On the Combinatorial Symmetries of Curvature Tensors and Kulkarni-Nomizu Tensors

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Abstract

The algebraic symmetries of the *Riemann curvature tensor* are well-known, and can be replicated on a finite-dimensional vector space as *algebraic* curvature tensors. In this paper we investigate the symmetries of the Riemann curvature tensor using methods from combinatorics and finite group theory. We motivate the use of *representation theory* to study and generalize these symmetries. In Section (3) we study the role of *idempotent* elements of the group algebra of a finite group, and how they provide elegant ways to project onto spaces of tensors having certain symmetries. We then shift attention to studying a certain generalization of curvature tensor symmetries to higher rank tensors called the *Kulkarni-Nomizu algebra,* which has the structure of a graded, commutative algebra. We generalize the construction of the Riemann curvature tensor from the metric, to provide a homogeneous map on the Kulkarni-Nomizu algebra which raises degree, and takes into account the interesting symmetries of these tensors. Finally, we examine the Kulkarni-Nomizu bundle on a smooth manifold, and the possibility of connecting these findings to a variant of de Rham cohomology on the Kulkarni-Nomizu bundle.

1 Geometric Background

One goal of this paper is to classify *algebraic curvature tensors.* These are the algebraic analogies to the *Riemann curvature tensor* on a Riemannian manifold. The source used for all of the geometric background provided here is Lee's [6]. The Riemann curvature tensor is constructed on a manifold (M, g) with smooth metric g, and is a $(0, 4)$-tensor on each tangent space. Its explicit form is

$$Rm(X, Y, Z, W) = g(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, W),$$

Where ∇ is an *affine connection* on (M, g).

Equivalently, we define a $(1, 3)$-tensor $\mathcal{R}$ by

$$\mathcal{R}(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

and then write

$$Rm(X, Y, Z, W) = g(\mathcal{R}(X, Y)Z, W).$$

Meanwhile, *algebraic curvature tensors* on a vector space V are tensors $R \in V^{\otimes 4}$ that satisfy all of the symmetries of the Riemann curvature tensor on a manifold.

The first symmetry of the tensorfield Rm is that

$$Rm(X, Y, Z, W) = -Rm(Y, X, Z, W).$$

This is simply because the term

$$\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

is *alternating* in X and Y, since the *Lie bracket,* $[X, Y] = -[Y, X]$ is also alternating in X, Y, and the terms involving ∇_X, ∇_Y acquire a negative sign when we reverse the order of these variables. So our *algebraic curvature tensor* R should also satisfy:

$$R(x, y, z, w) = -R(y, x, z, w).$$

This is the only symmetry of the tensor field Rm that doesn't exploit extra facts about the Levi-Civita connection. The properties of the Levi-Civita connection are what uniquely determine it with respect to the metric g on M, and give us two additional symmetries.

Definition 1.1. A connection ∇ on a manifold M is called **torsion-free** if for all vectorfields X, Y on M,

$$\nabla_X Y - \nabla_Y X = [X, Y].$$

Definition 1.2. A connection ∇ on a manifold M with metric g is called **g−compatible** if

$$\nabla g = 0$$

$$\iff \nabla_X g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z),$$

for all vector fields X, Y, Z.

Definition 1.3. With these definitions in mind, a **Levi-Civita connection** on (M, g) is a torsion-free, $g-$compatible connection ∇.

In fact, the *Fundamental Theorem of Pseudo-Riemannian Geometry* says that for any inner product g on M, there is a **unique** Levi-Civita connection with respect to g.

When ∇ is *g-compatible*, we get the following symmetry of the Riemann curvature tensor:

$$Rm(X, Y, Z, W) = -Rm(X, Y, W, Z),$$

so that Rm is *alternating* in Z and W.

When ∇ is *torsion-free*, we get the following symmetry:

$$Rm(X, Y, Z, W) + Rm(Y, Z, X, W) + Rm(Z, X, Y, W) = 0,$$

which we call the the *Bianchi Identity in X, Y, and Z.*

Note that both of these symmetries are actually *equivalent* to each of the respective conditions on ∇. The proof of the Bianchi identity is not too hard to show once you know the *Jacobi identity* for the Lie bracket:

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

The full symmetries of the Riemann curvature tensor are thus all the symmetries generated by

$Rm(X, Y, Z, W) = -Rm(Y, X, Z, W) = -Rm(X, Y, W, Z),$

$Rm(X, Y, Z, W) + Rm(Y, Z, X, W) + Rm(Z, X, Y, W) = 0.$

This includes a few other important symmetries:

Lemma 1.1. *The Riemann curvature tensor has **swap symmmetry**:*

$$Rm(X, Y, Z, W) = Rm(Z, W, X, Y)$$

This *swap symmetry* is actually a very useful symmetry of Rm and something we might want to salvage even in the absence of the Bianchi identity, but more on that later. See [6] for a proof of this fact.

Lemma 1.2. *The Riemann curvature tensor has **Bianchi identity** with respect to **any three variables**.*

This removes some of the arbitrariness of the Bianchi identity, namely the fact that it involves the first three variables.

An **algebraic curvature tensor** on a vector space V is then any $(0, 4)$-tensor with all three of the important symmetries of the Riemann curvature tensor:

$$R(x, y, z, w) = -R(y, x, z, w) = -R(x, y, w, z),$$

$$R(x, y, z, w) + R(y, z, x, w) + R(z, x, y, w) = 0.$$

Note that all the symmetries *generated* by these still apply. Namely, R has *swap symmetry* as well as Bianchi symmetry with respect to *any three variables.*

2 Representation Theory

It is useful to develop elegant language for talking about symmetries, especially symmetries of tensors and tensor fields. The language most mathematicians use to describe symmetry is group theory, and the language we use to describe symmetries of vector spaces is **representation theory**. A good reference for representation theory, from which much of this discussion is sourced, is Fulton and Harris' [3].

Definition 2.1 (Group Representation)**.** Let G be a group. A **representation of G** is a vector space V together with a homomorphism

$$\varrho : G \longrightarrow GL(V).$$

Equivalently, a representation of G is an action of G on a vector space by *linear automorphisms*, i.e. and action $G \curvearrowright V$, such that

$$g(v + w) = gv + gw,$$

$$g(cv) = cg(v), \text{forall} c \in \mathbb{F},$$

where $\mathbb{F}$ is the underlying field, which we will assume throughout is $\mathbb{R}$ or $\mathbb{C}$. We usually suppress the homomorphism ϱ and think of representations as actions. Even though a representation of G is really a pair (V, ϱ), we often simply say ϱ is the representation of G, and the underlying vector space V is sometimes called the **representation space** of ϱ, but is also called a representation of G itself. For abbreviation's sake, we sometimes refer to representations of G as "reps" of G or say that ϱ or V is a *"$G-$rep."*

Example 2.1 (Permutation Representations)**.** Let G be any finite group and X any finite set with a G-action, $G \curvearrowright X$. Let $V = \mathbb{F}^X$ be the vector space spanned freely by X, so that V has a basis of the form e_x for $x \in X$. Then V is called the **permutation representation over X**, and the $G-$rep structure is defined by:

$$ge_x = e_{gx}.$$

More generally, a **permutation representation** of G is a rep V such that G permutes a basis of V.

Permutation representations are important for constructing examples of representations of groups, but not every representation arises this way (The *alternating representation* of the symmetric group, which we will discuss later, is not of this form).

Example 2.2 (The Regular Representation)**.** Any finite group G naturally acts on a finite set, namely, $G \curvearrowright G$ by right multiplication. The permutation representation associated to this action is called the **regular representation** of G. Explicitly, this is $\mathbb{F}^G$ with the rep structure given by

$$ge_h = e_{g \cdot h}.$$

The regular representation, although it seems even more specialized permutation representation, is actually the **most** important representation of G in a few ways. Not only does it contain all the basic (irreducible) representations of G, but it also carries the structure of a **ring**, or really an **algebra** over $\mathbb{F}$. This algebra describes every representation of G through its universal property. We will present these facts in a moment.

Example 2.3 (Permuting Tensors)**.** Return to algebraic curvature tensors. These are tensors in $V^{\otimes 4}$ defined by a set of equations:

$$\begin{aligned} R(x, y, z, w) &= -R(y, x, z, w) = -R(x, y, w, z), \\ R(x, y, z, w) &+ R(y, z, x, w) + R(z, x, y, w) = 0. \end{aligned}$$

The space of these tensors is constructed out of a permutation representation as follows. Notice that each equation involves permuting the variables x, y, z, w in some way, then adding signs or summing over permutations. Permutations of these variables form a finite group, the *symmetric group on four letters* S_4. In general, the symmetric group on k letters S_k acts on $V^{\otimes k}$ in the same way: for $\sigma \in S_k$, define

$$(\sigma T)(v_1, v_2, ..., v_k) = T(v_{\sigma(1)}, v_{\sigma(2)}, ..., v_{\sigma(k)}).$$

Then $V^{\otimes k}$ forms a representation of the symmetric group S_k. Note that the representation space of this rep is not V, and S_k might not act on V itself, but it does act on $V^{\otimes k}$ in this manner. This is a **permutation rep** of S_k: If $e_1, ..., e_n \in V$ is a basis for V, then there is a basis for $V^{\otimes k}$ consisting of elements of the form

$$e_{i_1 \dots i_k} = e_{i_1} \otimes e_{i_2} \otimes ... \otimes e_{i_k},$$

where each $i_1, ... i_k$ ranges from 1 to n. It is easy to see that S_k permutes these basis elements:

$$\sigma e_{i_1 \dots i_k} = e_{i_{\sigma(1)} \dots i_{\sigma(k)}}.$$

Another way to express this as a permutation rep is to consider the set $X = [n]^{[k]}$ consisting of functions I from a set of k elements $[k]$ to a set of n elements $[n]$:

$$[n]^{[k]} = \{I : [1, 2, ..., k] \longrightarrow [1, 2, ..., n]\},$$

$$I = (i_1, i_2, ..., i_k) \in [n]^k.$$

Then S_k acts on $[n]^{[k]}$ by precomposition,

$$\sigma I = I \circ \sigma = (i_{\sigma(1)}, i_{\sigma(2)}, ..., i_{\sigma(k)}),$$

and if V has dimension n, then the basis for $V^{\otimes k}$ consisting of $e_{i_1 \dots i_k}$ is indexed over the set $[n]^{[k]}$, and the two representations $V^{\otimes k}$ and $\mathbb{F}^{([n]^{[k]})}$ agree.

Definition 2.2 (Subrepresentation)**.** Let V be a representation of a finite group G. A **subrepresentation of** V is a vector subspace $W \subseteq V$ which is carried into itself by the action of G. This means that for all $g \in G$,

$$w \in W \Longrightarrow gw \in W.$$

Notice that a subrep of V is actually a representation of G itself, with the $G-$rep structure being the one inherited from V. We consider the rep W to be a "smaller" representation hiding inside the rep V.

