Linear Dependence of Canonical Algebraic Curvature Tensors as Described by Weighted Directed Graphs

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Abstract

Our goal is to understand the role of kernels in linear dependence relationships between canonical algebraic curvature tensors. In this paper, we study such linear dependence relationships by analyzing different types of weighted directed graphs. We present the background information necessary for motivating our work. We then state our conclusions about linear dependence with respect to the different weighted directed graphs. Lastly, we present directions for future study.

1 Introduction and Motivation

First and foremost, in order to fully understand this study, there are a handful of definitions that one must know. Those definitions are provided here.

Definition 1.1. Let V be a finite-dimensional, real vector space. A multilinear function $R : V \times V \times V \times V \to \mathbb{R}$ is an *algebraic curvature tensor* if it satisfies the following properties:

1. $R(x, y, z, w) = -R(y, x, z, w)$,

2. $R(x, y, z, w) = R(z, w, x, y)$, and

3. $R(x, y, z, w) + R(z, x, y, w) + R(y, z, x, w) = 0$.

The third property is known as the Bianchi Identity, and the vector space of all algebraic curvature tensors on V is denoted by $\mathcal{A}(V)$.

Definition 1.2. Let V be a real vector space. An *inner product* is a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ which satisfies the following properties:

1. $\langle \cdot, \cdot \rangle$ *is symmetric:* for all $\vec{v}, \vec{w} \in V, \langle \vec{v}, \vec{w} \rangle = \langle \vec{w}, \vec{v} \rangle$,

2. $\langle \cdot, \cdot \rangle$ *is bilinear:* for all $\vec{v}, \vec{w}, \vec{u} \in V,$ and for all $a, b \in \mathbb{R},$ we have $\langle a\vec{v} + b\vec{w}, \vec{u} \rangle = a\langle \vec{v}, \vec{u} \rangle + b\langle \vec{w}, \vec{u} \rangle$, and

3. $\langle \cdot, \cdot \rangle$ *is nondegenerate:* for all $\vec{v} \in V$ such that $\vec{v} \neq \vec{0},$ there exists $\vec{w} \in V$ such that $\langle \vec{v}, \vec{w} \rangle \neq 0$.

In this paper, we will only be working with inner products which are *positive-definite.* That is, for all $\vec{v} \in V$, we have $\langle \vec{v}, \vec{v} \rangle \geq 0$, and $\langle \vec{v}, \vec{v} \rangle = 0$ if and only if $\vec{v} = \vec{0}$. Thus, from now on, we will assume that all inner products are positive-definite.

Definition 1.4 describes an algebraic curvature tensor that is defined in terms of a linear transformation A. This algebraic curvature tensor is denoted R_A. It is also possible to define an algebraic curvature tensor in terms of a symmetric bilinear form (such as an inner product), but for this paper, we only consider algebraic curvature tensors that are defined in terms of linear transformations [3]. However, before we can state such a definition, we need to understand some more terminology regarding the linear transformations.

Definition 1.3. Let $\langle \cdot, \cdot \rangle$ be the inner product on the vector space V and let $A : V \to V$ be a linear transformation. The *adjoint* of A, denoted A^*, is characterized by the following equation:

$$\langle Ax, y\rangle = \langle x, A^*y\rangle.$$

A is called *self-adjoint* when $A = A^*$, and A is called *skew-adjoint* when $A = -A^*$. For this paper, we will only consider self-adjoint linear operators.

We can now define R_A.

Definition 1.4. Let $A : V \to V$ be a linear operator on the vector space V. Let $\langle\cdot,\cdot\rangle$ be the inner product on V. A *canonical algebraic curvature tensor* R_A is defined as follows:

$$R_A(x, y, z, w) = \langle Ax, w\rangle\langle Ay, z\rangle - \langle Ax, z\rangle\langle Ay, w\rangle.$$

It is crucial to note here that there are two types of builds of a canonical algebraic curvature tensor: symmetric and anti symmetric. In fact, the build of R_A is dependent on whether $A = A^*$ or $A = -A^*$. Specifically, when R_A is symmetrically built, then R_A is an algebraic curvature tensor if $A = A^*$, and when R_A is anti symmetrically built, then R_A is an algebraic curvature tensor if $A = -A^*$ [1]. In this paper, we will only consider symmetrically built R_A.

Definition 1.5. Let V be a vector space, and let R be an algebraic curvature tensor on V. If $V = V_1 \oplus V_2$, and $R(v_1, v_2, \cdot, \cdot) = 0$ for $v_1 \in V_1$ and $v_2 \in V_2$, then R is *decomposable*, meaning that $R = R_1 \oplus R_2$ where $R_1 \in \mathcal{A}(V_1)$ and $R_2 \in \mathcal{A}(V_2)$.

Let V be an $n-$dimensional vector space, and let R be an algebraic curvature tensor on V. If $\{e_1, \ldots, e_n\}$ is a basis for V, then R is determined by $R(e_i, e_j, e_k, e_l) = R_{ijkl}$. These e_i, e_j, e_k, e_l are called the *curvature entries* of R, and the number of independent combinations of curvature entries is dependent on n. Specifically, If $dimV = n$, then $dim\mathcal{A}(V) = \frac{n^2(n^2-1)}{12}$.

Definition 1.6. Let R be an algebraic curvature tensor. The *kernel* of R is defined as follows:

$$Ker(R) = \{x \in V | R(x, y, z, w) = 0 \text{ for all } y, z, w \in V\}.$$

In fact, it has been proven in [2] that $Ker(R)$ does not depend on entry position. That is,

$$\begin{aligned} Ker(R) &= \{y \in V | R(x, y, z, w) = 0 \text{ for all } x, z, w \in V\} \\ &= \{z \in V | R(x, y, z, w) = 0 \text{ for all } x, y, w \in V\} \\ &= \{w \in V | R(x, y, z, w) = 0 \text{ for all } x, y, z \in V\}. \end{aligned}$$

The kernel of an algebraic curvature tensor is an especially crucial component of our study. In the next definition, we begin to see how the objects described above relate to one another in the context of linear dependence. But first, we will define linear dependence itself.

