just multiple knots that are connected together. An FAL is a link where every twist of two strands in the knot circle are augmented via a trivial component called a crossing circle. Figure 1 illustrates this notion of augmenting all the twist regions and being left over with either half twist or no twist in all the twist regions.

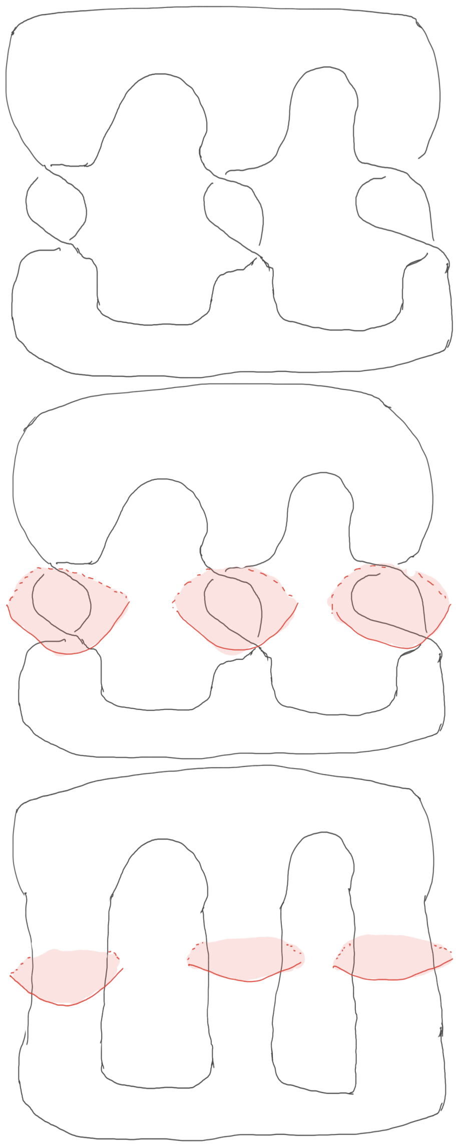

Figure 1. The original link diagram (Top). Placing the trivial crossing circle components on each twist region (Middle). Aftermath of untwisting each twist region (Bottom).

2 FALs, Generalized FALs, and Nested Links

One can view a link complement as the stuff that is left over if a worm entered an apple and ate a maze within the apple. If a link has a hyperbolic metric, it is said to be a hyperbolic link. FALs is one example of a hyperbolic link. Another example of a hyperbolic link is a generalized FAls, also know as GFALs. GFALs are similar to FALs in the sense that they both are fully augmented. However, GFALs are different from FALs because they can have more than two strands per twist region. This paper will focus on a specific type of GFALs called nested links. For more detail concerning FALs and how augmentation works, refer to John Harnios' 2015 paper [5]. Nested links get their name from the idea of nesting one crossing circle into the same plane as another crossing circle. The nested crossing circle becomes a crossing circle puncture for the larger crossing disk. When dealing with nested links, there are exactly 3 ways for any crossing disk to have its two punctures:

- Two knot circle punctures
- Two crossing circle punctures
- One knot circle puncture and one crossing circle puncture

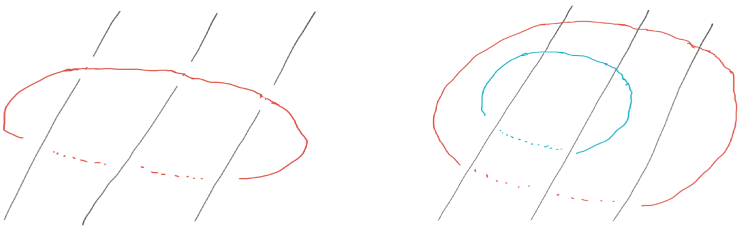

Figure 2. GFALs (Left). Nested Link (Right).
More detail can be found in Harios' paper [5].

3 Belted Sum Decomposition

A BSD occurs when a link diagram can be split into two separate parts.

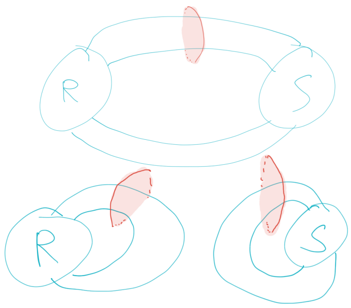

Figure 3. Link diagram with one crossing circle (Top). The two separate components that result from the act of slicing and gluing along the crossing disk (Bottom).

Link diagrams can be useful to view in some scenarios, but at other times it can be better to have a different perspective. Cell decompositions offer different perspective on the link diagram that can potentially shed light on something that would have otherwise been difficult to notice. Cell decompositions first work by creating what is known as a "pita-bread" slice and "bow-tie" configuration. Then, shrink each knot circle into their own point and notice where the regions are at. Changing the unshaded regions into circle packings and shaded regions into shaded triangles is known as the circle packing. Regions should be more clear with their circular shapes and the last step is determining the crushtacean. The crushtacean can be determined by putting a vertex in the center of each unshaded region in the circle packing. Then connect each vertices through points of tangency. The final product is the crushtacean. Shaded edges in the crushtacean correspond to crossing circles in the link diagram. These steps can all be seen in Figure 4-6.