Lemma 2.1 (Complete Reducibility)**.** *Let V be a representation of G, and let W be any subrepresentation of V. Then there is a **unique** complementary subrepresentation W' of V such that $W \cap W' = \{0\}$ and $W + W' = V$. Equivalently, $V = W \oplus W'$ splits as a direct sum of $G-$reps.*

Proof. There are two standard proofs of this fact, both of which rely on the idea of $G-$*averaging*, and can be found in [3] but are paraphrased here to emphasize some of the ideas used. The first proof uses this to find a $G-$invariant inner product: this is an inner product $\langle \cdot, \cdot \rangle : V \times V \longrightarrow \mathbb{R}$ such that

$$\langle gv, gw \rangle = \langle v, w \rangle$$

For all $v, w \in V$. To do this, first take any inner product $\langle \cdot, \cdot \rangle$ on V. Now consider its $G-$average, defined by

$$\langle v, w \rangle^G = \frac{1}{|G|} \sum_{g \in G} \langle gv, gw \rangle.$$

After checking this is indeed a positive definite inner product (whenever the original inner product is positive definite), it is easy to show it is also $G-$invariant:

$$\langle hv, hw \rangle^G = \frac{1}{|G|} \sum_{g \in G} \langle ghv, ghw \rangle = \frac{1}{|G|} \sum_{g' \in G} \langle g'v, g'w \rangle = \langle v, w \rangle^G,$$

since multiplying by h simply permutes terms in the sum and the total is unaffected. Now with respect to any $G-$invariant inner product on V, and for W any subrep, take

$$W' = W^{\perp} = \{w' \in V | \langle w', w \rangle = 0 \forall w \in W\}$$

to be the orthogonal complement with respect to this inner product. Then $V = W \oplus W'$ as vector spaces, and W' is a subrep of V: for all $w' \in W'$ and $w \in W$,

$$\langle gw', w \rangle = \langle g^{-1}gw', g^{-1}w \rangle = \langle w', g^{-1}w \rangle = 0$$

because $g^{-1}w \in W$, and so $gw' \in W'$.

The other proof of this fact relies on a statement from ordinary linear algebra: If W is simply a subspace of a vector space V, then there is a (non-unique) function $\mathbb{P} : V \longrightarrow V$ called a *projection onto* W that satisfies any one of the following equivalent properties:

- The image of $\mathbb{P}$ is all of W and $\mathbb{P} \circ \mathbb{P} = \mathbb{P}$.
- $\mathbb{P} : V \longrightarrow W$ and if $i_W : W \hookrightarrow V$ is the inclusion of W into V, then $\mathbb{P} \circ i_W = \mathbb{1}_W : W \longrightarrow W$ is the identity function on W.
- The image of $\mathbb{P}$ is W and the restriction of $\mathbb{P}$ to W is $\mathbb{1}_W$.
- $\mathbb{P}(v) = v$ if and only if $v \in W$, and if W' is any complementary vector subspace, then $\mathbb{P}(W') = \{0\}$.

The important algebraic equation to notice here is that $\mathbb{P} \circ \mathbb{P} = \mathbb{P}$. When we consider $\mathbb{P} \in \mathrm{End}(V)$ as a matrix, we can rewrite this as $\mathbb{P}^2 = \mathbb{P}$. Elements of an algebra satisfying the equation $e^2 = e$ are called **idempotent** and play an important role in the theory.

Now with some $\mathbb{P}$ defined, consider $\mathbb{1} - \mathbb{P}$. This element is also idempotent, and is a projection onto some complementary subspace $W', V = W \oplus W'$. Averaging $\mathbb{1} - \mathbb{P}$ over G, we obtain a $G-$ **equivariant** projection onto some subspace W''. Now we can check that this new subspace is also subrep, and is complementary to W. $\square$

There are two important things to notice in this proof:

- Sums of the form
$$\frac{1}{|G|} \sum_{g \in G} gv, \text{ and}$$
- Idempotent (projection) operators.

In the next section, we elaborate on these averaging methods and their significance. First, a corollary.

Definition 2.3. An **irreducible** representation is one which has no proper nonzero subrepresentations.

Corollary 2.1.1. *Every representation of G splits as a direct sum of* ***irreducible*** *subrepresentations.*

Proof. Let V be a vector space of dimension n. If V contains a nonzero proper subrep W, let $W' \leq V$ be the complementary subrep. Then $V = W \oplus W'$. Now since both W, W' are nonzero and $\dim(V) = \dim(W) + \dim(W')$, we can apply induction on the dimension of the representation. The trivial one-dimensional representation is already irreducible. $\square$

3 The Group Algebra

We have seen that "averaging" over elements of G in some representation can prove useful. These averages are always well-defined in characteristic zero: if V is any representation of any finite group G, and $v \in V$ any vector, the quantity

$$\frac{1}{|G|} \sum_{g \in G} gv$$

always makes sense. In fact, expressions that look like

$$\sum_{g \in G} a_g g,$$

for any coefficients $a_g \in \mathbb{F}$, always make sense, without even appealing to elements of V, since these are just linear combinations of endomorphisms of V, and have a well-defined action as an endomorphism of V. Now return to

$$\mathbb{F}^G = \{a : G \longrightarrow \mathbb{F}\}.$$

Adopting the basis $\{\delta_g \mid g \in G\}$ defined by

$$\delta_g(h) = 0, h \neq g,$$
$$\delta_g(g) = 1,$$

we can say that any expression of the form

$$\sum_{g\in G} a_g g$$

is really a vector in the regular representation:

$$\sum_{g\in G} a_g \delta_g.$$

The multiplicative structure on $\mathbb{F}^G$ is defined to agree with the multiplicative structure on G:

$$\delta_g \delta_h(v) = \delta_{gh}(v)$$

And it should extend by linearity to any elements of the regular representation:

$$\left(\sum_{g\in G} a_g \delta_g\right)\left(\sum_{h\in G} b_h \delta_h\right) = \sum_{k\in G}\left(\sum_{gh=k} a_g b_h\right)\delta_k.$$

This agrees with the action of the regular representation on $G-$reps: For any $a, b \in \mathbb{F}^G$,

$$a(b(v)) = ab(v).$$

Definition 3.1. The regular representation, together with the multiplicative structure

$$\left(\sum_{g\in G} a_g \delta_g\right)\left(\sum_{h\in G} b_h \delta_h\right) = \sum_{k\in G}\left(\sum_{gh=k} a_g b_h\right)\delta_k,$$

is called the **group algebra** of G, and is denoted $\mathbb{F}[G]$

Lemma 3.1 (Universal Property)**.** *There is a natural correspondence between **group homomorphisms** from G to the group of invertible linear transformations $GL(V)$ and **algebra homomorphisms** from the group algebra $\mathbb{F}[G]$ to the algebra of linear transformations $\mathrm{End}(V)$.*

Proof. Any vector space V with the structure of a $G-$rep can be extended linearly to a $\mathbb{F}[G]-$module as described:

$$\left(\sum_{g\in G} a_g g\right) v = \sum_{g\in G} a_g g(v),$$

and conversely any $\mathbb{F}[G]-$module V becomes a $G-$rep by restricting to the multiplicative subgroup $\{\delta_g\}$: define $g(v) = \delta_g(v)$. One can check that these operations are mutually inverse.

□

In fact, everything we can say about $G-$reps translates into a statement about $\mathbb{F}[G]-$modules. For example, subrepresentations are simply $\mathbb{F}[G]-$submodules.

This statement is actually true replacing $\mathrm{End}(V)$ with any $\mathbb{F}$-algebra A, and $\mathrm{GL}(V)$ with its group of units $A^\times$. This more general result is the universal property of **group rings**. Thus there is a natural bijection (or more fancily, *equivalence of categories*):

$$G-reps \longleftrightarrow \mathbb{F}[G]-modules.$$

We know that any G-rep decomposes as a direct sum of irreps. Thus if V is finite dimensional, there are irreps $V_1, ..., V_r$ and multiplicities $d_1, ..., d_r$ such that

$$V \cong \bigoplus_{i=1}^{r} V_i^{\oplus d_i}.$$

Here is how the group algebra decomposes.

Theorem 3.2. *Every irreducible representation of G is a subrepresentation of the group algebra $\mathbb{F}[G]$. If V is an irreducible representation, then V appears in $\mathbb{F}[G]$ with multiplicity $dim(V)$. This means that there are only finitely many finite-dimensional irreps of G, and if $V_1, ..., V_r$ is a complete list of irreducible representations of G, then*

$$\mathbb{F}[G] \cong \bigoplus_{i=1}^{r} V_i^{\oplus dim(V_i)}$$

as G-reps.

This theorem says that any **irrep** of G is actually a **submodule** of $\mathbb{F}[G]$. But the submodules of a ring (considered as a module over itself) are precisely the **ideals** of that ring (where we must specify left or right submodules to arrive at left or right ideals, respectively). Thus we arrive at the statement that *any irrep of G is a* ***minimal ideal*** *of* $\mathbb{F}[G]$. We must specify *minimal* ideals because the sum of two distinct irreps of G is also an ideal in $\mathbb{F}[G]$, for example.

We can actually be more specific about the decomposition above. More than just finding how $\mathbb{F}[G]$ decomposes as a G-rep, we can describe its *algebra structure* in terms of the irreps of G.

Theorem 3.3. *If $V_1, ..., V_r$ is a complete list of irreducible representations of G, then there is an isomorphism of algebras*

$$\mathbb{F}[G] \cong \bigoplus_{i=1}^{r} \mathrm{End}(V_i).$$

Proof. See [3]. □

This tells us about the ring structure of $\mathbb{F}[G]$.

Corollary 3.3.1. $\mathbb{F}[G]$ *is a* ***semisimple algebra****.*

This is a somewhat technical condition (either being a direct sum of simple algebras, or having a Jacobson radical equal to zero), but it is important because it highlights the role of **idempotents**. The following result describes this, and for now, it can function as a definition of a semisimple algebra.

Lemma 3.4. *In any finite-dimensional semisimple algebra A, every left or right ideal is generated by an* ***idempotent*** *element $e \in A$:*

$$\mathfrak{l} = Ae, \ \ \textit{or } \mathfrak{r} = eA, \ \ \textit{for right ideals.}$$

This lemma is very important to our work. These idempotent generators of ideals are also idempotent endomorphisms of any representation of the group. Thus they are projection operators, and since they correspond to ideals, the spaces they project onto tell us interesting things about the representation. We should remark now that $\mathrm{ACT}(V)$ *is one of these spaces.*

4 Algebraic Curvature Tensors

We should now apply some of the language of representation theory to the study of algebraic curvature tensors. More generally, we will explore how the symmetries of tensors can be described through idempotents in the group algebra.

Consider the equation

$$R(x, y, z, w) = -R(y, x, z, w).$$

The second term in this equation is in the *orbit* of the first term under the action of the symmetric group. The permutation that takes one to the other is $\sigma = (1, 2)$ in cycle notation. Thus we can rewrite this equation as

$$\sigma R = -R.$$

The second important symmetry of algebraic curvature tensors is *swap symmetry*:

$$R(x, y, z, w) = R(z, w, x, y).$$

The permutation here is given by $\tau = (1, 3)(2, 4)$ because we are sending $x \mapsto z \mapsto x$ and $y \mapsto w \mapsto y$. We can rewrite this equation as

$$\tau R = R.$$

To unify these somewhat different-looking symmetries, consider the *sign* of each permutation: σ is a transposition ($2-$cycle) and thus has sign -1. Meanwhile τ is the product of two transpositions so it has sign 1. We can thus rewrite these equations as:

$$\sigma R = -R = \mathrm{sign}(\sigma)R,$$

$$\tau R = R = \mathrm{sign}(\tau)R.$$

The trick now is to consider this as a *different* rep of S_4 derived in a particular manner from $V^{\otimes 4}$. This is a general construction that works with any representation of any symmetric group:

Definition 4.1 (Conjugate Representation)**.** Let (W, ϱ) be a representation of the symmetric group S_k. Then the **conjugate** of (W, ϱ) is another representation $(W, \bar{\varrho})$ with the same rep space W but with rep structure given by

$$\sigma \bar{\cdot} w = \mathrm{sign}(\sigma)\sigma w,$$

or equivalently,

$$\bar{\varrho}(\sigma) = \mathrm{sign}(\sigma)\varrho(\sigma).$$

The conjugate representation "twists" the action of S_k by the sign homomorphism.