Definition 1.7. Let V be a real vector space and let $A_1, \ldots, A_n : V \to V$ be linear operators on V. We say that the algebraic curvature tensors $R_{A_1}, \ldots, R_{A_n}$ are *linearly dependent* if the following equation holds:

$$\sum_{i=1}^{n} \epsilon_i R_{A_i} = 0,$$

where $\epsilon_i \in \{1, -1\}$.

At a first glance, this definition may not appear correct, because ϵ_i cannot be just any real number and thus $\sum_{i=1}^{n} \epsilon_i R_{A_i}$ does not represent just any arbitrary linear combination. However, Diaz and Dunn explain that, for any real number c, the multilinearity of algebraic curvature tensors gives us $cR_A = \epsilon R_{\sqrt{c}A}$, where $\epsilon = \text{ sign}(c) = \pm 1$. Thus, if we let c_i be any real number and $\epsilon_i = \text{ sign}(c_i)$, we can simply let $B_i = \sqrt{c}A_i$, which means that the following is true. [3]

$$\sum_{i=1}^{n} c_i R_{A_i} = \sum_{i=1}^{n} \epsilon_i R_{B_i}.$$

Remark 1. Note that, because $\epsilon_i \in \{-1, 1\}$, then without loss of generality, we can multiply any linear dependence equation through by any ϵ_i so that the corresponding R_{B_i} need not have any coefficient (because $(\pm 1)^2 = 1$). For the sake of simplicity, we will always multiply through by ϵ_i so that the first tensor in our linear dependence equation can be written with no coefficient. Thus, without loss of generality, we can rewrite any linear dependence equation as follows:

$$R_{B_1} + \sum_{i=2}^{n} \epsilon_i R_{B_i} = 0.$$

Some previous results about linear dependence of algebraic curvature tensors come from the study of chain complexes. We now provide the definition of a chain complex.

Definition 1.8. Let V be a vector space, and let $A_1, \ldots, A_n : V \to V$ be linear transformations on V. Suppose that for $1 \leq i \leq n-1$, we have $Im(A_i) \subseteq Ker(A_{i+1})$. Then the following diagram is called a *chain complex*:

$$V \overset{A_1}{\longrightarrow} \ldots \overset{A_i}{\longrightarrow} V \overset{A_{i+1}}{\longrightarrow} \ldots \overset{A_k}{\longrightarrow} V.$$

It is important to note here that, given a chain complex, since $Im(A_i) \subseteq Ker(A_{i+1})$, then $Rank(A_{i+1}A_i) = 0$ for all i such that $1 \leq i \leq n-1$. In their studies of linear dependence using chain complexes, McMahon and Williams both examined *compound chain complexes*, which are collections of chain complexes [5]. Some of the compound chain complexes they used had diagrams that were shaped as n-gons. For example, McMahon studied the linear dependence relation $R_A + \epsilon_1 R_B + \epsilon_2 R_C + \epsilon_3 R_D = 0$ with respect to the following diagram:

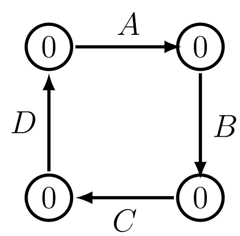
$$\begin{array}{ccccc} & & A & & \\ & V & \longrightarrow & V & \\ D & \uparrow & & \downarrow & B \\ & V & \longleftarrow & V & \\ & & C & & \end{array}$$

By Definition 1.7, This diagram yields the following information:

$$\begin{aligned} &Im(A) \subseteq Ker(B) \Rightarrow Rank(BA) = 0, \\ &Im(B) \subseteq Ker(C) \Rightarrow Rank(CB) = 0, \\ &Im(C) \subseteq Ker(D) \Rightarrow Rank(DC) = 0, \text{ and} \\ &Im(D) \subseteq Ker(A) \Rightarrow Rank(AD) = 0. \end{aligned}$$

While numerous interesting results about linear dependence arise from the study of chain complexes, one might observe that the fact that $Rank(A_{i+1}A_i) = 0$ for all i such that $1 \leq i \leq n-1$ is actually quite restrictive. In other words, studying linear dependence relations using chain complexes illustrates only one specific type of linear dependence relation: a relation which requires $Rank(A_{i+1}A_i) = 0$ in all cases. In this paper, we entertain the more general case where, for one or more i such that $1 \leq i \leq n-1$, $Rank(A_{i+1}A_i) > 0$. This vast collection of scenarios can be illustrated by what are known as *weighted directed graphs*. Weighted directed graphs are quite similar in appearance to chain complexes, but the information they provide is much more specific. In fact, there does not exist a linear dependence relation that cannot be modeled using a weighted directed graph. Hence, by studying linear dependence relations using weighted directed graphs, not only are we generalizing some of the linear dependence results that arise from the use of chain complexes, but we are generalizing the concept of linear dependence as a whole.

The key difference between a chain complex and a weighted directed graph is that, instead of V connecting the arrows, there is a circle with a number in it. This circle is known as a weight, and the number is the upper bound on $Rank(A_{i+1}A_i)$. Thus, for example, if we were to translate McMahon's chain complex above into a weighted directed graph, we would get the following diagram.



Additionally, in this paper, just as McMahon did, we will only be considering cyclic weighted directed graphs. The vast majority of this paper focuses on the following diagram, which illustrates a relation between R_A, R_B, and R_C :

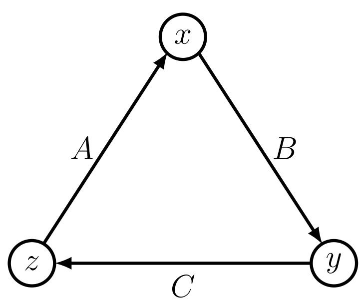


Figure 1: A weighted directed graph that illustrates a relation between linear operators A, B, and C and can be used to find out information about linear dependence relations between R_A, R_B, and R_C.

where x, y, and z can each take on any nonnegative value. This diagram is meant to hypothesize the following information:

$$Rank(BA) \leq x, \; Rank(CB) \leq y, \; \text{and } Rank(AC) \leq z.$$

Note that because x, y, and z can take on positive values, the assumptions that $Im(A) \subseteq Ker(B)$, $Im(B) \subseteq Ker(C)$, and $Im(C) \subseteq Ker(A)$ are no longer necessarily assumed.