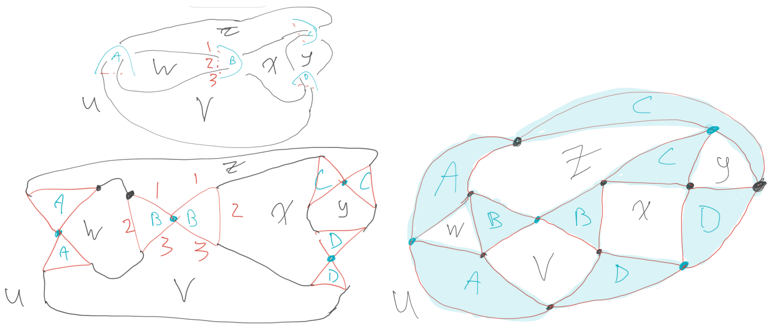

Figure 4. Original link diagram (Top). Aftermath of pita-bread slice and bow-tie configuration (Bottom Left). Shrinking all edges to a single point (Bottom Right).

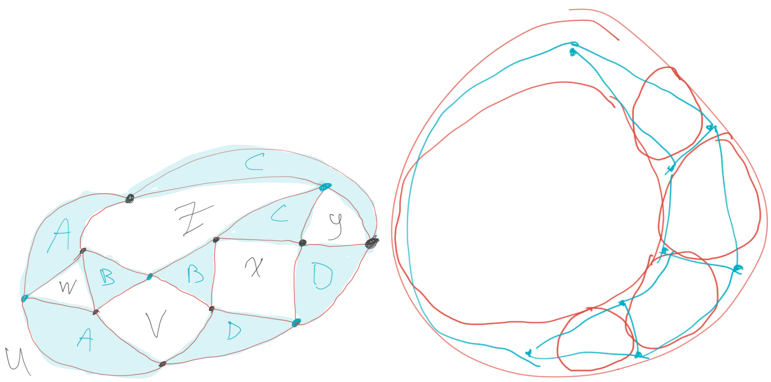

Figure 5. Bottom right of figure 4 (Left). Vertex in center of all shaded regions and placing an edge piece through all points of tangency (Right).

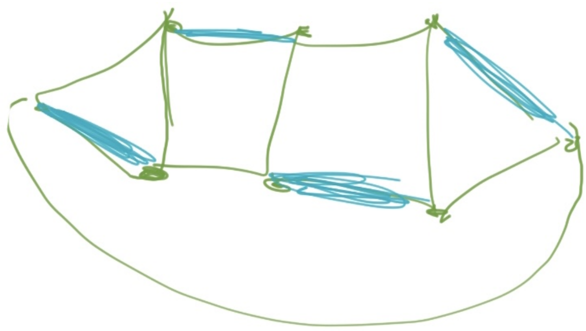

Figure 6. This is the painted crushtacean of our original link diagram. Note that the blue painted edges correspond to the crossing circles in our original link diagram.

Olson defines a specific type of crushtacean called a edge symmetric forest as "An edge-symmetric graph is a connected graph such that for one edge a, each edge b1 is related to one edge b2 by symmetry about a. An edge-symmetric spanning forest is a spanning forest such that each tree is an edge-symmetric graph" [4].

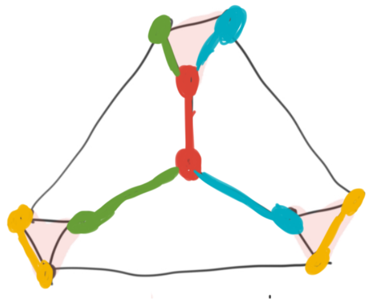

Figure 7. Example of a edge symmetric forest.

The red painted edge is called the edge of symmetry as everything can be thought as revolving around it. Both the green and blue painted edges are called leaves of the tree. The two yellow painted edges are simply two separate crossing circles. If there was a painted edge between either the red painted edge and the green or blue painted edges, that newly added painted edge is said to be interior to the tree.

Lemma 3.1 The edges in a non-trivial 3-edge-cut are incident with distinct vertices.

Non-trivial implies all can't go to same vertex or else a trivial, non-hyperbolic 3-edge-cut occurs. If two vertices are connected there now exist a 2-edge-cut which is also not hyperbolic. □

Lemma 3.2 In a trivalent graph,

$$2e = \sum degrees\ of\ vertices = 3v$$

Make the RHS into 1 vertex. This is a trivalent graph so there are even number of total vertices. Since RHS is odd, LHS must also have an odd total number of vertices. Mirror proof to show why RHS must have an odd total number of vertices. Thus there must be a odd total number of vertices in both RHS and LHS. □

To understand the similarities and differences of having two separate components in either the link diagram or crushtacean, it is important to know what is going on behind the scene.