Returning to the equations defining the first two algebraic curvature tensor symmetries, we now write

$$\sigma \bar{\cdot} R = \tau \bar{\cdot} R = R,$$

That is, these permutations *preserve* the tensor R, but in the *conjugate* representation. We can show that if elements g, h preserve some vector v in a representation, then so does any expression involving g and h, and so do the inverses of g and h. In particular, since all these expressions fix v, the entire *subgroup generated by* these elements must also fix v.

Definition 4.2 ($G-$fixed subrep)**.** Let V be any representation of any group G. The subspace consisting of vectors fixed by every element of G is called the $G-$*fixed subspace* and is a subrep of V. We use the notation

$$V^G = \{v \in V \mid gv = v \ \forall g \in G\}.$$

The elements σ and τ defined above generate a subgroup D of S_4. The statement that σ, τ fix R (in the conjugate representation) is equivalent to R being D-fixed.

Remark 4.1. *The subgroup $D \leq S_4$ is isomorphic to the dihedral group D_4. One can check that $\sigma^2 = \tau^2 = (\sigma\tau)^4 = 1$, so at the very least D is a factor of D_4, and it is not hard to find more than four elements, showing that it is the whole dihedral group. In fact, this is the largest subgroup of S_4 which acts on* $\mathrm{ACT}(V)$ *in the permutation representation. This result, and its generalization, will be shown in the section on the Kulkarni-Nomizu algebra.*

We describe the projection onto the space of vectors fixed by a group.

Lemma 4.1. *Let V be any rep of any finite group G. Then*

$$\mathrm{Av}_G = \frac{1}{|G|}\sum_{g \in G} g$$

is projection $V \longrightarrow V^G$.

Proof. This fact is straightforward once we consider that:

- If $v \in V^G$ then $\mathrm{Av}_G(v) = v$, because each term in the above sum is equal to v, and
- For any $v \in V$, $\mathrm{Av}_G(v)$ is in V^G. In particular,Av_G is **idempotent**:

$$\mathrm{Av}_G(\mathrm{Av}_G(v)) = \mathrm{Av}_G(v),$$

$$\mathrm{Av}_G^2 = \mathrm{Av}_G.$$

□

So the two symmetries of R are equivalent to the statement that R is in the image of Av_D, where D is the subgroup generated by σ, τ as above. In fact, Av_D is the projection of $V^{\otimes 4}$ onto the space of tensors with alternating and swap symmetries. We can now rewrite both of these symmetries simultaneously as

$$\mathrm{Av}_D(R) = R.$$

The Bianchi identity is different from this kind of symmetry. Instead of the tensor R being *fixed* by some subgroup (which we've shown is equivalent to it being fixed by *averaging* over this subgroup), R is instead *annihiliated* by averaging over some subgroup. The permutation involved in the Bianchi identity cyclically permutes the first three variables, so it is given by $h = (1, 2, 3)$. Let

$$H = \langle h \rangle = \{\mathbb{1},\ (1, 2, 3),\ (1, 3, 2)\}$$

By the theory we've developed, Av_H is idempotent. But if R is an algebraic curvature tensor, then instead of being fixed by Av_H, we have

$$\mathrm{Av}_H(R) = 0,$$

so R is in the kernel of Av_H. However, we can still rewrite this so that R is fixed by an idempotent.

Lemma 4.2. *If $\mathbb{P}$ is an idempotent, so is $\mathbb{1} - \mathbb{P}$, where $\mathbb{1}$ denotes the identity. If $\mathbb{P}$ is projection of V onto W, then $\mathbb{1} - \mathbb{P}$ is projection of V onto some complementary subspace W'.*

Proof. Let

$$W' = \mathrm{Ker}\ \mathbb{P}.$$

Then $\mathbb{1} - \mathbb{P}$ restricted to W' is just $\mathbb{1}$ since $\mathbb{P}$ vanishes. Also, $\mathbb{1} - \mathbb{P}$ restricted to W is zero since $\mathbb{P} = \mathbb{1}$ on W. The proof that $\mathbb{1} - \mathbb{P}$ is idempotent is a simple calculation:

$$(\mathbb{1} - \mathbb{P})^2 = \mathbb{1}^2 - 2\mathbb{P} + \mathbb{P}^2 = \mathbb{1} - 2\mathbb{P} + \mathbb{P} = \mathbb{1} - \mathbb{P}.$$

□

Now the Bianchi identity is equivalent to R being fixed by the idempotent

$$\mathbb{1} - \mathrm{Av}_H.$$

In order to *combine this* with the other two symmetries of R, we could naively say that R is fixed by the *product* of these two idempotents Av_D and $\mathbb{1} - \mathrm{Av}_H$. This would mean that $\mathrm{Av}_D \circ (\mathbb{1} - \mathrm{Av}_H)$ is the projection of $V^{\otimes 4}$ onto the space of algebraic curvature tensors. In order to verify that this works, we prove a lemma regarding averaging operators over subgroups of a finite group G.

Lemma 4.3. *Let G be a finite group and A be the group algebra. Let H, K be subgroups of G and $\mathrm{Av}_H, \mathrm{Av}_K \in A$ be the averages over H, K. Then Av_H commutes with Av_K if and only if $HK = KH$ are commuting subgroups. This in turn is true if and only if $HK = L$ is also a subgroup of G, and we have*

$$\mathrm{Av}_H \circ \mathrm{Av}_K = \mathrm{Av}_K \circ \mathrm{Av}_H = \mathrm{Av}_L.$$

Proof. Write these averaging operators out explicitly and then compose them using linearity:

$$\mathrm{Av}_H \circ \mathrm{Av}_K = (\frac{1}{|H|}\sum_{h\in H} h) \circ (\frac{1}{|K|}\sum_{k\in K} k)$$

$$= \frac{1}{|H||K|}\sum_{k\in K}(\sum_{h\in H} h\circ k),$$

$$\mathrm{Av}_K \circ \mathrm{Av}_H = \frac{1}{|H||K|}\sum_{h\in H}(\sum_{k\in K} kh).$$

These two operators are equal if and only if each of the elements in each sum appears with the same multiplicity in the other sum. But this would imply that

$$HK = KH$$

as sets, since each element of the form hk for $h \in H, k \in K$ also appears in the form kh. Thus if the two sums commute, then H, K are commuting subgroups, and so the product $HK = L$ is also a subgroup, equivalently. The converse is true if and only if each element hk appears with the same *multiplicity* in either sum. However, from the general theory of finite groups, if H, K are commuting subgroups, then there are precisely $|H \cap K|$ ways to write any element $g \in HK$ as a product $g = hk$ with $h \in H$, $k \in K$. Thus

$$\frac{1}{|H||K|}\sum_{h\in H}(\sum_{k\in K} hk) = \frac{1}{|H||K|}\sum_{g\in HK}|H\cap K|g = \frac{|H\cap K|}{|H||K|}\sum_{g\in HK} g = \frac{1}{|L|}\sum_{g\in L} g = \mathrm{Av}_L,$$

Where the last equality is true because

$$|HK| = |H||K|/|H \cap K|$$

For commuting subgroups H, K.

□

Corollary 4.3.1. *The above lemma is true for signed averages as well:*

$$\bar{\mathrm{Av}}_H \circ \bar{\mathrm{Av}}_K = \bar{\mathrm{Av}}_K \circ \bar{\mathrm{Av}}_H \iff HK = KH.$$

Proof. This is true because sign is a homomorphism and it distributes over products in the group algebra. It can also be derived by considering these averages as taking place over the conjugate representation of the regular representation, and then using the lemma. □

Now returning to the space of algebraic curvature tensors, we can apply the preceding lemma and deduce that $\bar{\mathrm{Av}}_D$ commutes with $\bar{\mathrm{Av}}_H$, where $D \leq S_4$ is generated by antisymmetry $(1,2)$ and swap symmetry $(1,3)(2,4)$, and H is generated by the $3-$cycle $(1,2,3)$, because these two groups commute; in fact:

$$DH = S_4 = HD,$$

because D has order 8, H has order 3, and $D \cap H$ is trivial, so their product DH is a subset with 24 elements, which is the order of the symmetric group S_4, and thus must be the whole group S_4. So the composition

$$\bar{\mathrm{Av}}_D \circ (\mathbb{1} - \bar{\mathrm{Av}}_H) = \bar{\mathrm{Av}}_D - \bar{\mathrm{Av}}_{S_4}$$

is idempotent, and is the projection of $V^{\otimes 4}$ onto the space $\mathrm{ACT}(V)$ of algebraic curvature tensors. This comes from the following lemma:

Lemma 4.4. *Suppose $\mathbb{P}_1$ and $\mathbb{P}_2$ are idempotent projection operators $V \longrightarrow V$ which project onto subspaces W_1 and W_2 respectively. Suppose also that $\mathbb{P}_1$ and $\mathbb{P}_2$ commute. Then $\mathbb{P}_1 \circ \mathbb{P}_2$ is also idempotent, and is the projection onto $W_1 \cap W_2$.*

Proof. This composition is idempotent when these commute:

$$(\mathbb{P}_1 \circ \mathbb{P}_2) \circ (\mathbb{P}_1 \circ \mathbb{P}_2) = \mathbb{P}_1 \circ (\mathbb{P}_2 \circ \mathbb{P}_1) \circ \mathbb{P}_2 = \mathbb{P}_1 \circ (\mathbb{P}_1 \circ \mathbb{P}_2) \circ \mathbb{P}_2 = \mathbb{P}_1^2 \circ \mathbb{P}_2^2 = \mathbb{P}_1 \circ \mathbb{P}_2.$$

Thus this composition defines some projection operator. Since the image of $\mathbb{P}_1$ is W_1 and the image of $\mathbb{P}_2$ is W_2, the image of $\mathbb{P}_1 \circ \mathbb{P}_2$ must be contained in $W_1 \cap W_2$. Since $\mathbb{P}_1$ fixes pointwise every element of W_1, and $\mathbb{P}_2$ fixes pointwise every element of W_2, their composition is the identity on $W_1 \cap W_2$. By the characterization of projection operators from earlier, this shows that $\mathbb{P}_1 \circ \mathbb{P}_2$ is the projection onto $W_1 \cap W_2$.

$\square$

Corollary 4.4.1. *The element*

$$\bar{\mathrm{Av}}_D \circ (\mathbb{1} - \bar{\mathrm{Av}}_H) = \bar{\mathrm{Av}}_D - \bar{\mathrm{Av}}_{S_4}$$

Is indeed projection onto $\mathrm{ACT}(V)$.