2 Preliminaries

Now that we have defined all of the terminology we need to know, we now state a handful of lemmas that will help us understand and prove our main results. The first result that we state is crucial to our understanding of how algebraic curvature tensors behave with respect to the linear operators that define them.

Lemma 2.1. *[4] Let V be a real, finite-dimensional vector space, and let $A : V \to V$ be a linear transformation on V. Then $Rank(A) \leq 1$ if and only if $R_A = 0$.*

Lemma 2.2. *[4] Let V be a real, finite-dimensional vector space, and let $A : V \to V$ be a linear transformation on V. If $Rank(A) \geq 2$, then $Ker(R_A) = Ker(A)$.*

When considering the linear dependence relation $R_A + \epsilon_B R_B + \epsilon_C R_C = 0$ illustrated by Figure 1, Lemma 2.1 gives rise to the question of replacing all of x, y, and z in our weighted directed graph with 1. In this case, we know immediately that

$$\begin{aligned} &Rank(BA) \leq 1 \Rightarrow R_{BA} = 0, \\ &Rank(CB) \leq 1 \Rightarrow R_{CB} = 0, \text{ and} \\ &Rank(AC) \leq 1 \Rightarrow R_{AC} = 0, \end{aligned}$$

but what does this tell us about the individual algebraic curvature tensors R_A, R_B, and R_C? It is the linear dependence relation $R_A + \epsilon_B R_B + \epsilon_C R_C = 0$ that we are most interested in, after all. Is it the case that these algebraic curvature tensors are all trivial? Because if so, not only would the scenario where all of x, y, and z are 0 yield only trivial solutions, but so would the scenario where all x, y, and z are 1 (because of Lemma 2.1). How can we use Rank information about the matrix combinations BA, CB, and AC to derive more specific information about A, B, and C by themselves? The next definition provides a useful method for doing so.

Definition 2.3. Let V be a vector space and let $A, B : V \to V$ be linear transformations on V. Let R_A be the algebraic curvature tensor defined in terms of A. *Precomposition by* B, written B^*R_A, is defined as follows:

$$B^*R_A(x, y, z, w) = R_A(Bx, By, Bz, Bw).$$

Precomposition allows us to see how we can relate algebraic curvature tensors to one another. The following lemma clarifies this concept.

Lemma 2.4. *[5]* $B^*R_A = R_{BAB}$.

Proof. Given that $B^*R_A(x, y, z, w) = R_A(Bx, By, Bz, Bw)$, we can then apply Definition 1.4 to get

$$R_A(Bx, By, Bz, Bw) = \langle ABx, Bw\rangle\langle ABy, Bz\rangle - \langle ABx, Bz\rangle\langle ABy, Bw\rangle.$$

In fact, since we are making the additional assumption in this paper that $A^* = A$ and $B^* = B$, we can apply Definition 1.3 to the above, which yields

$$\begin{aligned} R_A(Bx, By, Bz, Bw) &= \langle ABx, Bw\rangle\langle ABy, Bz\rangle - \langle ABx, Bz\rangle\langle ABy, Bw\rangle \\ &= \langle B^*ABx, w\rangle\langle B^*ABy, z\rangle - \langle B^*ABx, z\rangle\langle B^*ABy, w\rangle, \end{aligned}$$

which, again by Definition 1.4, is equal to

$$R_{B^*AB}(x, y, z, w),$$

which equals

$$R_{BAB}(x, y, z, w).$$

Hence, $B^* R_A = R_{BAB}$. □

Thus, with precomposition, we can begin to see how we can relate the tensors R_{BA}, R_{CB}, and R_{AC} to R_A, R_B, and R_C. Precomposition is a very powerful tool for understanding the algbraic curvature tensors with which we are working. But before we can answer our question about x, y, and z all being equal to 1, we must state two more results.

Lemma 2.5. *[6] Let V be a vector space and let $A : V \to V$ be a self-adjoint linear operator on V. If $Rank(A^k) = p$, then $Rank(A) = p$.*

Lemma 2.6. *Let V be a vector space and let $A, B, : V \to V$ be two linear operators on V. Suppose $Rank(AB) \leq p$. Then $Rank(ABA) \leq p$.*

We can now state and prove the following theorem.

Theorem 2.7. *Given that $R_A + \epsilon_B R_B + \epsilon_C R_C = 0$ and $x = y = z = 1$, there are no solutions such that R_A, R_B, and R_C are all nonzero.*

Proof. we have:

$$\begin{aligned}
&Rank(BA) \leq 1 \Rightarrow R_{BA} = 0, \\
&Rank(CB) \leq 1 \Rightarrow R_{CB} = 0, \text{ and} \\
&Rank(AC) \leq 1 \Rightarrow R_{AC} = 0.
\end{aligned}$$

First, we will precompose our linear dependence equation with $A:$

$$A^* R_A + \epsilon_B A^* R_B + \epsilon_C A^* R_C = 0.$$

By Lemma 2.3, we get

$$R_{A^3} + \epsilon_B R_{ABA} + \epsilon_C R_{ACA} = 0.$$

But since $Rank(BA) \leq 1$ and $Rank(AC) \leq 1$, then by Lemma 2.6, we know that $Rank(ABA) \leq 1$ and $Rank(ACA) \leq 1$. Thus, by Lemma 2.1, we have $R_{ABA} = R_{ACA} = 0$. Thus, we are left with

$$R_{A^3} = 0.$$

By Lemma 2.1, we have $Rank(A^3) = 0$, which by Lemma 2.5, means that $Rank(A) \leq 1$, which again by Lemma 2.1 means that $R_A = 0$. If we repeat this process and precompose the equation by B, we get $R_B = 0$, and precomposing by C will yield $R_C = 0$ by the same logic. Therefore, we have confirmed that not only does having all vertices of our weighted directed graph equal to 0 yield trivial results, but so too does having all vertices equal to 1. □

Thus, we wish to examine weighted directed graphs in which at least one of x, y, and z is strictly greater than 1, so as to have a chance of finding nontrivial linear dependence.