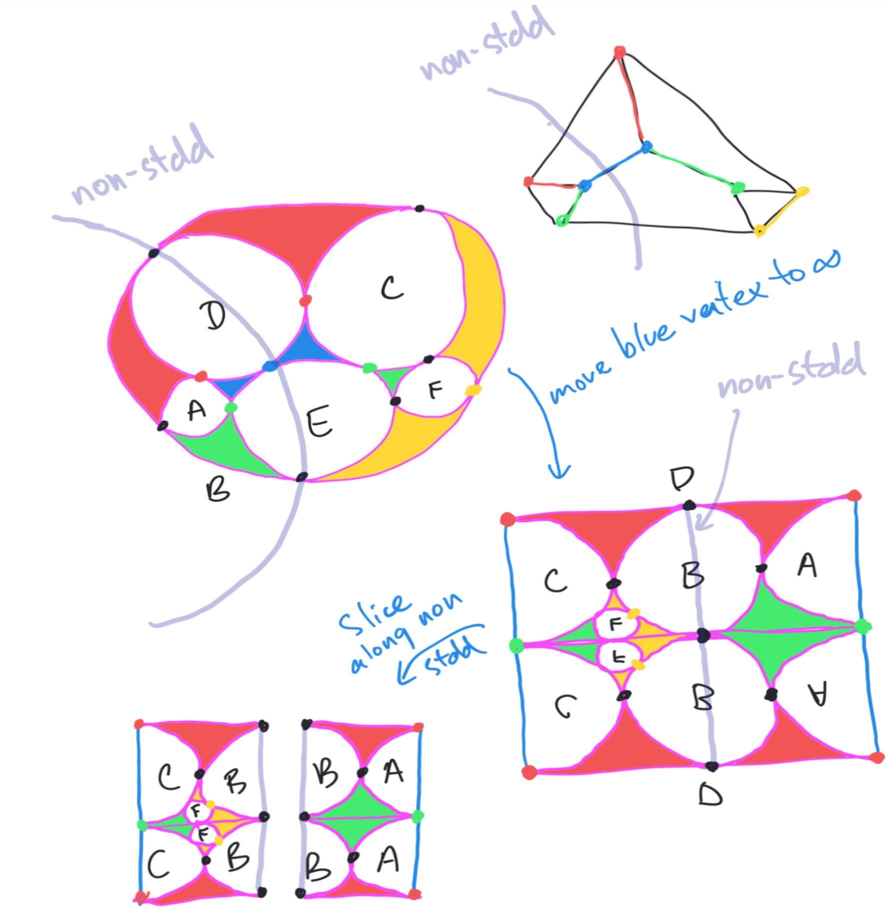

Figure 8.

Given the crushtacean, it is possible to go straight to the circle packing. By sending the blue vertex to infinity it forces the outer edges of the $P_+/-$ diagram to be blue. The non-standard 3-edge-cut results in two pieces. However, when gluing the shaded triangles to their corresponding colors, it connects the two together and thus only one component. The P_+ green triangle on the left glues to the P_+ green triangle on the right. Ultimately resulting in one piece and is not a valid BSD. Connecting this back to the crushtacean, this did not form a valid SD because there was more than one pair of associated vertices in different sides of the non-standard 3-edge-cut.

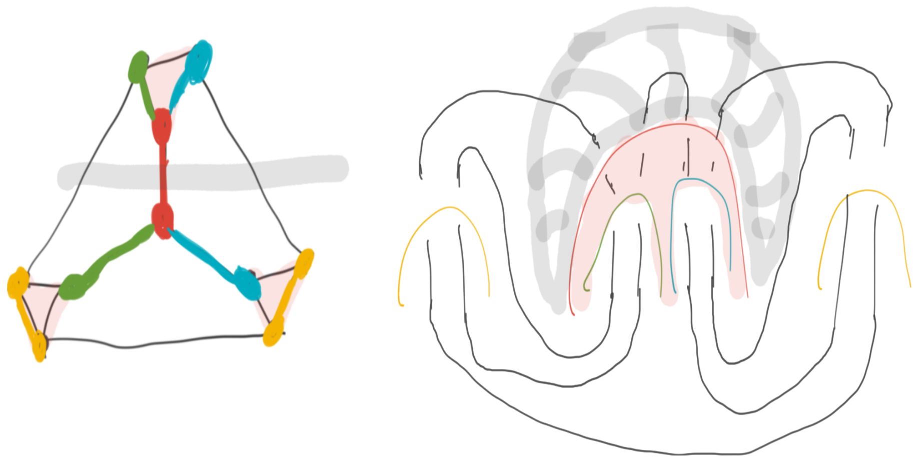

Figure 9.