5 Forcing Symmetries of Tensors

We have seen how the space $\mathrm{ACT}(V)$ is constructed from the two idempotent symmetry operators

$$\bar{\mathrm{Av}}_D,\ \mathbb{1} - \mathrm{Av}_H$$

by taking their *product,*

$$\bar{\mathrm{Av}}_D \circ (\mathbb{1} - \mathrm{Av}_H) = (\mathbb{1} - \mathrm{Av}_H) \circ \bar{\mathrm{Av}}_D = \bar{\mathrm{Av}}_D - \bar{\mathrm{Av}}_{S_4}.$$

This is done because we want to consider the subspace of $V^{\otimes 4}$ which is *fixed* by both of these idempotent operators. Since these two operators happen to *commute*, we can say that the projection onto this space is simply the product (in the group algebra) or composition (as endomorphisms of $V^{\otimes 4}$) of the projection onto either space. The following subsection describes methods to find the space fixed by two or more idempotents that may or may not commute.

Consider the group algebra $\mathbb{F}[S_k]$. Recall that this algebra is *semisimple*; this implies that every ideal is generated by an *idempotent*, and that result is true for left or right ideals: any right ideal $\mathfrak{r} \subseteq \mathbb{F}[S_k]$ is one of the form $\mathfrak{r} = e\mathbb{F}[S_k]$ for some idempotent $e \in \mathbb{F}[S_k]$, and any left ideal $\mathfrak{l} \subseteq \mathbb{F}[S_k]$ is one of the form $\mathfrak{l} = \mathbb{F}[S_k]e$. The converse is also true: any idempotent element e generates a right ideal $e\mathbb{F}[S_k]$ and a left ideal $\mathbb{F}[S_k]e$ (really, any element generates such an ideal, but the important thing is that every ideal is *principal,* or generated on the left or right by a *single* element, and we can choose that element to be *idempotent*). In the language of *projection operators,* this says that given any semisimple algebra A, and any ideal $I \subseteq A$, the projection $\mathbb{P} : A \longrightarrow I$ can be realized as an element of the algebra A, so that $\mathbb{P}(a) = ea$ or $\mathbb{P}(a) = ae$ (depending on whether the ideal is left or right) for some element of the algebra $e \in A$. The fact that e is idempotent corresponds to the fact that $\mathbb{P}$ is a projection operator (which are really just idempotents in an endomorphism ring).

However, these idempotent generators are *not unique!* As we will see in a moment, there are certain finite subsets of idempotents which can describe all ideals by taking multiplicity-free linear combinations of these idempotents, so that any ideal is generated by a sum of some collection of these; but not every idempotent can be described in this form.

What is the purpose of all this? Consider again the space of **algebraic curvature tensors** as a subspace $\mathrm{ACT}(V) \subset V^{\otimes 4}$. Recall that this subspace has the projection operator

$$e := \bar{\mathrm{Av}}_D \circ (\mathbb{1} - \bar{\mathrm{Av}}_H) = \bar{\mathrm{Av}}_D - \bar{\mathrm{Av}}_{S_4} \in \mathbb{F}[S_4].$$

As a linear map $V^{\otimes 4} \longrightarrow V^{\otimes 4}$, e is a projection operator; as an element of the group algebra, e is idempotent. This means it corresponds to some right ideal

$$I = e\mathbb{F}[S_4] = \{ea \mid a \in \mathbb{F}[S_4]\}.$$

We constructed $ACT(V)$ earlier as the space of elements *fixed* by this idempotent, because those are precisely the elements invariant under the symmetries we described. However, because this is an *idempotent,* this is equivalent to the space of elements in the *image* of this idempotent:

$$eV^{\otimes 4} = \{ev \mid \ \in V^{\otimes 4}\}.$$

But since $\mathbb{F}[S_4]$ is a *unital algebra,* meaning it contains an identity element $\mathbb{1}$, this is equivalent in turn to the image of $V^{\otimes 4}$ under the *whole ideal* I,

$$IV^{\otimes 4} = \{i(v) \mid i \in I,\ v \in V^{\otimes 4}\}$$

$$= e\mathbb{F}[S_4]V^{\otimes 4} = \{ea(v) \mid a \in \mathbb{F}[S_4],\ v \in V^{\otimes 4}\},$$

because the image of all of $V^{\otimes 4}$ under all of $\mathbb{F}[S_4]$ is still all of $V^{\otimes 4}$, since the element $\mathbb{1}(v) = v$ is in this subspace for every $v \in V^{\otimes 4}$.

We have constructed the idempotent e as the *product* of two commuting idempotents. The image of this product is the intersection of the images of these two idempotents; that is, the image of e is precisely all the tensors fixed both by $\bar{\mathrm{A}}\mathrm{v}_D$ as well as $\mathbb{1} - \mathrm{Av}_H$. In order to find the intersection of the fixed sets of noncommuting idempotents, we must develop some more theory.

Let $A = \mathbb{F}[S_k]$. Really, any finite group works in place of S_k, and for most purposes, any finite-dimensional semisimple algebra works in place of A, but we will only prove the results when A is the group algebra of a finite group. For now, we will work only with left ideals, including possibly two-sided ideals (which are both left and right ideals), and call them "ideals" with the understanding that they are left ideals. Everything that we might say about left ideals can also be paraphrased about right ideals. This is because of the anti-involutive map x^* which extends linearly from the map

$$\delta_g \mapsto \delta_{g^{-1}}.$$

This turns $\mathbb{F}[G]$ into a $*$-*algebra.* This means that the operation x^* satisfies

$$(x^*)^* = x,$$

$$(xy)^* = y^* x^*,$$

as well as being linear. Thus $*$ interchanges left and right ideals bijectively,

$$(Ae)^* = e^* A$$

and is self-inverse. The ambiguity involved in exclusively discussing left ideals is resolved by this operator.

Now we develop some terminology and results about idempotents.

Two idempotents e_1, e_2 are **commuting** if $e_1 e_2 = e_2 e_1$. It is easy to see that if this is the case, then their **product** $e_1 e_2$ is also idempotent. In fact, if the commuting idempotents e_1, e_2 correspond to ideals I_1, I_2 respectively, then the idempotent $e_1 e_2$ corresponds to the ideal $I_1 \cap I_2$.

As a special case, two idempotents are **orthogonal** if $e_1 e_2 = e_2 e_1 = 0$, so that they are *commuting* and have product equal to zero. If this is the case, then their **sum** $e_1 + e_2$ is also idempotent, and corresponds to the *sum of ideals*

$$I_1 + I_2 = \{i_1 + i_2 \mid i_1 \in I_1,\ i_2 \in I_2\},$$

And in fact, this sum is *direct*:

$$I_1 \cap I_2 = \{0\}$$

Because $e_1 e_2 = 0$ is the generator of this ideal.

An idempotent e is **primitive** if it cannot be written as

$$e = e_1 + e_2,$$

for two *nonzero, orthogonal* idempotents e_1, e_2. Let $I \subseteq A$ be the ideal generated by e. Because the sum of orthogonal idempotents gives the *direct sum* of the ideals they generate, this means that the ideal I cannot be written

$$I = I_1 \oplus I_2,$$

for two nonzero ideals $I_1, I_2 \subset I$. Suppose there were a nonzero proper ideal $I' \subseteq I$. Because I is a *representation* of the group algebra, we can apply complete reducibility and say that there is a complementary subrepresentation $I'' \subseteq I$, such that $I' \oplus I'' = I$. But this would imply that e is not primitive, which is a contradiction. Thus the ideal I contains no proper nonzero ideals, and is called a *minimal ideal.* So *primitive idempotents* correspond to **minimal ideals**.

Finally, a set $\{e_1, ..., e_m\}$ of idempotents is called a **complete** if each idempotent is primitive, every pair of idempotents is orthogonal, and every ideal is generated by a linear combination of its elements, where all the coefficients are 1 or 0.

Proposition 5.1. *Let $\{e_1, ..., e_m\}$ be a set of primite, pairwise orthogonal idempotents. Then it is complete if, and only if,*

$$e_1 + e_2 + ... + e_m = \mathbb{1},$$

where $\mathbb{1} \in \mathbb{F}[S_k]$ is the identity.

Proof. The algebra $A = \mathbb{F}[S_k]$ is an ideal of itself, and is generated by the idempotent $\mathbb{1}$. But as a representation, it splits into a direct sum of irreducible subrepresentations,

$$A = \bigoplus I_i,$$

Where the I_i are all irreducible subrepresentations. Subrepresentations of the group algebra are equivalently submodules, which are equivalently ideals of the group algebra. Thus the group algebra A is the direct sum of its *minimal ideals.* These are generated by primitive idempotents $e_i, I_i = Ae_i$. These primitive idempotents are pairwise orthogonal, $e_i e_j = e_j e_i = 0$, because the ideals satisfy $I_i \cap I_j = 0$. Finally, because the sum of orthogonal idempotents generates the direct sum of their ideals, we must have

$$e_1 + e_2 + ... + e_m = \mathbb{1},$$

since

$$A = I_1 \oplus I_2 \oplus ... \oplus I_m.$$

$\square$

A complete set of orthogonal, primitive idempotents $\{e_1, ..., e_m\}$ will be called an **idemsystem.**

Let $\{e_1, ..., e_m\}$ be an idemsystem. Let $I \subseteq A$ be an ideal. Then I is generated by some element of the form

$$e(I) = e_{I_1} + ... + e_{I_k},$$

Where we decompose I as a direct sum of mininal subideals

$$I = I_1 \oplus ... \oplus I_k,$$

And then find the corresponding generators in our idemsystem, so that

$$I_1 = Ae_1, \ \ldots, \ I_k = Ae_k.$$

Thus there is a correspondence

$$\{\text{subsets of the idemsystem } e_1,\ e_2,\ ...,\ e_m\} \longleftrightarrow \{\text{ideals in the algebra } A\}$$

given by the composition

$$S \mapsto e(S) = \sum_{e_s \in S} e_s,$$

$$e(S) \mapsto I(S) = Ae(S) = \bigoplus_{e_s \in S} Ae_s.$$

Lemma 5.1. *This map is a bijection. Furthermore, the intersection of subsets maps to the intersection of ideals, the union of subsets maps to the sum of ideals, and the complement of subsets maps to the orthogonal complement of ideals (which exist thanks to complete reducibility).*

Proof. The fact that this is a bijection has already been proved: we can decompose any ideal I uniquely into a direct sum of minimal ideals I_i and then find the generators e_i for these ideals in the idemsystem, which gives a subset of $\{e_1,\ ...,\ e_m\}$. The inverse to this map sends a subset S of $\{e_1,\ ...,\ e_m\}$ to the ideal $I(S)$ generated by $e(S)$.