We are just about ready to move onto the main results section of this paper. But before we do so, we state a few more general results.

Theorem 2.8. ***(The Spectral Theorem)*** *Let V be a vector space, let $\langle \cdot, \cdot \rangle$ be a positive-definite inner product on V, and let $A : V \to V$ be a self-adjoint linear operator on V. Then there exists an orthonormal basis $\mathcal{B}$ of eigenvectors of A. That is, there exists $\mathcal{B} = \{e_1, \ldots, e_n\}$ such that for every $e_i \in \mathcal{B}$, we have $Ae_i = \lambda_i e_i$, where λ_i is the associated eigenvalue for eigenvector e_i.*

Lemma 2.9. *Let V be a vector space, let $\langle \cdot, \cdot \rangle$ be a positive-definite inner product on V, and let $T : V \to V$ be a self-adjoint linear transformation on V. Let A be the $n \times n$ matrix that represents T with respect to an orthonormal basis for V. Then A is symmetric.*

Proof. Observe that by the Spectral Theorem, we know that there exists an orthonormal basis $\mathcal{B}$ of eigenvectors of T. Suppose that the matrix A is the matrix representation of T with respect to the basis $\mathcal{B}$. Because $\mathcal{B}$ is an eigenbasis for T and is orthornomal, we know that the matrix A with respect to $\mathcal{B}$ is a diagonal matrix. Thus, because A is a square, diagonal matrix, we know that $A = A^T$, and is therefore a symmetric matrix. □

Hence, because we are assuming throughout this paper that all linear operators are self-adjoint, all inner products are positive-definite, and all bases we will use are orthonormal, then we know that the matrices representing those linear operators are all symmetric. This brings us to our next preliminary result.

Lemma 2.10. *Let V be a vector space and let $A, B : V \to V$ be linear operators on V. In this paper, $Rank(BA) = Rank(AB)$.*

Proof. By assumption in this paper, we know that A and B are self-adjoint and that the matrices A and B are symmetric. Thus, we have

$$\begin{aligned} Rank(BA) &= \text{ dimension of the column space of } BA \\ &= \text{ dimension of the row space of } BA \\ &= \text{ dimension of the column space of } (BA)^T \\ &= \text{ dimension of the column space of } A^T B^T \\ &= \text{ dimension of the column space of } AB \\ &= Rank(AB). \end{aligned}$$

□

Lemma 2.11. *Let V be a vector space and let A and B be linear transformations on V such that $Rank(A) \geq 2$ and $Rank(B) \geq 2$. Then $Ker(A) \cap Ker(B) \subseteq Ker(R_A \pm R_B)$.*

Proof. Let $\vec{v} \in V$ such that $\vec{v} \in Ker(A) \cap Ker(B)$. Then $A(\vec{v}) = 0$ and $B(\vec{v}) = 0$. By Lemma 2.2, we know that $Ker(A) = Ker(R_A)$ and $Ker(B) =$

$Ker(R_B)$. Therefore, $\vec{v} \in Ker(R_A) \cap Ker(R_B)$. Observe:

$$\begin{aligned} R_A(\vec{v}, x, y, z) \pm R_B(\vec{v}, x, y, z) \\ = 0 \pm 0 \\ = 0. \end{aligned}$$

Thus, $\vec{v} \in Ker(R_A \pm R_B)$. □

3 Main Results

3.1 The 2-0-0 Diagram

In this section, we only alter the weighted directed graph slightly to see if we can find a nontrivial solution to the linear dependence relation $R_A + \epsilon_B R_B + \epsilon_C R_C = 0$. Consider the following diagram.

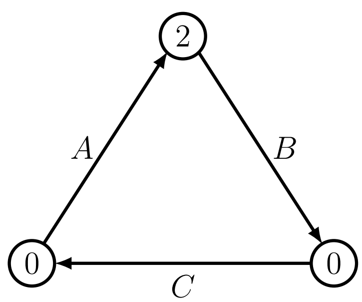


Figure 2: The 2-0-0 Diagram.

Theorem 3.1. *If the 2-0-0 diagram holds, then there do not exist any non-trivial R_A, R_B, and R_C that satisfy the linear dependence relation $R_A + \epsilon_B R_B + \epsilon_C R_C = 0$.*

Proof. The 2-0-0 diagram allows us to deduce the following information:

$$Rank(BA) \leq 2 \Rightarrow Rank(ABA) = Rank(BAB) \leq 2 \text{ (by Lemma 2.6)},$$
$$Rank(CB) = 0 \Rightarrow CB = 0 \Rightarrow Rank(BCB) = Rank(CBC) = 0, \text{ and}$$
$$Rank(AC) = 0 \Rightarrow AC = 0 \Rightarrow Rank(CAC) = Rank(ACA) = 0.$$

To gather more information about R_A, R_B, and R_C, we will begin by pre-composing the equation with C :

$$\begin{aligned} C^* R_A + \epsilon_B C^* R_B + \epsilon_C C^* R_C &= 0 \\ \underbrace{R_{CAC}}_{0} + \epsilon_B \underbrace{R_{CBC}}_{0} + \epsilon_C R_{C^3} &= 0 \\ R_{C^3} &= 0. \end{aligned}$$

By Lemmas 2.1 and 2.5, this means that $Rank(C^3) = Rank(C) \leq 1$. Thus, $R_C = 0$, and there are no nontrivial solutions to this linear dependence equation in the 2-0-0 case. □

3.2 The 2-0-1 Diagram

To see if anything changes when we replace one of the 0's in the above case to a 1, we examine the following diagram.

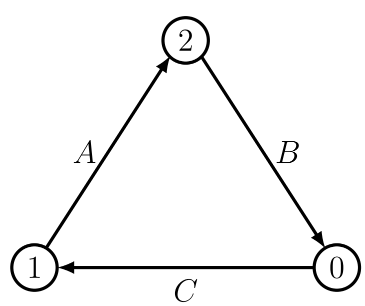


Figure 3: The 2-0-1 Diagram.