Figure 9 offers a visual of slicing along a crushtacean with a non-standard 3-edge-cut and failing to form a valid BSD. In regards to the crushtacean, it is not a valid BSD because there is more than one pair of associated vertices in two separate components. In regards to the link diagram, it is not a valid BSD because it does not separate the link diagram into two components. The opening from the green and blue crossing disks prevent it from being a valid BSD because one could travel through the path that is left open with these two crossing circles.

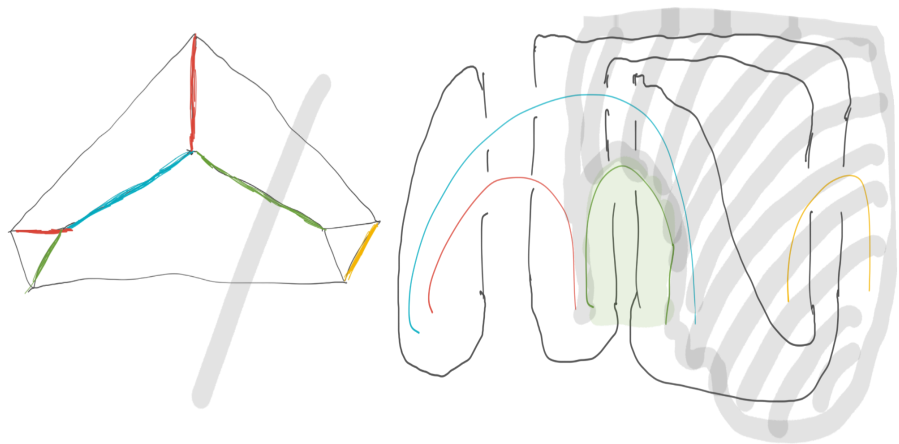

Figure 10.

Figure 10 offers a visual of slicing along a crushtacean with a non-standard 3-edge-cut and succeeding to form a valid BSD. In regards to the crushtacean, it is a valid BSD because there is exactly one pair of associated vertices in two separate components. In regards to the link diagram, it is a valid BSD because it does separate the link diagram into two components. The two components are the outside and inside of the non-standard 3ps.

4 Belted Sum Decompositions of Nested Links

Pairing Non-Standard 3PS with a Standard 3PS

4.1 Once-Painted 3-Edge-Cut

Theorem 4.1 A 3ps corresponding to a once-painted 3-cut pairs with a standard 3ps to form a valid BSD iff it is not interior to a tree.

Let G be a planar trivalent graph where one 3-edge-cut has exactly one painted edge. Then there are 3 ways for any graph to have a painted edge:

CASE 1: The painted edge is the entire tree.

CASE 2: It is a leaf of a larger tree.

CASE 3: It is interior to the tree, not a leaf.

We will look at each case one at a time, starting with case 1. Now consider the link that is associated to this graph.

The only possible pair of vertices that could be on two different components are the corresponding vertices on the painted edge. Thus, the only gap that could ever form must be made from the painted crossing disk. However, we can simply choose our standard 3ps to be the crossing disk associated with the painted edge. This will close the gap created from the non-standard 3ps and will create a valid belted sum decomposition.

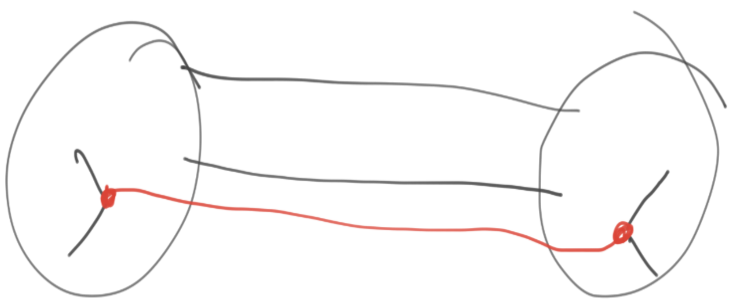

Figure 11. The painted edge is the entire tree.

Case 2 looks at if the painted edge is a leaf in a tree. Now consider the link that is associated to this graph. By definition, since the painted edge is a leaf in a tree then the painted edge is a 3ps with one longitudinal crossing disk puncture and two knot circle punctures. This means that there cannot be any other crossing disk nested within the painted edge. Thus it follows, the only possible pair of vertices that could be on two different components are the corresponding vertices on the painted edge. Thus, the only gap that could ever form must be made from the painted crossing disk. However, we can simply choose our standard 3ps to be the crossing disk associated with the painted edge. This will close the gap created from the non-standard 3ps and will create a valid belted sum decomposition.