Now we show that the intersection $S \cap T$ maps to the intersection $I(S) \cap I(T)$. We know that for any two ideals I, J generated by commuting idempotents e_I, e_J, an idempotent generator for $I \cap J$ is given by $e_I e_J = e_J e_I$. Thus it suffices to show that

$$e(S)e(T) = e(T)e(S) = e(S \cap T).$$

The fact that the elements $e(S), e(T)$ commute is clear: all linear combinations of $\{e_1, \ ..., \ e_m\}$ commute, because the elements $e_1, \ ..., \ e_m$ are orthogonal and thus commute. Now writing $e(S), e(T)$ explicitly,

$$e(S)e(T) = (\sum_{i \in S} e_i)(\sum_{j \in T} e_j)$$

$$= \sum_{i \in S, \ j \in T} e_i e_j,$$

but the product $e_i e_j$ vanishes unless $i = j$, so that this sum is equal to

$$\sum_{i \in S, \ i \in T} e_i e_i$$

$$= \sum_{i \in S \cap T} e_i^2 = \sum_{i \in S \cap T} e_i = e(S \cap T),$$

where the middle equality holds because $e_i = e_i^2$ are idempotent.

Now we prove that the complement of a subset maps to the orthogonal complement of its corresponding ideal. Let S be a subset of $\{e_1, \ ..., \ e_m\}$, and let I be the ideal it generates. It suffices to show this on the level of idempotents, i.e.,

$$e(S^c) = \mathbb{1} - e(S),$$

Because if $I^\perp$ is the unique ideal of A complementary to I, then

$$A = I \oplus I^\perp,$$

And A is generated by $\mathbb{1}$, I is generated by $e(S)$, and the sum of orthogonal idempotents generates the direct sum of their ideals, so that $I^\perp$ is generated by $\mathbb{1} - e(S)$. The equation

$$e(S^c) = \mathbb{1} - e(S)$$

is easy to derive from the completeness equation,

$$\mathbb{1} = e_1 + e_2 + \ ... \ + e_m$$

$$= \sum_{i=1, \ ..., \ m} e_i = (\sum_{i \in S} e_i) + (\sum_{j \notin S} e_j) = e(S) + e(S^c).$$

Finally, it remains to show the correspondence between unions of subsets and sums of ideals. It is not true in general that

$$e(S \cup T) = e(S) + e(T),$$

Because this holds if and only if S and T are *disjoint*: Any element $i \in S \cap T$ appears with multiplicity 1 in $e(S \cup T)$ and with multiplicity 2 in $e(S) + e(T)$. However, the following holds for any two subsets:

$$e(S \cup T) = e(S) + e(T) - e(S)e(T),$$

Because

$$e(S) = e(S \backslash T) + e(S \cap T),$$

$$e(T) = e(T \backslash S) + e(S \cap T),$$

$$e(S) + e(T) = e(S \backslash T) + e(T \backslash S) + 2e(S \cap T),$$

$$e(S) + e(T) - e(S \cap T) = e(S \backslash T) + e(T \backslash S) + e(S \cap T) = e((S \backslash T) \cup (T \backslash S) \cup (S \cap T)) = e(S \cup T),$$

Where the last and second-to-last equalities in the last line follow by decomposing $S \cup T$ into the three disjoint subsets $S \backslash T$, $T \backslash S$, and $S \cap T$. Because the last three terms are orthogonal,

$$Ae(S \cup T) = Ae(S \backslash T) \oplus Ae(T \backslash S) \oplus Ae(S \cap T)$$

$$= (\bigoplus_{s \in S \backslash T} I_s) \oplus (\bigoplus_{t \in T \backslash S} I_t) \oplus (\bigoplus_{i \in S \cap T} I_i)$$

$$= ((\bigoplus_{s \in S \backslash T} I_s) \oplus (\bigoplus_{i \in S \cap T} I_i)) + (\bigoplus_{i \in S \cap T} I_i) \oplus (\bigoplus_{t \in T \backslash S} I_t)$$

$$= (\bigoplus_{s \in S} I_s) + (\bigoplus_{t \in T} I_t) = I(S) + I(T),$$

as desired.

□

Corollary 5.1.1. *The set of all multiplicity-free combinations of an idemsystem* $\{e_1, \ ..., \ e_m\}$ *is a* ***Boolean algebra*** *under the operations*

$$a \wedge b = ab,$$

$$a \vee b = a + b - ab,$$

$$a^c = \mathbb{1} - a.$$

This is isomorphic to the algebra of subsets of $\{e_1, \ ..., \ e_m\}$ *under intersection, union and complement, and this in turn is isomorphic to the algebra of ideals of* A *under taking intersection, sums, and orthogonal complements.*

We can also introduce **partial orders** on each of these sets. The partial ordering on the algebra of subsets of a finite set is simply

$$S \leq T \iff S \subseteq T.$$

The partial ordering on the algebra of ideals of A is the same,

$$I \leq J \iff I \subseteq J.$$

The third partial ordering is on the set of *all* idempotents, not just the ones generated as multiplicity-free sums of an idemsystem. We say

$$e_1 \leq e_2$$

if there is another idempotent e' such that $e_1 e' = e' e_1 = 0$ and $e_2 = e_1 + e'$.

Lemma 5.2. *The following are equivalent:*

1. $e_1 \leq e_2$
2. $e_1 e_2 = e_2 e_1 = e_1$
3. e_1 *and* e_2 *commute and* $e_2 - e_1$ *is idempotent.*
4. *There is an idemsystem* $\{f_1,\ ...,\ f_m\}$ *such that* $e_1 = e(S)$ *and* $e_2 = e(T)$ *for subsets* $S, T \subseteq \{f_1,\ ...,\ f_m\}$ *such that* $S \subseteq T$.

Proof. First notice that the orthogonal idempotent e' is unique, since it must equal $e_2 - e_1$.

We show (1) implies (2). Since e_1 and e' are orthogonal, they commute, and so e_1 commutes with $e_2 = e' + e_1$. Then by orthogonality,

$$e_1 e_2 = e_1(e' + e_1) = e_1 e' + e_1^2 = e_1.$$

Now we show (2) implies (3). Obviously if (2) holds then e_1 and e_2 commute. Now we compute

$$(e_2 - e_1)^2 = e_2^2 - 2e_1 e_2 + e_1^2 = e_2 - 2e_1 + e_1 = e_2 - e_1$$

so that $e_2 - e_1$ is idempotent. Now we show (3) implies (2). Since e_1, e_2 commute,

$$(e_2 - e_1)^2 = e_2^2 - 2e_1 e_2 + e_1^2 = e_2 - 2e_1 e_2 + e_1 = e_2 - e_1,$$

since this element is idempotent, and this is true if and only if

$$-2e_1 e_2 = -2e_1.$$

Now we show (2) implies (1). Let

$$e' = e_2 - e_1.$$

It suffices to check that e' is orthogonal to e_1 because it obviously holds that $e_2 = e_1 + e'$, and e' is idempotent since (2) is equivalent to (3). But

$$e_1 e' = e_1(e_2 - e_1) = e_1 e_2 - e_1^2 = e_1 - e_1 = 0,$$

so these idempotents are orthogonal.

The proof of (4) is as follows. First notice that $e \le \mathbb{1}$ for any idempotent because of the characterization (2). Thus we decompose orthogonally

$$\mathbb{1} = e_2 + e_3, e_3 = \mathbb{1} - e_2.$$

Now decompose orthogonally

$$e_2 = e_1 + e'.$$

If the idempotents e_1, e', e_3 are primitive, then they are also orthogonal and complete and form an idemsystem. If not, keep decomposing them orthogonally until we can write each as a sum of orthogonal primitive idempotents. Then this new set of orthogonal primitive idempotents is complete since $e_1 + e' + e_3 = \mathbb{1}$, and we are done. □

We can actually adjust this partial ordering as follows. Instead of considering e_1 and e_2 on the level of the group algebra, consider them on the level of *ideals*. Then define $e_1 \le' e_2$ whenever the ideal $I_1 = Ae_1$ is contained in the ideal $I_2 = Ae_2$.

Lemma 5.3. *Suppose $e_1 \le' e_2$. Then $e_1 \le e_2$ if and only if e_1 commutes with e_2.*

Proof. e_2 is the projection onto some subspace, e_1 is the projection onto some smaller subspace. If these two projection operators commute, then $e_1 e_2$ is the projection onto their intersection, which is just the projection onto the image of e_1 since this is contained in the image of e_2. But this means that $e_1 e_2 = e_2 e_1 = e_1$ and then part (2) of the above lemma shows that $e_1 \le e_2$. The converse is straightforward. □

The convenient thing about this new ordering is that it is more general than the old ordering, and encodes everything on the level of ideals in terms of idempotent generators.

The purpose of doing all this is as follows. Consider the action of S_k on $V^{\otimes k}$. We would like to study subspaces that generalize $\mathrm{ACT}(V)$ in their construction. More specifically, say we have idempotents $e_1, ..., e_r$ in the group algebra $\mathbb{F}[S_k]$. Then what is the subspace of $V^{\otimes k}$ *fixed* by all of these idempotents? And what is the projection of $V^{\otimes k}$ onto this space?

In the special case when all these idempotents commute, the projection operator is simply

$$e = e_1 e_2 ... e_r.$$

When these idempotents do not commute, however, we must take care in constructing the projection operator out of these.

For any two elements a, b in a partially ordered set, define a **least upper bound** to be any element c with $a \le c$ and $b \le c$, such that for any other $d \ge a, b$, we have $d \ge c$. Similarly define a **greatest lower bound** to be an element c with $c \le a$ and $c \le b$, such that for any other $d \le a, b$, we have $d \le c$. In general, greatest lower bounds and least upper bounds need not exist, nor be unique.

In our particular Boolean algebra, however, these elements do exist. Consider this on the level of ideals of the group algebra. A lower bound of I, J is an ideal K such that $K \subseteq I, K \subseteq J$. Thus $K \subseteq I \cap J$. Since $I \cap J \subseteq I, I \cap J \subseteq I$, and $I \cap J$ is an ideal, we conclude that $I \cap J$ is a greatest lower bound of I and J, and is in fact unique. Similarly, the least upper bound of I and J is the ideal $I + J$. On the level of the subset algebra of a finite set (really, of a set consisting of an idemsystem), the greatest lower bound of S and T is the subset $S \cap T$, while the least upper bound is the subset $S \cup T$.

In the Boolean algebra of multiplicity-free combinations of elements of an idemsystem, the least upper bound and greatest lower bound operation correspond to intersections and unions, which can be written in terms of the algebraic structure as $e_1 e_2$ for the greatest lower bound and $e_1 + e_2 - e_1 e_2$ for the least upper bound, by our correspondence from earlier.

However, greatest lower bounds and least upper bounds also exist for any two idempotents, but are not in general unique. Let e and f be idempotents, and suppose these generate ideals I and J. Then a greatest lower bound of e and f is an idempotent generating $I \cap J$, and a least upper bound is an idempotent generating $I + J$. Notice that in the case where e, f are *commuting,* these idempotents can be chosen as ef and $e + f - ef$ respectively, but this process still works when e, f do not commute.

Now finally we can return to the question of **"forcing"** tensors to have symmetries. The definition of greatest lower bounds and least upper bounds can be generalized to make sense for an arbitrary finite number of elements of the poset: the greatest element which they all cover, or the least element which covers all of them, respectively. In our setup, we are given several idempotents $e_1, ..., e_r$ and we want to find the projection operator from $V^{\otimes k}$ onto the subspace **fixed by all of these**. Notice that when all the e_i *commute*, their greatest lower bound is given by their *product*

$$e = e_1 e_2 ... e_r,$$

and this projects onto the subspace fixed by all of these idempotents. When the e_i fail to commute, let

$$I_i = e_i A$$

be the ideals generated by these, and let

$$I = \bigcap_{i=1}^{r} I_i$$

be their intersection, and then let

$$I = eA,$$

for some idempotent generator e. The the idempotent element e is the projection of $V^{\otimes k}$ onto the subspace fixed by all of $e_1, ..., e_r$.