Theorem 3.2. *If the 2-0-1 diagram holds, then there do not exist any nontrivial R_A, R_B, or R_C that satisfy the linear dependence relation $R_A+\epsilon_B R_B+\epsilon_C R_C = 0$.*

Proof. The 2-0-1 diagram allows us to deduce the following information:

$$Rank(BA) \leq 2 \Rightarrow Rank(ABA) = Rank(BAB) \leq 2 \text{ (by Lemma 2.6)},$$
$$Rank(CB) = 0 \Rightarrow CB = 0 \Rightarrow Rank(BCB) = Rank(CBC) = 0, \text{ and}$$
$$Rank(AC) \leq 1 \Rightarrow Rank(CAC) = Rank(ACA) \leq 1 \Rightarrow R_{CAC} = R_{ACA} = 0.$$

We precompose the equation with C :

$$C^* R_A + \epsilon_B C^* R_B + \epsilon_C C^* R_C = 0$$
$$\underbrace{R_{CAC}}_{0} + \epsilon_B \underbrace{R_{CBC}}_{0} + \epsilon_C R_{C^3} = 0$$
$$\epsilon_C R_{C^3} = 0.$$

By Lemmas 2.1 and 2.5, we know that $R_C = 0$. Thus, the 2-0-1 case fails to yield any nontrivial solutions. □

Hence, in order to find nontrivial solutions to our linear dependence equation, we must consider weighted directed graphs with two or more values that are strictly greater than 1.

3.3 The 2-2-0 Diagram

In this section, we attempt to find a nontrivial solution to the linear dependence relation $R_A + \epsilon_B R_B + \epsilon_C R_C = 0$ by examining the weighted directed graph below.

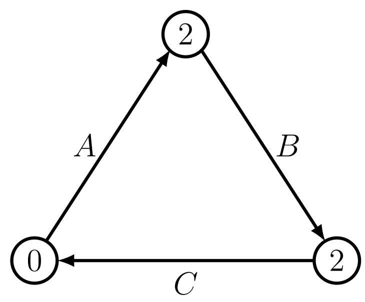


Figure 4: The 2-2-0 Diagram.

This gives us the following information:

$$Rank(BA) \leq 2 \Rightarrow Rank(ABA) = Rank(BAB) \leq 2 \text{ (by Lemma 2.6)},$$
$$Rank(CB) \leq 2 \Rightarrow Rank(CBC) = Rank(BCB) \leq 2 \text{ (by Lemma 2.6), and}$$
$$Rank(AC) = 0 \Rightarrow AC = 0.$$

Theorem 3.3. *If the 2-2-0 diagram holds, then there do not exist any nonzero $R_A, R_B,$ or R_C that satisfy $R_A + \epsilon_B R_B + \epsilon_C R_C = 0$.*

We will prove this theorem by contradiction. That is, we will assume to the contrary that none of R_A, R_B, or R_C are the zero tensor. For the remainder of this section, we will be operating under that assumption. Thus, before we prove this theorem, we state and prove some helpful lemmas.

Lemma 3.4. *Given the 2-2-0 diagram, $Rank(B) \geq 2$.*

Proof. Precomposing $R_A + \epsilon_B R_B + \epsilon_C R_C = 0$ with A yields:

$$A^* R_A + \epsilon_B A^* R_B + \epsilon_C A^* R_C = 0$$
$$R_{A^3} + \epsilon_B R_{ABA} + \epsilon_C \underbrace{R_{ACA}}_{0} = 0$$
$$R_{A^3} = -\epsilon_B R_{ABA}.$$

Given that we are working in an n-dimensional vector space V and $Rank(ABA) \leq$

2, we have:

$$\begin{aligned} Rank(ABA) \leq 2 &\Rightarrow dimKer(ABA) \geq n-2 \\ &\Rightarrow dimKer(R_{ABA}) \geq n-2 \text{ (by Lemma 2.2)} \\ &\Rightarrow dimKer(R_{A^3}) \geq n-2 \\ &\Rightarrow dimKer(A^3) \geq n-2 \text{ (by Lemma 2.2)} \\ &\Rightarrow Rank(A^3) \leq 2 \\ &\Rightarrow Rank(A) \leq 2 \text{ (by Lemma 2.5).} \end{aligned}$$

We will now precomspose the original equation with C :

$$\begin{gathered} C^* R_A + \epsilon_B C^* R_B + \epsilon_C C^* R_C = 0 \\ \underbrace{R_{CAC}}_{0} + \epsilon_B R_{CBC} + \epsilon_C R_{C^3} = 0 \\ \epsilon_C R_{C^3} = -\epsilon_B R_{CBC}. \end{gathered}$$

We can now operate as follows:

$$\begin{aligned} Rank(CBC) \leq 2 &\Rightarrow dimKer(CBC) \geq n-2 \\ &\Rightarrow dimKer(R_{CBC}) \geq n-2 \text{ (by Lemma 2.2)} \\ &\Rightarrow dimKer(R_{C^3}) \geq n-2 \\ &\Rightarrow dimKer(C^3) \geq n-2 \text{ (by Lemma 2.2)} \\ &\Rightarrow Rank(C^3) \leq 2 \\ &\Rightarrow Rank(C) \leq 2 \text{ (by Lemma 2.5).} \end{aligned}$$

Thus, since we have $Rank(BA) \leq 2$ and $Rank(CB) \leq 2$, it follows that $Rank(B) \geq 2$. □

Notice that in order for none of our tensors to equal the zero tensor, we would need $Rank(A) = 2$ and $Rank(C) = 2$. But does this stricter assumption contradict any previous known information about this weighted directed graph? For instance, if $Rank(A) = 2, Rank(C) = 2$, and $Rank(B) \geq 2$, we certainly have $Rank(BA) = 2$, but do we still have $Rank(ABA) = 2$? It turns out that we do.