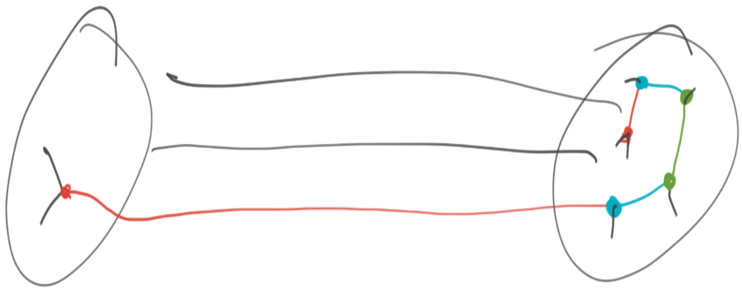

Figure 12. The painted edge is a leaf of a larger tree.

Now let us consider case 3. The painted edge is interior to the tree, not a leaf. Now consider the link that is associated to this graph. Since the painted edge is not a leaf on a tree, that means that it has at least one crossing circle puncture. This one crossing disk will have two vertices, one in each component. It is now known that a valid BSD must have one pair of associated vertices that are in different components. However, we know that the interior painted edge will give us one pair of associated vertices that are in two different components and we get at least one more pair of associated vertices in two different components from the one crossing circle punctures in the interior painted edge of the tree.

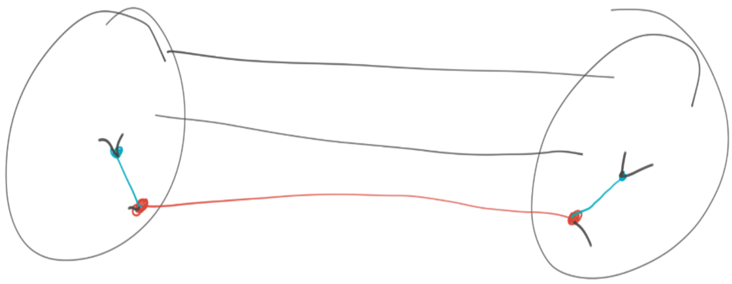

Figure 13. The painted edge is the edge of symmetry.

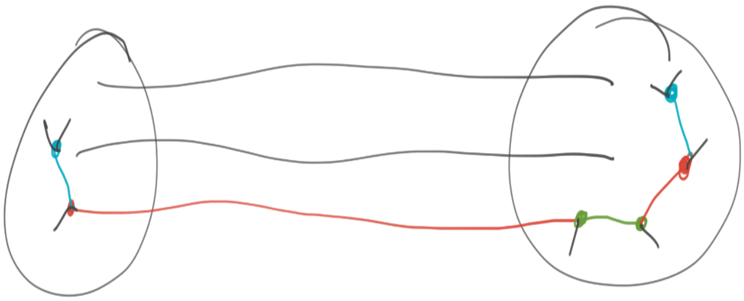

Figure 14. The painted edge is interior to the tree.

4.2 Twice-Painted 3-Edge-Cut

Theorem 4.2 A 3ps corresponding to a twice-painted 3-cut pairs with a standard 3ps to form a valid BSD iff the two painted edges are adjacent to each other.

Recall that there exist a 2-edge-cut when at least two edges in a 3-edge-cut share a vertex. This is bad because it allows for either a trivial cut to occur or it forces the link to be non-hyperbolic. Thus, we continue with this in mind and force all edges to have distinct vertices. Since we won't consider any cases where the edges in 3-edge-cut are connected, we only need to look at the cases when the edges in 3-edge-cut are not connected. We are then split into two scenarios. Either both painted edges in our twice-painted 3-edge-cut are the same color or they are different colors.

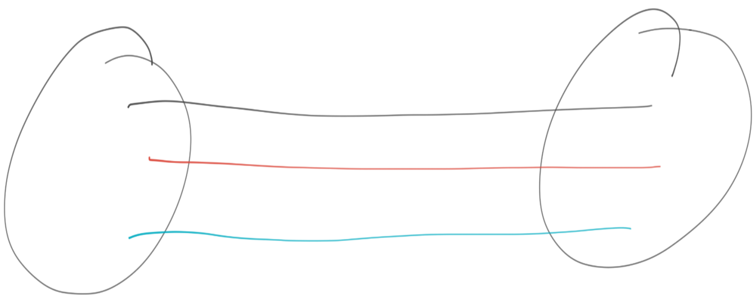

Figure 15. Twice-painted 3-edge-cut with two different colors.

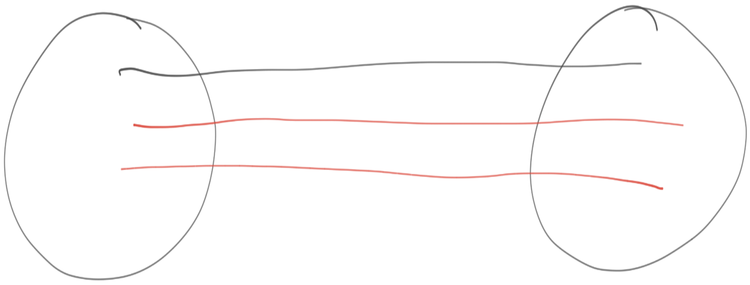

Figure 16. Twice-painted 3-edge-cut with same color.