Now we derive some results specifically about averaging over subgroups, using the language of the group algebra and idempotents that we have developed. We have seen an important results from earlier that will be stated again here now:

Lemma. *Let G be a finite group and A be the group algebra. Let H, K be subgroups of G and $\mathrm{Av}_H, \mathrm{Av}_K \in A$ be the averages over H, K. Then Av_H commutes with Av_K if and only if $HK = KH$ are commuting subgroups. This in turn is true if and only if $HK = L$ is also a subgroup of G, and we have*

$$\mathrm{Av}_H \circ \mathrm{Av}_K = \mathrm{Av}_K \circ \mathrm{Av}_H = \mathrm{Av}_L.$$

When G is the symmetric group, this result also holds when we replace all of the averages by conjugate (signed) averages.

We use this lemma to state and prove a result about the partial ordering of idempotents.

Corollary 5.3.1. *Let H, K be subgroups of some finite group G and let $A = \mathbb{F}[G]$ be the group algebra. Let $\mathrm{Av}_H, \mathrm{Av}_K \in A$ be the averages over H, K. Then $\mathrm{Av}_H \leq \mathrm{Av}_K$ if, and only if, $K \leq H$ as subgroups. Thus the map*

$$\mathrm{Av} : \{\textit{subgroups of } G\} \longrightarrow A,$$

$$H \mapsto \mathrm{Av}_H$$

*Is an **order-reversing inclusion.***

The same result hold when replacing all averages by conjugate averages, when G is a symmetric group. The proof is simple.

Proof. As we have seen from earlier, two idempotents e_1, e_2 satisfy $e_1 \leq e_2$ if and only if

$$e_1 e_2 = e_2 e_1 = e_1.$$

The two idempotents $\mathrm{Av}_H, \mathrm{Av}_K$ commute if and only if $HK = KH = L$ is a subgroup of G. When this is true,

$$\mathrm{Av}_H \circ \mathrm{Av}_K = \mathrm{Av}_K \circ \mathrm{Av}_H = \mathrm{Av}_L.$$

Thus $\mathrm{Av}_H \leq \mathrm{Av}_K$ if and only if $HK = KH = H$. But this is equivalent to $K \subseteq H$, as desired. The proof with signed averages is identical.

□

Example 5.1. Consider again the space of algebraic curvature tensors. This has idempotent generator

$$\mathrm{A\bar{v}}_D \circ (\mathbb{1} - \mathrm{A\bar{v}}_H) = \mathrm{A\bar{v}}_D - \mathrm{A\bar{v}}_{S_4}.$$

We can see easily that this idempotent is "built" out of two other idempotents, namely,

$$\bar{\mathrm{A}}\mathrm{v}_D,\ \bar{\mathrm{A}}\mathrm{v}_{S_4}.$$

Now $D \leq S_4$ is a subgroup of S_4, and so our general theory tells us that

$$\bar{\mathrm{A}}\mathrm{v}_{S_4} \leq \bar{\mathrm{A}}\mathrm{v}_D,$$

and this is exactly why the element

$$\bar{\mathrm{A}}\mathrm{v}_D - \bar{\mathrm{A}}\mathrm{v}_{S_4}$$

is also idempotent. The image of $V^{\otimes 4}$ under $\bar{\mathrm{A}}\mathrm{v}_D$ is precisely the space

$$\mathrm{Sym}^2(\Lambda^2(V)).$$

These are all the $(0,4)-$tensors on V that satisfy

$$R(x, y, z, w) = -R(y, x, z, w),$$

$$R(x,y,z,w) = -R(x,y,w,z),$$

$$R(x,y,z,w) = R(z,w,x,y).$$

However, this space also contains the space of all **alternating** tensors. These in addition satisfy

$$R(x,y,z,w) = -R(x,z,w,y),$$

or more generally,

$$R(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)}) = \operatorname{sign}(\sigma) R(x_1, x_2, x_3, x_4).$$

Then because

$$\Lambda^4(V) \subseteq \operatorname{Sym}^2(\Lambda^2(V)),$$

there must be an idempotent projection operator $e : \operatorname{Sym}^2(\Lambda^2(V)) \longrightarrow \Lambda^4(V)$. We already know that the idempotent projecting onto $\operatorname{Sym}^2(\Lambda^2(V))$ is $\bar{\mathrm{Av}}_D$, while the idempotent projecting onto $\Lambda^4(V)$ is $\bar{\mathrm{Av}}_{S_4}$. The gap between the two groups $D \leq S_4$ is precisely the **complementary subgroup** $H = \langle (1,2,3) \rangle$ generated by cyclic permutations of the first three variables, because

$$HD = DH = S_4$$

$$\iff \bar{\mathrm{Av}}_H \bar{\mathrm{Av}}_D = \bar{\mathrm{Av}}_D \bar{\mathrm{Av}}_H = \bar{\mathrm{Av}}_{S_4},$$

and so $e = \bar{\mathrm{Av}}_H$ is the projection of $\operatorname{Sym}^2(\Lambda^2(V))$ onto $\Lambda^4(V)$. Because of the general theory, we know there must be a **complementary subspace** of $\operatorname{Sym}^2(\Lambda^2(V))$. This can be realized as either

- A subspace W of $\operatorname{Sym}^2(\Lambda^2(V))$ such that $W + \Lambda^4(V) = \operatorname{Sym}^2(\Lambda^2(V))$ satisfying additional properties (W is a subrepresentation of the group $\mathrm{GL}(V)$),
- The kernel of $\bar{\mathrm{Av}}_H$ in $\operatorname{Sym}^2(\Lambda^2(V))$,
- The image of the complementary idempotent $\mathbb{1} - \bar{\mathrm{Av}}_H$ in $\operatorname{Sym}^2(\Lambda^2(V))$.

From the third characterization, the projection onto this space must be the idempotent

$$\bar{\mathrm{Av}}_D - \bar{\mathrm{Av}}_{S_4}.$$

From the first characterization, we derive the following result:

Theorem 5.4. $\operatorname{Sym}^2(\Lambda^2(V))$ *splits as the direct sum of* $\mathrm{GL}(V)$*-reps:*

$$\operatorname{Sym}^2(\Lambda^2(V)) = \mathrm{ACT}(V) + \Lambda^4(V).$$

6 Kulkarni-Nomizu Products and the Algebra $\mathrm{KN}(V)$

We now might want to ask how to generalize the symmetries of the Riemann curvature tensor, or really, how to generalize the space of algebraic curvature tensors. One approach that has been studied thoroughly is to characterize the *covariant derivatives* $\nabla^i Rm$. The symmetries of the first covariant derivative are well-known: ∇R satisfies all the symmetries of R in the first 4 variables, plus the *(differential) Bianchi identity* in the last three variables:

$$\nabla R(x,y,z,w;u) = -\nabla R(y,x,z,w;u) = \nabla R(x,y,w,z;u) = \nabla R(z,w,x,y;u),$$

$$\nabla R(x,y,z,w;u) + \nabla R(y,z,x,w;u) + \nabla R(z,x,y,w;u) = 0,$$

$$\nabla R(x,y,z,w;u) + \nabla R(x,y,w,u;z) + \nabla R(x,y,u,z;w) = 0.$$

This actually implies that the space ∇R over V forms an *irreducible* $\mathrm{GL}(V)$ *subrepresentation of* $V^{\otimes 5}$, which in the language of [2] is called a *symmetry class* and can be described using more language from the representation theory of symmetric groups.

However, this approach presents obstacles when we get to the second covariant derivative $\nabla^2 R(x,y,z,w;u,v)$. This tensor still has all the symmetries of ∇R in the first 5 variables. But symmetries of $\nabla^2 R$ in the last two variables *involve the tensor* R. This is due to the **Ricci identities**:

$$\begin{aligned}\nabla^2 R(x,y,z,w;u,v) - \nabla^2 R(x,y,z,w;v,u) &= R(\mathcal{R}(u,v)x,y,z,w) + R(x,\mathcal{R}(u,v)y,z,w)\\ &+R(x,y,\mathcal{R}(u,v)z,w) + R(x,y,z,\mathcal{R}(u,v)w).\end{aligned}$$

These symmetries might not be intractible, but there are certainly not any **visible** symmetries to exploit here; that is, any symmetries coming from the Ricci identities are hard to deduce from this, and it appears that the current literature does not have much to say on whether there are algebraic symmetries of $\nabla^2 R$ which do not involve R itself.

We now consider an alternative approach to generalizing the symmetries of algebraic curvature tensors.

The swap symmetry of the Riemann curvature tensor is an important feature that we might want to recover in the absence of Bianchi symmetry. The geometric reasons for this symmetry come from how we construct an algebraic curvature tensor out of *canonical* algebraic curvature tensors. These are tensors of the form

$$R_\phi(x,y,z,w) = \phi(x,z)\phi(y,w) - \phi(x,w)\phi(y,z),$$

where $\phi \in \mathrm{Sym}^2(V)$ is a symmetric tensor. There is a theorem due to Gilkey that every algebraic curvature tensor can be realized as a sum of these canonical tensors, that is, these span the space $\mathrm{ACT}(V)$, see [4]. This operation taking ϕ to R_ϕ can be extended to a product of two symmetric tensors, which is called the **Kulkarni-Nomizu product**, and is written with a circle-wedge symbol:

$$(\phi \owedge \psi)(x,y,z,w) = \phi(x,z)\psi(y,w) - \phi(x,w)\psi(y,z) + \phi(y,w)\psi(x,z) - \phi(y,z)\psi(x,w).$$

Notice that $\phi \owedge \psi$ is antisymmetric in x, y and in z, w, and also has swap symmetry in $(x,y) \longleftrightarrow (z,w)$, which we can tell from its symmetries. Thus $\phi \owedge \psi \in \mathrm{Sym}^2(\Lambda^2(V))$.