Lemma 3.5. *Given the 2-2-0 diagram,*

1. *If* $Rank(BA) = 2,$ *then* $Rank(ABA) = Rank(BAB) = 2,$ *and*
2. *if* $Rank(CB) = 2,$ *then* $Rank(BCB) = Rank(CBC) = 2.$

Proof. 1. Suppose for contradiction that $Rank(BA) = 2$ and $Rank(ABA) < 2$. By precomposing our linear dependence equation with $A,$ we get $Rank(A^3) < 2$. By Lemma 2.5, it follows that $Rank(A) < 2$. This implies that $Rank(BA) < 2,$ which is a contradiction.

2. Suppose for contradiction that $Rank(CB) = 2$ and $Rank(BCB) < 2$. By precomposing our linear dependence equation with $B,$ we get $Rank(C^3) < 2$. By Lemma 2.5, it follows that $Rank(C) < 2$. This implies that $Rank(CB) < 2,$ which is a contradiction.

□

Therefore, we can continue to assume that $Rank(A) = Rank(C) = 2$, and $Rank(B) \geq 2$. This leads us to our next result.

Lemma 3.6. *Given the 2-2-0 diagram,* $dimV \geq 4$.

Proof. Suppose for contradiction that $\dim V = 3$. Then both A and C are 3×3 matrices. However, we know from the 2-2-0 diagram that $Rank(A) = Rank(C) = 2$, and $Im(C) \subseteq Ker(A)$. Since $Rank(A) = 2$ and $dimV = 3$, this means that $dimKer(A) = 1$, which implies that $Rank(C) \leq 1$. This contradicts the fact that $Rank(C) = 2$. Thus, $dimV$ cannot be less than 4. □

In fact, we can generalize this result.

Theorem 3.7. *Given that A and C are linear transformations on V such that $Im(C) \subseteq Ker(A)$, we have $dimV \geq Rank(A) + Rank(C)$.*

Proof. Since $Im(C) \subseteq Ker(A)$, then, $Rank(C) \leq dimKer(A)$, which means that $Rank(C) \leq dimV - Rank(A)$. Adding $Rank(A)$ to both sides gets us $Rank(A) + Rank(C) \leq dimV$, as desired. □

We are now ready to prove the main theorem of this section, Theorem 3.3. But first, we must state the following lemma.

Lemma 3.8. *[4] Let V be a vector space and let B be a linear transformation on V. Given that $Rank(B) \geq 2$, if there is a decomposition $V = V_1 \oplus V_2$ with $R_B = R_1 \oplus R_2$ (where $R_1 \in \mathcal{A}(V_1)$ and $R_2 \in \mathcal{A}(V_2)$), then either $V_1 \subseteq Ker(B)$ or $V_2 \subseteq Ker(B)$.*

We now prove Theorem 3.3.

Proof. Under our assumption that none of R_A, R_B, or R_C are the zero tensor, by Lemmas 3.4 and 3.5, we know that $Rank(A) = Rank(C) = 2$, and that $Rank(B) \geq 2$. Because Lemma 3.6 tells us that $dimV \geq 4$, we will let $dimV = 4$. From here, we can see that there exists a decomposition $V = V_1 \oplus V_2$ where $V_1 = Ker(C)$ and $V_2 = Ker(A)$. This is because the 2-2-0 diagram tells us that $Im(C) \subseteq Ker(A)$, so $AC = 0$, but because all of the matrices we are considering in this paper are symmetric, we know that $CA = 0$ and thus $Im(A) \subseteq Ker(C)$. Now, observe that

$$R_A + \epsilon_B R_B + \epsilon_C R_C = 0$$

implies that

$$R_B = -\epsilon_B R_A - \epsilon_B \epsilon_C R_C.$$

So, if we let $v_1 \in V_1$ and $v_2 \in V_2$, then by Definition 1.6, we know that $R_A(v_1, v_2, \cdot, \cdot) = 0$ (because $v_2 \in V_2 = Ker(A)$), and $R_C(v_1, v_2, \cdot, \cdot) = 0$

(because $v_1 \in V_1 = Ker(C)$). Thus,

$$\begin{aligned} R_B(v_1, v_2, \cdot, \cdot) &= -\epsilon_B R_A(v_1, v_2, \cdot, \cdot) - \epsilon_B \epsilon_C R_C(v_1, v_2, \cdot, \cdot) \\ R_B(v_1, v_2, \cdot, \cdot) &= 0. \end{aligned}$$

By Definition 1.5, this means that $R_B = R_1 \oplus R_2$, where $R_A \in \mathcal{A}(V_1)$ and $R_2 \in \mathcal{A}(V_2)$. Thus, by Lemma 3.8, we can let $V_1 \subseteq Ker(B)$. This means that V_2 cannot be contained in $Ker(B)$. And since $dim V = 4$ and $V = V_1 \oplus V_2$ where $V_1 = Ker(C)$ and $V_2 = Ker(A)$, and $dim V_1 = dim V_2 = 2$, then $Rank(B) \leq 2$. So, since $Rank(B) \geq 2$ by Lemma 3.4, we know that $Rank(B) = 2$. Note that because A and C commute, then they are simultaneously diagonalizable on an orthonormal basis $\{e_1, e_2, e_3, e_4\}$, which we can arrange so that $V_1 = span\{e_1, e_2\}$ and $V_2 = span\{e_3, e_4\}$. This means that $R_A(e_1, e_2, e_2, e_1) = a_1 a_2$, where a_1 and a_2 are the nonzero eigenvalues of A (this arises from the Spectral Theorem). However, observe that

$$R_A + \epsilon_B R_B + \epsilon_C R_C = 0$$

implies

$$R_A = -\epsilon_B R_B - \epsilon_C R_C.$$

So, because V_1 is contained in both $Ker(B)$ and $Ker(C)$, then we have

$$\begin{aligned} R_A(e_1, e_2, e_2, e_1) &= -\epsilon_B \underbrace{R_B(e_1, e_2, e_2, e_1)}_{0} - \epsilon_C \underbrace{R_C(e_1, e_2, e_2, e_1)}_{0} \\ a_1 a_2 &= 0. \end{aligned}$$

This is a contradiction, because we had arranged our basis for V so that $a_1 a_2 \neq 0$. Therefore, given the 2-2-0 diagram, there do not exist any nonzero R_A, R_B, or R_C that satisfy the linear dependence equation $R_A + \epsilon_B R_B + \epsilon_C R_C = 0$. □

3.4 The 2-2-1 Diagram

In this section, we examine the equation $R_A + \epsilon_B R_B + \epsilon_C R_C = 0$ alongside the following diagram.