Let us first tackle on the scenario when both painted edges are the same color. Since both painted edges are the same color, we know that color cannot be the edge of symmetry. Since we know the that cannot be the edge of symmetry, the edge of symmetry must be in one of the two components. WLOG, assume it is on the left component. Since the edge of symmetry is on the left side we have at least 2 vertices on the left. The only way we can add more vertices is by either having more leaves, we can add more vertices is by either having more leaves, interiors, and/or other trees in the form of crossing circles. However, both painted edges in our twice painted 3-edge-cut are taken. This means there cannot be a painted edge that travels from one component to the other. Thus, any other vertex on the left must also have its corresponding vertex on the left side as well. This means we have a minimal of 2 the left side as well. This means we have a minimal of 2 vertices and we add in vertices in sets of 2, thus the total number of vertices on the left component will always be even. However, this is a contradiction to **Lemma 3.2** and thus a valid BSD cannot form.

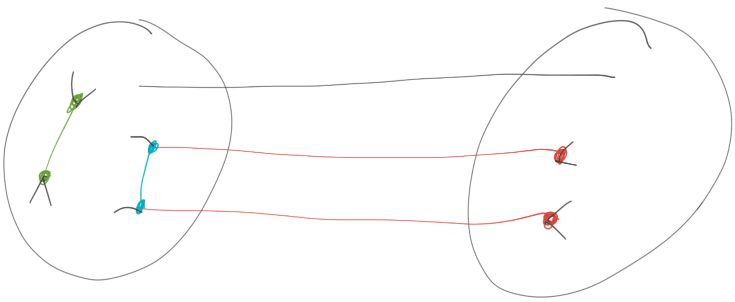

Figure 17. Both painted edges are the same color.

Now consider the scenario when the two painted edges in the 3-edge-cut are different colors. Recall that different colors corresponds to different crossing circles. If it is possible to form a VBSD, both crossing circles are in the same nesting which implies both edges are in the same tree. Since we know they are in the same tree there are now two possible sub-cases to look into. The first sub-case is when the vertices that corresponds to each of the painted edges in the 3-edge-cut are in the same component. The second sub-case is when they are in different components. However, the only way for this to happen is if each painted edge in our 3-edge-cut is in its own tree. However, this is impossible for the reasons we stated earlier. Thus, the only sub-case we shall look at is when the vertices that correspond to the painted edges in the 3-edge-cut are in the same component. There are now two things to consider with this sub-case, whether or not the painted edges in the 3-edge-cut are edges of one another.

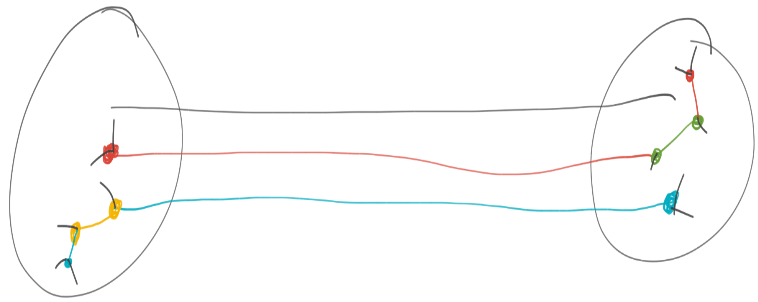

Figure 18. Painted edges are different colors and on separate trees

Assume the painted edges in the 3-edge-cut are not edges of one another. We already know that neither of the two can be the edge of symmetry so they are each have a choice of being interior or a leaf. Since neither of these two painted edges can be the edge a symmetry that means the edge of symmetry must be in one of the components. WLOG, assume it is on the right. We also know that one of these painted edges must be further away from the edge of symmetry in terms of number of edges. WLOG, assume the middle painted edge from the 3-edge-cut is further away from the edge of symmetry. Since the edge of symmetry is on the right component and the middle painted edge in our 3-edge-cut is further away from the edge of symmetry, we know that the top painted edge in the 3-edge-cut has at least one painted edge that is placed directly after it and before the painted- painted in the 3-edge-cut that is further away from the edge of symmetry. This fact forces it to be impossible to form a VBSD because we get at least two pairs of corresponding vertices that are in two different components. Once from the painted edge in the 3-edge-cut that is closer to the edge of symmetry and one from the painted edge that separates the two painted edges in the 3-edge-cut.