Another important fact is that the space of algebraic curvature tensors is equivalently generated by all Kulkarni-Nomizu products of symmetric tensors, not just KN-products of a tensor with itself. This is a general fact about commutative algebras: the subspace spanned by all elements of the form x^2 is equal to the subspace spanned by all elements of the form $x \cdot y$, because the elements

$$x \cdot y = \frac{1}{4}((x+y)^2 - (x-y)^2)$$

are all in the linear span of the set of squares. Because each R_ϕ has *Bianchi symmetry*, and these are precisely the squares of elements, each KN-product has *Bianchi symmetry* as well. So the subspace of $\mathrm{Sym}^2(\Lambda^2(V))$ spanned by the KN-products of symmetric tensors is precisely the space $\mathrm{ACT}(V)$ consisting of algebraic curvature tensors:

$$\mathrm{Sym}^2(V) \owedge \mathrm{Sym}^2(V) = \mathrm{ACT}(V) \subseteq \mathrm{Sym}^2(\Lambda^2(V)).$$

The Kulkarni-Nomizu product is even more general than this. It extends to a product taking $\phi \in \mathrm{Sym}^2(\Lambda^k(V))$ and $\psi \in \mathrm{Sym}^2(\Lambda^\ell(V))$ to $\phi \owedge \psi \in \mathrm{Sym}^2(\Lambda^{k+\ell}(V))$ as follows. When both ϕ and ψ can be written as symmetric products of alternating tensors, this product takes the form

$$(\alpha \cdot \beta) \owedge (\gamma \cdot \delta) = (\alpha \wedge \gamma) \cdot (\beta \wedge \delta),$$

where $\cdot$ denotes the symmetric product, and $\wedge$ is the exterior product; and the general product is defined by extending this bilinearly (see [1] for this description of the KN-product, and [5] for Kulkarni's original construction, which makes it clear that this product is well-defined and removes some of the ambiguity in taking the wedge product in the order written). Notice that this turns the direct sum

$$\mathrm{KN}(V) = \bigoplus \mathrm{Sym}^2(\Lambda^k(V))$$

into a *graded algebra*, called the **Kulkarni-Nomizu algebra** over V, where the term *graded* means that the algebra is a direct sum of the factors

$$\mathrm{KN}^k(V) = \mathrm{Sym}^2(\Lambda^k(V))$$

and that the multiplication takes the *graded factors* of degree k and ℓ into the factor of degree $k+\ell$:

$$\owedge : \mathrm{KN}^k(V) \otimes \mathrm{KN}^\ell(V) \longrightarrow \mathrm{Sym}^2(\Lambda^{k+\ell}(V)).$$

The result that each product $\phi \owedge \psi$ has *Bianchi symmetry* implies that the subspace

$$\mathrm{KN}^1 \owedge \mathrm{KN}^1 \subset \mathrm{KN}^2$$

is contained *properly*, so not all elements of the higher graded factors are generated by products from lower graded factors (in particular, $\mathrm{Sym}^2(\Lambda^2(V))$ contains $\Lambda^4(V)$, which is orthogonal to $\mathrm{ACT}(V)$ and is the subspace *fixed* by the Bianchi average, instead of killed by it).

Now we describe this algebra rep-theoretically. Consider the group of permutations generating the symmetries of $\mathrm{KN}^k(V) = \mathrm{Sym}^2(\Lambda^k(V))$. If $R \in \mathrm{KN}^k(V)$ is a $(0, 2k)-$tensor in the Kulkarni-Nomizu factor of degree k, and $\sigma, \tau \in S_k$ are any two permutations, then these symmetries can be written as

$$R(x_1, x_2, ..., x_k; z_1, z_2, ...z_k) = \mathrm{sign}(\sigma) R(x_{\sigma(1)}, ...x_{\sigma(k)}; z_1, ..., z_k) = \mathrm{sign}(\tau) R(x_1, ..., x_k; z_{\tau(1)}, ..., z_{\tau(k)}),$$

$$R(x_1, ..., x_k; z_1, ..., z_k) = R(z_1, ..., z_k; x_1, ..., x_k).$$

The permutations fixing R (in some representation of the symmetric group S_{2k}) can be written as a (signed) permutation of the x_i, followed by a (signed) permutation of the z_j, then possibly followed by the permuation that takes $x_i \mapsto z_i$ for all i. Call this last permutation α:

$$\alpha(x_i) = z_i,$$

$$\alpha(z_i) = x_1.$$

The group generated by the first two kind of permutations is $S_k \times S_k$, since R is invariant under permutations of the x_i alone and the z_j alone. The whole group is a *semidirect product* $(S_k \times S_k) \rtimes S_2$, where the permutation α described above acts on a pair of permutations σ, τ via

$$\alpha(\sigma, \tau) = (\tau, \sigma).$$

Call this the **Kulkarni-Nomizu group** and write it as $\mathrm{KN}_k \leq S_{2k}$. Although it will not be used in the remainder of this paper, we now provide two alternative realizations of the Kulkarni-Nomizu group.

Lemma 6.1. *The Kulkarni-Nomizu group is the **wreath product** of S_k with S_2 over $S_2 \curvearrowright [2]$.*

Proof. This follows directly from the definition of the wreathe product, since in the specified semidirect product $(S_k \times S_k) \rtimes S_2$, the group S_2 acts by taking the pair (σ_1, σ_2) to the pair $(\sigma_{\alpha(1)}, \sigma_{\alpha(2)})$.

□

Lemma 6.2. *The Kulkarni-Nomizu group is the graph automorphism group* $\mathrm{Aut}(K_{k,k})$ *of the complete bipartate graph $K_{k,k}$ on the set of vertices $[k] \times [k]$.*

Remark 6.1. *Notice how this generalizes the group D when $k = 2$. The complete bipartate graph on $[2] \times [2]$ is just a square, and the group of graph automorphisms of a square is D_4.*

Proof. When $k \neq m$, the automorphism group of $K_{k,m} = K([k] \times [m])$ is just the direct product $S_k \times S_m$, since we can permute any vertices in $[k]$ and any vertices in $[m]$ and preserve the graph structure, but we cannot send a vertex in $[k]$ to a vertex in $[m]$ or vice versa, because the vertices in $[k]$ have degree m (they all are connected to every element of $[m]$) while the vertices in $[m]$ have degree k and $k \neq m$. So the automorphism group of $K_{k,k}$ should at least contain the subgroup $S_k \times S_k$.

However, when $k = m$ there is an additional automorphism of the complete bipartate graph $K_{k,k} = K([k] \times [k]')$, where we use the notation $[k]' = \{1', 2', ..., k'\}$ to distinguish from elements of $[k] = \{1, 2, ..., k\}$. This is the automorphism

$$\alpha : [k] \times [k]' \longrightarrow [k]' \times [k],$$

$$\alpha(i) = i', \; \alpha(i') = i,$$

Which *interchanges* the two subsets of vertices.

□

7 The Map $\mathcal{D}$

In this section, we move from finite-dimensional vector spaces to vector bundles on manifolds. We will use the Kulkarni-Nomizu algebra of the last section extensively. On a geometric level, the spaces

$$\mathrm{KN}_p^k(M) := \mathrm{Sym}^2(\Lambda^k(T_p^*M)) = \mathrm{Sym}^2(\Omega_p^k(M)),$$

Defined over the cotangent space at every point, form a **vector bundle** over M, which we simply denote $\mathrm{KN}^k(M)$. The direct sum

$$\mathrm{KN}(M) = \bigoplus \mathrm{KN}^k(M)$$

Is also a vector bundle, which we call the **Kulkarni-Nomizu bundle** or **KN-bundle** over M. This carries the structure of an associative, commutative unital graded algebra, defined by taking the Kulkarni-Nomizu product at every point. The KN-bundle with this structure will be called the **Kulkarni-Nomizu algebra of** M or simply the **KN-algebra of M**.

We will construct a degree-raising map on the KN-bundle of M which carries geometric significance. In order to motivate this construction, consider how we construct Rm out of $\mathcal{R}$ and g on the manifold level:

$$Rm(X,Y,Z,W) = g(\mathcal{R}(X,Y)Z,W).$$

The tensor field Rm then is alternating in X,Y because $\mathcal{R}(X,Y)$ has this property. The other two symmetries (the Bianchi identity and swap symmetry) follow from geometric properties of ∇ (zero torsion and compatible with g). But this is not enough for us: We would like for Rm to have full Kulkarni-Nomizu symmetry, regardless of the torsion-free connection ∇ and its compatibility with the symmetric tensor g. Using the theory of *averaging*, Rm is alternating in Z,W if and only if

$$Rm(X,Y,Z,W) = \frac{1}{2}(Rm(X,Y,Z,W) - Rm(X,Y,W,Z)),$$

so if we instead define the curvature tensor as

$$Rm(X,Y,Z,W) = \frac{1}{2}(g(\mathcal{R}(X,Y)Z,W) - g(\mathcal{R}(X,Y)W,Z)),$$

Then Rm is automatically alternating in Z,W as well, and this agrees with the usual definition of Rm in the special case that ∇ is $g-$compatible. Similarly, Rm is symmetric in $(X,Y) \longleftrightarrow (Z,W)$ if and only if

$$Rm(X,Y,Z,W) = \frac{1}{2}(Rm(X,Y,Z,W) + Rm(Z,W,X,Y).$$

Now combining these two symmetries, we arrive at the new definition

$$Rm(X,Y,Z,W) = \frac{1}{4}(g(\mathcal{R}(X,Y)Z,W) - g(\mathcal{R}(X,Y)W,Z) + g(\mathcal{R}(Z,W)X,Y) - g(\mathcal{R}(Z,W)Y,X)),$$

Where the first two terms force Rm to be antisymmetric in Z,W, and the second two terms force Rm to be symmetric in $(X,Y) \longleftrightarrow (Z,W)$.

This is a more useful definition of Rm for a number of reasons. First we should notice that this agrees with the usual definition of Rm when ∇ is Levi-Civita with respect to g. But this definition highlights all of the important symmetries of Rm - in fact, even when ∇ is neither torsion-free nor $g-$compatible, Rm still lives in $\mathrm{Sym}^2(\Omega^2(M))$, because we have forced this to be true.

In fact, even when g is not a metric, but an arbitrary symmetric tensor, we still have $Rm \in \mathrm{Sym}^2(\Omega^2(M))$. In other words, fixing an affine connection ∇ and corresponding curvature operator $\mathcal{R}$, then for any symmetric bilinear form $g \in \mathrm{Sym}^2(M)$, we can define

$$(\mathcal{D}g)(X,Y,Z,W) = Rm(X,Y,Z,W)$$
$$= \frac{1}{4}(g(\mathcal{R}(X,Y)Z,W) - g(\mathcal{R}(X,Y)W,Z) + g(\mathcal{R}(Z,W)X,Y) - g(\mathcal{R}(Z,W)Y,X)),$$

Then $\mathcal{D}$ is a well-defined map

$$\mathcal{D} : \mathrm{Sym}^2(\mathrm{T}^*M) \longrightarrow \mathrm{Sym}^2(\Omega^2(M)),$$

Taking the degree 1 elements of the KN-algebra to the degree 2 elements of the KN-algebra. In the special case where g is a metric, and ∇ is its Levi-Civita connection, we have

$$Rm = \mathcal{D}g.$$

We would like to generalize this construction so that we can perform the same kind of operation on algebraic curvature tensors and produce $6-$tensors in $\mathrm{Sym}^2(\Omega^3(M))$. That is, we would like to construct a map that takes tensors in the degree k graded factor of the KN-algebra, to tensors in the degree $k+1$ factor, or a map

$$\mathcal{D} : \mathrm{Sym}^2(\Omega^k(M)) \longrightarrow \mathrm{Sym}^2(\Omega^{k+1}(M)).$$

This map $\mathcal{D}$ should somehow involve replacing one variable of a tensor R with $\mathcal{R}(A,B)C$, since this gives the right rank; and then forcing symmetries using idempotents.