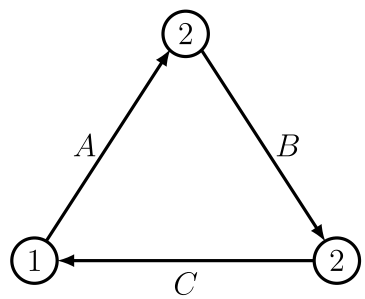


Figure 5: The 2-2-1 Diagram.

This gives us the following information:

$$
\begin{aligned}
&Rank(BA) \leq 2 \Rightarrow Rank(ABA) = Rank(BAB) \leq 2 \text{ (by Lemma 2.6)},\\
&Rank(CB) \leq 2 \Rightarrow Rank(CBC) = Rank(BCB) \leq 2 \text{ (by Lemma 2.6), and}\\
&Rank(AC) \leq 1 \Rightarrow Rank(CAC) = Rank(ACA) \leq 1 \Rightarrow R_{CAC} = R_{ACA} = 0.
\end{aligned}
$$

Lemma 3.9. *Given the 2-2-1 diagram, $Rank(B) \geq 2$.*

Proof. Precomposing $R_A + \epsilon_B R_B + \epsilon_C R_C = 0$ with A yields:

$$
\begin{aligned}
A^* R_A + \epsilon_B A^* R_B + \epsilon_C A^* R_C &= 0\\
R_{A^3} + \epsilon_B R_{ABA} + \epsilon_C \underbrace{R_{ACA}}_{0} &= 0\\
R_{A^3} &= -\epsilon_B R_{ABA}.
\end{aligned}
$$

Given that we are working in an n-dimensional vector space V and $Rank(ABA) \leq$

2, we have:

$$\begin{aligned}
Rank(ABA) \leq 2 &\Rightarrow dimKer(ABA) \geq n-2 \\
&\Rightarrow dimKer(R_{ABA}) \geq n-2 \text{ (by Lemma 2.2)} \\
&\Rightarrow dimKer(R_{A^3}) \geq n-2 \\
&\Rightarrow dimKer(A^3) \geq n-2 \text{ (by Lemma 2.2)} \\
&\Rightarrow Rank(A^3) \leq 2 \\
&\Rightarrow Rank(A) \leq 2 \text{ (by Lemma 2.5).}
\end{aligned}$$

We will now precomspose the original equation with C :

$$\begin{aligned}
C^* R_A + \epsilon_B C^* R_B + \epsilon_C C^* R_C = 0 \\
\underbrace{R_{CAC}}_{0} + \epsilon_B R_{CBC} + \epsilon_C R_{C^3} = 0 \\
\epsilon_C R_{C^3} = -\epsilon_B R_{CBC}.
\end{aligned}$$

We can now operate as follows:

$$\begin{aligned}
Rank(CBC) \leq 2 &\Rightarrow dimKer(CBC) \geq n-2 \\
&\Rightarrow dimKer(R_{CBC}) \geq n-2 \text{ (by Lemma 2.2)} \\
&\Rightarrow dimKer(R_{C^3}) \geq n-2 \\
&\Rightarrow dimKer(C^3) \geq n-2 \text{ (by Lemma 2.2)} \\
&\Rightarrow Rank(C^3) \leq 2 \\
&\Rightarrow Rank(C) \leq 2 \text{ (by Lemma 2.5).}
\end{aligned}$$

Thus, since we have $Rank(BA) \leq 2$ and $Rank(CB) \leq 2$, it follows that $Rank(B) \geq 2$. □

Theorem 3.10. *Given the 2-2-1 diagram, there do not exist any nonzero R_A, R_B, or R_C that satisfy the equation $R_A + \epsilon_B R_B + \epsilon_C R_C = 0$.*

Proof. Notice that if we let $dimV \geq 4$, we could run into the same contradiction that we found in the previous theorem, which involves decomposability. So, given that $Rank(AC) \leq 1$ in this case, we need not have $Im(C) \subseteq Ker(A)$ and $Im(A) \subseteq Ker(C)$. Thus, we may let $dimV = 3$. So, given that $dimV = 3$ and all of our matrices A, B, and C are symmetric by assumption, we can visualize the matrices as follows.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}, B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{12} & b_{22} & b_{23} \\ b_{13} & b_{23} & b_{33} \end{bmatrix}, C = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{12} & c_{22} & c_{23} \\ c_{13} & c_{23} & c_{33} \end{bmatrix}.$$

In fact, without loss of generality, we can choose a basis for V such that A is diagonalizable. Thus, we have

$$A = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ (letting } Rank(A) = 2).$$

Then, we have

$$AC = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{12} & c_{22} & c_{23} \\ c_{13} & c_{23} & c_{33} \end{bmatrix} = \begin{bmatrix} a_{11}c_{11} & a_{11}c_{12} & a_{11}c_{13} \\ a_{22}c_{12} & a_{22}c_{22} & a_{22}c_{23} \\ 0 & 0 & 0 \end{bmatrix}.$$

From this, we can see that $Rank(AC) = 1$. We now have the following cases for C.

Case 1. $c_{11} \neq 0$, $c_{12} = 0$, $c_{22} = 0$, and $c_{23} = 0$. This means that

$$C = \begin{bmatrix} c_{11} & 0 & c_{13} \\ 0 & 0 & 0 \\ c_{13} & 0 & c_{33} \end{bmatrix}.$$

Now, given that $dimV = 3$, we are considering six different combinations of curvature entries: 1221, 1331, 2332, 1231, 2132, and 3123. We now construct the following table, which provides a method of organizing the values of R_A, R_B, and R_C with respect to each of the six curvature entry combinations.