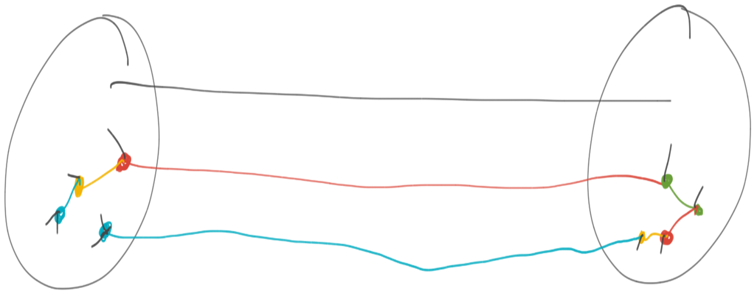

Figure 19. Different colors and on same tree but are not adjacent to each other.

Now assume the painted edges in the 3-edge-cut are edges of each other. This means the painted edge from the 3-edge-cut that is further away from the edge of symmetry has both of its vertices on the left component. The painted edge from the 3-edge-cut that is closer to the edge of symmetry will have one edge in each component. Now we can simply choose our standard 3ps to be the crossing disk that is associated with the painted edge in the 3-edge-cut that is closer to the edge of symmetry.

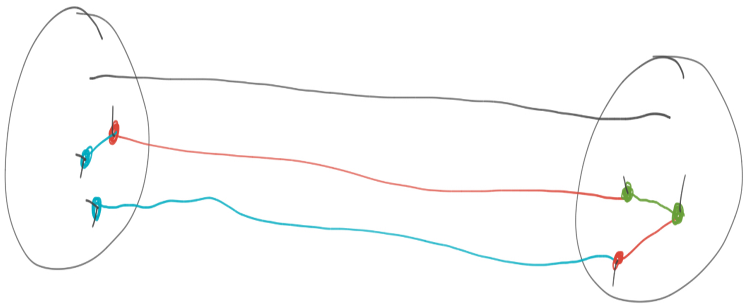

Figure 20. Different colors, on same tree, and are adjacent to each other.

4.3 Thrice-Painted 3-Edge-Cut

Theorem 4.3 A 3ps corresponding to a thrice-painted 3-cut pairs with a standard 3ps to form a valid BSD iff the 3-edge-cut has two different colors (2 same and 1 diff) and the single different colored edge is a leaf.

Thrice-painted 3-edge-cut has exactly two main cases that each have multiple sub-cases. The first main case is when two of the painted edges in the 3-edge-cut are the same color and the last painted edge in the 3-edge-cut is a different color. The second main case is when all three painted edges in the 3-edge-cut are different colors. Note that it is impossible to have all three painted edges in the 3-edge-cut be the same color because there are only two sides to each crossing disk, front or back.

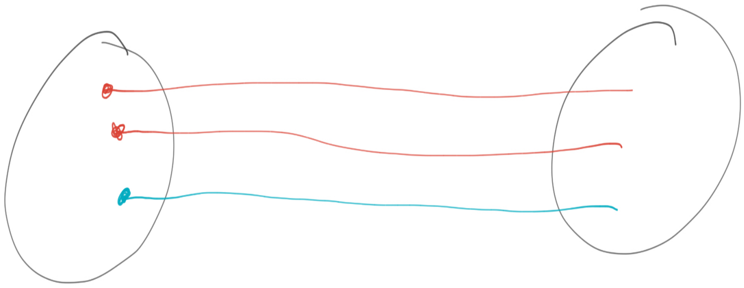

Figure 21. Thrice-painted 3-edge-cut with two same colors and one different color.

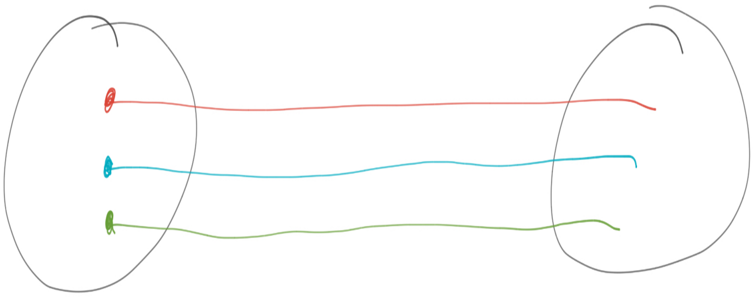

Figure 22. Thrice-painted 3-edge-cut with three different colors.

The first main case forces one condition. The edge of symmetry must be on either one of the components and cannot be the last painted edge in the 3-edge-cut. This now breaks down the first main case into two sub-cases. The first sub-case is when the last painted edge in the 3-edge-cut is not connected to the tree associated with the two same painted edges in the 3-edge-cut. The second sub-case is when it is connected.

Starting off with the first sub-case, it is known that the edge of symmetry for the tree associated with the two same colored painted edges in the 3-edge-cut is in one of the two components. WLOG, let the edge of symmetry be on the left component. Any vertices before the two same colored painted edges in the 3-edge-cut will have its mate also on the left component. Any vertices after the same colored painted edges in the 3-edge-cut will have its mate also on the right component. Now that the different colored painted edge in the 3-edge-cut has its own tree it is important to note that a valid BSD will only form if the different colored painted edge in the 3-edge-cut is a honest to goodness FAL crossing disk. When this occurs, we can simply choose the standard 3ps to be the crossing disk associated with the different colored painted edge in the 3-edge-cut. If the different colored painted edge in the 3-edge-cut is not an honest to goodness FAL crossing disk that means that it has at least extra one pair of painted edges on both sides that would total everything up at least 2 pairs of corresponding vertices that are in different components.