The general construction will follow quickly once we figure out the case $k=2$,

$$\mathcal{D} : \mathrm{Sym}^2(\Omega^2(M)) \longrightarrow \mathrm{Sym}^2(\Omega^3(M)).$$

Let $R \in \mathrm{Sym}^2(\Omega^2(M))$ so that R satisfies

$$R(A,B,C,D) = -R(B,A,C,D) = -R(A,B,D,C) = R(C,D,A,B).$$

Notice that any two variables we put in the last two positions must be alternating because of the symmetries of R, and any two variables we put into $\mathcal{R}(-,-)$ must also be antisymmetric. Now $\mathcal{D}R$ is a $6-$tensor $(\mathcal{D}R)(X,Y,U,Z,W,V)$ which is alternating in X,Y,U, alternating in Z,W,V, and symmetric in $(X,Y,U) \longleftrightarrow (Z,W,V)$. We want to construct this new tensor out of terms of the form

$$R(\mathcal{R}(-,-)-,-,-,-)$$

Where we replace dashes with variables. Since the arguments of $\mathcal{R}$ must be alternating, they must both be in X,Y,U or both in Z,W,V, and similarly for the last two arguments of R. Start by defining

$$\mathcal{D}R(X,Y,U,Z,W,V) = R(\mathcal{R}(X,Y)U,Z,W,V).$$

While this appears to be the most logical choice of ordering variables, this is not the correct way to proceed. The reason is that once we apply full alternating symmetries in X,Y,U we get a sum of the form

$$R(\mathcal{R}(X,Y)U,Z,W,V) + R(\mathcal{R}(U,X)Y,Z,W,V) + R(\mathcal{R}(Y,U)X,Z,W,V)$$

$$-R(\mathcal{R}(Y,X)U,Z,W,V)+R(\mathcal{R}(X,U)Y,Z,W,V)+R(\mathcal{R}(U,Y)X,Z,W,V).$$

Then applying symmetries of $\mathcal{R}$, this is

$$2(R(\mathcal{R}(X,Y)U,Z,W,V)+R(\mathcal{R}(U,X)Y,Z,W,V)+R(\mathcal{R}(Y,U)X,Z,W,V))$$

$$=2R(\mathcal{R}(X,Y)U+\mathcal{R}(U,X)Y+\mathcal{R}(Y,U)X,Z,W,V)$$

$$=2R(0,Z,W,V)=0,$$

because $\mathcal{R}(X,Y)U$ satisfies the Bianchi identity. So *instead* we start by defining

$$\mathcal{D}R(X,Y,U,Z,W,V)=R(\mathcal{R}(X,Y)V,U,Z,W).$$

Now we simply *force* full KN-symmetries. This means we permute all of X,Y,U and apply signs, then permute all of Z,W,V and apply signs, then permute $(X,Y,U)\longleftrightarrow(Z,W,V)$, and finally add these together. This gives a sum with 18 terms: first apply alternating symmetry in Z,W,V:

$$R(\mathcal{R}(X,Y)V,U,Z,W)-R(\mathcal{R}(X,Y)Z,U,V,W)-R(\mathcal{R}(X,Y)W,U,Z,V)$$

Now combine this with alternating symmetry in X,Y,Z:

$$-R(\mathcal{R}(U,Y)V,X,Z,W)+R(\mathcal{R}(U,Y)Z,X,V,W)+R(\mathcal{R}(U,Y)W,X,Z,V)$$

$$-R(\mathcal{R}(X,U)V,Y,Z,W)-R(\mathcal{R}(X,U)Z,Y,V,W)-R(\mathcal{R}(X,U)W,Y,Z,V)$$

This produces a tensor in $\Omega^3(M)\otimes\Omega^3(M)$. Finally, we swap the triple of variables $(X,Y,U)\longleftrightarrow(Z,W,V)$ in each of these terms:

$$R(\mathcal{R}(Z,W)U,V,X,Y)-R(\mathcal{R}(Z,W)X,V,U,Y)-R(\mathcal{R}(Z,W)Y,V,X,U)$$

$$-R(\mathcal{R}(V,W)U,Z,X,Y)+R(\mathcal{R}(V,W)X,Z,U,Y)+R(\mathcal{R}(V,W)Y,Z,X,U)$$

$$-R(\mathcal{R}(Z,V)U,W,X,Y)+R(\mathcal{R}(Z,V)X,W,U,Y)+R(\mathcal{R}(Z,V)Y,W,X,U).$$

Why are there 18 terms? The reason is that the Kulkarni-Nomizu group has order $2(k!)^2$ which for $k=3$ is equal to 72. Averaging over the full KN-group should produce a sum with 72 terms. However, only 1 in 4 of these terms is explicitly written, because there are already symmetries in the tensor $R(\mathcal{R}(X,Y)V,U,Z,W)$, which is alternating in X,Y and in Z,W. These symmetries generate a group of order 4, the *symmetry group* of R, which is a subgroup of the KN-group, and

$$72/4=18,$$

which explains the number of terms.

This generalizes to higher degrees k as follows. Let $R\in\mathrm{Sym}^2(\Omega^k(M))$. Then

$$\mathcal{D}R(x_1, x_2, ..., x_k, x_{k+1}; z_1, z_2, ..., z_k, z_{k+1}) \in \mathrm{Sym}^2(\Omega^{k+1}(M))$$

Is constructed by first replacing the first variable of R with $\mathcal{R}(x_1, x_{k+1})z_{k+1}$, then applying full KN-symmetries:

$$\mathcal{D}R(x_1, ..., x_k, x_{k+1}; z_1, ..., z_k, z_{k+1}) = \mathcal{KN}R(\mathcal{R}(x_1, x_{k+1})z_{k+1}, x_2, ..., x_k; z_1, ..., z_k),$$

Where $\mathcal{KN}$ denotes the projection onto $\mathrm{Sym}^2(\Omega^{k+1}(M))$.

8 Cohomology, Areas for Further Study

One very special feature of the Kulkarni-Nomizu bundle is that it has a natural structure of a **cochain complex**. There is a **coboundary map**

$$S\mathrm{d} : \mathrm{Sym}^2(\Omega^k(M)) \longrightarrow \mathrm{Sym}^2(\Omega^{k+1}(M))$$

Defined by taking the symmetric square of the *Cartan differential* (also known as the *exterior derivative*), which is the unique map

$$\mathrm{d} : \Omega(M) \longrightarrow \Omega(M)$$

satisfying the following four properties:

- d is homogeneous of degree 1. This means that for each k, d takes

$$\mathrm{d} : \Omega^k(M) \longrightarrow \Omega^{k+1}(M).$$

- d agrees with the *differential of smooth functions,* where we consider functions $f \in \mathrm{C}^\infty(M, \mathbb{R})$ as $0-$forms, and their differentials $\mathrm{d}f \in T^*M$ as covectorfields or $1-$forms. In other words,

$$\mathrm{d}f(X) = X(f)$$

  for any smooth $f \in C^\infty(M)$ and vector field X on M.

- d is a *coboundary,* so that

$$\mathrm{d}(\mathrm{d}(\alpha)) = 0$$

  For any differential form α.

- d is an *anti-derivation,* that is, if $\alpha \in \Omega^k(M)$ is a $k-$form and $\beta \in \Omega(M)$ is any differential form, then

$$\mathrm{d}(\alpha \wedge \beta) = \mathrm{d}(\alpha) \wedge \beta + (-1)^k \alpha \wedge \mathrm{d}(\beta).$$

(For more on the de Rham differential including proofs of these properties and explicit construction, see [7].)

The map $S\mathrm{d}$ is constructed from d via *universal properties.* In particular, for any linear transformation $T : V \longrightarrow W$ of vector spaces, there are induced linear transformations $S^kT : \mathrm{Sym}^k(V) \longrightarrow \mathrm{Sym}^k(W)$ defined by

$$S^kT(v_1v_2...v_k) = T(v_1)T(v_2)...T(v_k).$$

Thus $S\mathrm{d}$ is, explicitly,

$$S\mathrm{d}(\alpha \cdot \beta) = \mathrm{d}(\alpha) \cdot \mathrm{d}(\beta).$$

This makes it clear from the first property of d that $S\mathrm{d}$ is homogeneous of degree 1 on the KN-algebra of M, and from the third property of d, it is clear that

$$S\mathrm{d} \circ S\mathrm{d} = 0,$$

That is to say, $S\mathrm{d}$ is a **coboundary** on the KN-algebra of M.

Notice that $S\mathrm{d}$ is **not a derivation** on the KN-algebra of M. However, we can compute the coboundary of a KN-product to get a relation between the coboundaries of the factors. Let $\alpha, \beta \in \Omega^k(M)$, and $\gamma, \delta \in \Omega^\ell(M)$. Then

$$S\mathrm{d}[(\alpha \cdot \beta) \owedge (\gamma \cdot \delta)] = S\mathrm{d}[(\alpha \wedge \gamma) \cdot (\beta \wedge \delta)]$$

$$= \mathrm{d}(\alpha \wedge \gamma) \cdot \mathrm{d}(\beta \wedge \delta)$$

$$= (\mathrm{d}(\alpha) \wedge \gamma + (-1)^k \alpha \wedge \mathrm{d}\gamma) \cdot (\mathrm{d}(\beta) \wedge \delta + (-1)^k \beta \wedge \mathrm{d}(\delta))$$

$$= (\mathrm{d}(\alpha) \wedge \gamma) \cdot (\mathrm{d}(\beta) \wedge \delta) + (\alpha \wedge \mathrm{d}(\gamma)) \cdot (\beta \wedge \mathrm{d}(\delta)) + (-1)^k [(\mathrm{d}(\alpha) \wedge \gamma) \cdot (\beta \wedge \mathrm{d}(\delta)) + (\alpha \wedge \mathrm{d}(\gamma)) \cdot (\mathrm{d}(\beta) \wedge \delta)]$$

And the first two terms in this sum can be written as KN-products. They are equal to

$$(\mathrm{d}(\alpha) \cdot \mathrm{d}(\beta)) \owedge (\gamma \cdot \delta) + (\alpha \cdot \beta) \owedge (\mathrm{d}(\gamma) \cdot \mathrm{d}(\delta))$$

$$= S\mathrm{d}(\alpha \cdot \beta) \owedge (\gamma \cdot \delta) + (\alpha \cdot \beta) \owedge S\mathrm{d}(\gamma \cdot \delta).$$

This is exactly what we would expect to get if $S\mathrm{d}$ were a derivation. But the two terms

$$(\mathrm{d}(\alpha) \wedge \gamma) \cdot (\beta \wedge \mathrm{d}(\delta)) + (\alpha \wedge \mathrm{d}(\gamma)) \cdot (\mathrm{d}(\beta) \wedge \delta)$$

Are **not** expressible as KN-products in an obvious way. This is because $\mathrm{d}(\alpha)$ is of degree $(k+1)$ while β is of degree k, and γ is of degree ℓ while $\mathrm{d}(\delta)$ is of degree $\ell + 1$.

The cohomology of the chain complex $\mathrm{KN}(M)$ with the coboundary $S\mathrm{d}$ may yield interesting results. This cohomology theory has not yet been investigated by the author. The relationship to de Rham cohomology has not been investigated either, nor has the relationship to curvature. We conjecture that the map $\mathcal{D}$ play an important role in all of this. In particular, we conjecture either that the tensors $g, \mathcal{D}g = R, \mathcal{D}R, \mathcal{D}\mathcal{D}R, ...$ play an interesting role in describing cohomology classes as we let g vary over metrics on M; or that the map $\mathcal{D}$ is itself a coboundary or derivation, which may yield an interesting alternative cohomology theory. The algebraic and rep-theoretic symmetries of various cohomology classes, as well as the pure KN-products, should also play an interesting role.

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