	R_A	R_B	R_C
1221	$a_{11}a_{22}$	$b_{11}b_{22} - b_{12}^2$	0
1331	0	$b_{11}b_{33} - b_{13}^2$	$c_{11}c_{33} - c_{13}^2$
2332	0	0	0
1231	0	0	0
2132	0	0	0
3123	0	0	0

Observe that the information in this table yields the following system of equations. Our goal is to find a B which satisfies this system of equations:

$$\begin{aligned}
R_B(1221) &= b_{11}b_{22} - b_{12}^2 = -\epsilon_B a_{11}a_{22} \\
R_B(1331) &= b_{11}b_{33} - b_{13}^2 = -\epsilon_B\epsilon_C(c_{11}c_{33} - c_{13}^2) \\
R_B(2332) &= b_{22}b_{33} - b_{23}^2 = 0 \\
R_B(1231) &= b_{11}b_{23} - b_{13}b_{21} = 0 \\
R_B(2132) &= b_{22}b_{13} - b_{23}b_{12} = 0 \\
R_B(3123) &= b_{33}b_{12} - b_{23}b_{13} = 0.
\end{aligned}$$

However, solving this system of equations is a long and tedious process, the details of which are spared in this paper. It turns out that there is no solution to this system of equations.

Case 2. $c_{11} = 0, c_{12} = 0$, and $c_{22} = 0$. This means that

$$C = \begin{bmatrix} 0 & 0 & c_{13} \\ 0 & 0 & c_{23} \\ c_{13} & c_{23} & c_{33} \end{bmatrix}.$$

Using the same method as we did for Case 1, we construct the following table for this case.

	R_A	R_B	R_C
1221	$a_{11}a_{22}$	$a_{11}a_{22} + \epsilon_B(b_{11}b_{22} - b_{12}^2)$	0
1331	0	$\epsilon_B(b_{11}b_{33} - b_{13}^2) - \epsilon_C(c_{13}^2)$	$-c_{13}^2$
2332	0	$\epsilon_B(b_{22}b_{33} - b_{23}^2) - \epsilon_C(c_{23}^2)$	$-c_{23}^2$
1231	0	0	0
2132	0	0	0
3123	0	$\epsilon_B(b_{33}b_{12} - b_{23}b_{13}) - \epsilon_C(c_{23}c_{13})$	$-c_{23}c_{13}$

Again, observe that the information in this table yields a system of equations that we wish to solve. However, just as in Case 1, this system of equations is very tedious to solve, and it turns out that there are no solutions.

Case 3. $c_{11} = 0$, $c_{12} = 0$, and $c_{13} = 0$. This means that

$$C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & c_{22} & c_{23} \\ 0 & c_{23} & c_{33} \end{bmatrix}.$$

Case 4. $c_{11} \neq 0$ and $c_{12} \neq 0$.

Case 5. $c_{11} = 0$ and $c_{12} \neq 0$.

We continue analyzing cases 3, 4, and 5 in a similar manner, and we find that there is never a solution in any of these cases. □

4 Directions for Future Study

1. In addition to finding that there are no nonzero solutions to the equation $R_A + \epsilon_B R_B + \epsilon_C R_C = 0$ with respect to the 2-2-1 diagram, in the case where $Rank(AC) = 1$, this problem gives rise to a new and interesting concept called $k-$decomposability. That is, rather than $dimKer(AC) = n$, where $n = dimV$, we have $dimKer(AC) = n - k$, where $k > 0$. So, in the case where $Rank(AC) = 1$, we have $dimKer(AC) = n - 1$. But in the 2-2-0 case, when $Rank(AC) = 0$, we had complete decomposability. Now, we appear to have what we might call "1-decomposability." Problems for future study involve examining the concept of $k-$decomposability and its implications in the context of linear dependence.

2. In addition to the diagrams that we studied in this paper, I looked at the following two diagrams.

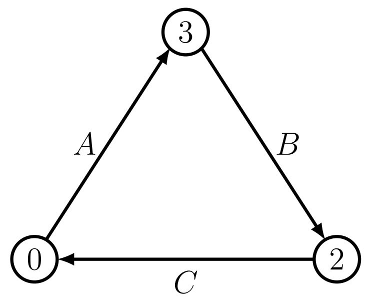


Figure 6: The 3-2-0 Diagram.

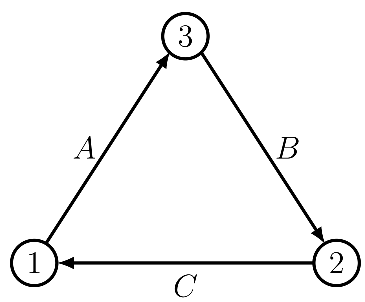


Figure 7: The 3-2-1 Diagram.

I conjecture that, so long as there is at least one 0 or 1 in the diagram, then there will always be no nonzero R_A, R_B, or R_C that satisfy the linear dependence equation $R_A + \epsilon_B R_B + \epsilon_C R_C = 0$. One problem for future study is to prove this conjecture, or to disprove it and find the number that causes this conjecture to break.

3. Another diagram that I examined but ran out of time to perform an in-depth study on is the following diagram.

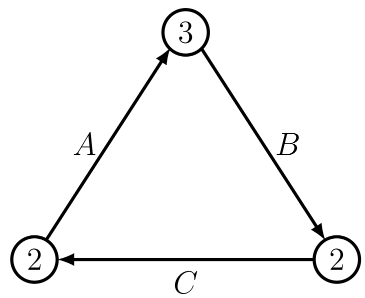


Figure 8: The 3-2-2 Diagram.

I was unable to arrive at any interesting conclusions about this diagram, but I suspect that because there is no 0 or 1 in this diagram, we may get nonzero R_A, R_B, and R_C that satisfy the equation $R_A + \epsilon_B R_B + \epsilon_C R_C = 0$. Another problem for future study involves examining this diagram and determining if anything changes about R_A, R_B, and R_C.

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