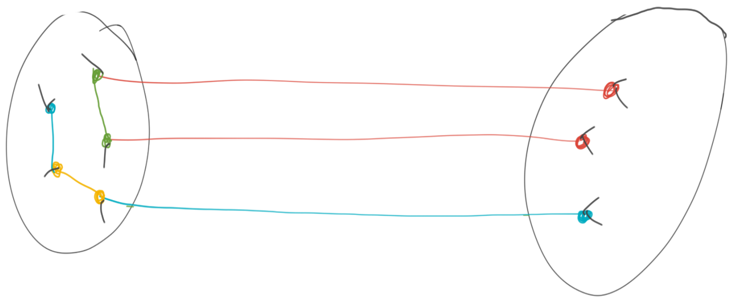

Figure 23. 2 same colored, 1 different colored, and separate trees.

Looking at the second sub-case the same condition of the edge of symmetry being in one of the components is also true. WLOG, assume the edge of symmetry is on the left component. A VBSD will only form if the different colored painted edge in the 3-edge-cut is an honest to goodness FAL crossing disk. This can be shown by the same reasoning from the previous sub-case.

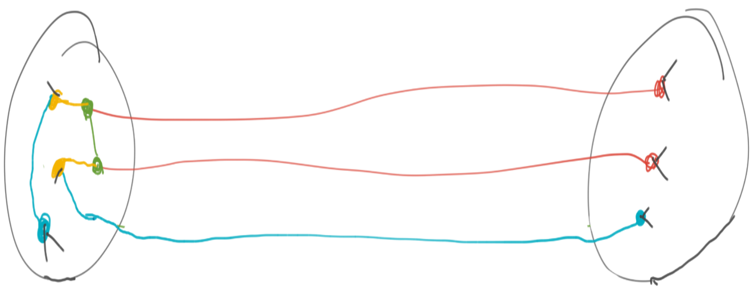

Figure 24. 2 same colored, 1 different colored, and same trees.

To start off on the case when all three painted edges in the 3-edge-cut are different color it is important to note that all painted edges must be on the same tree. It is also important to note that all the painted edges in the 3-edge-cut must have distinct vertices and cannot connect with each other. This forces the middle painted edge in the 3-edge-cut to be the edge of symmetry. However, it has been shown that there must be at least one more painted edge in between the middle painted edge in the 3-edge-cut and with each of the remaining painted edges in the 3-edge-cut. However, this forces at least two more pairs of corresponding vertices to be in two different components. The total number of corresponding vertices in different components is at least 3, one from the edge of symmetry and at least two from the painted edges that connect the middle painted edge in the 3-edge to the remained painted edges in the 30-edge-cut. This minimum of 3 clearly fails from the fact that there has to be exactly pair of corresponding vertices in two different components.

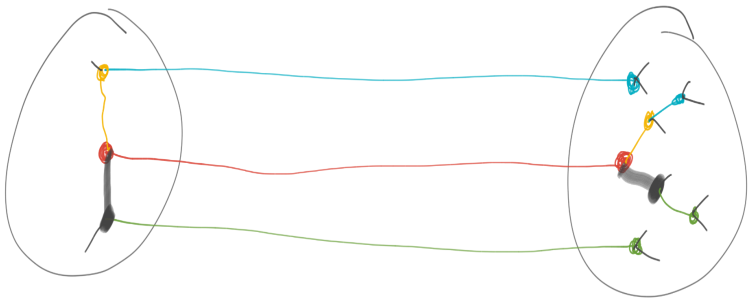

Figure 25. 3 different colored edges and same tree.

5 OPEN QUESTIONS

When will a once-painted non-standard 3-edge-cut pair with a non-standard 3ps to form a valid BSD? (Look at Madison Howard's paper)

When will a twice-painted non-standard 3-edge-cut pair with a non-standard 3ps to form a valid BSD? (Look at Madison Howard's paper)

When will a thrice-painted non-standard 3-edge-cut pair with a non-standard 3ps to form a valid BSD?

6 ACKNOWLEDGEMENTS

Youthful, amazing, zealous, meaningful, inspiring, and nice are just a few words that I could use to describe Dr. Rolland Trapp and Dr. Corey Dunn as. Thank you both for your constant support throughout the 2021 CSUSB Math Summer REU and beyond. The research was funded by NSF grant DMS-2050894 and by California State University, San Bernardino.

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Rolly sleeping next to me as I'm typing this paper. I bet he is dreaming of FALS and Nested Links!!! :)