Constructing Spanning Sets of Affine Algebraic Curvature Tensors

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Abstract

In this paper we construct two spanning sets for the affine algebraic curvature tensors. We then prove that every 2-dimensional affine algebraic curvature tensor can be represented by a single element from either of the two spanning sets. This paper provides a means to study affine algebraic curvature tensors in a geometric and algebraic manner similar to previous studies of "normal" algebraic curvature tensors.

1 Introduction

Let V be a n-dimensional manifold. We define an *algebraic curvature tensor* (ACT) to be an $R \in \bigotimes^4(V^*)$ such that R satisfies the following algebraic properties of a Riemannian curvature tensor:

1. $R(x, y, z, w) = -R(y, x, z, w)$,
2. $R(x, y, z, w) + R(y, z, x, w) + R(z, x, y, w) = 0$,
3. $R(x, y, z, w) = R(z, w, x, y)$.

There has been lots of research about algebraic curvature tensors because they allow algebraic methods to be used to understand geometric properties of surfaces. The current body of work on ACTs has mostly focused on linear dependence and decompositions of ACTs [1][5][6][8], but there has also been some investigation of ACTs in specific model spaces [7].

The commonality among all of these investigations has been the existence of canonical ACTs that span the space of of algebraic curvature tensors. It is natural to suspect that conclusions similar to those made about ACTs would also hold for the more general class of affine algebraic curvature tensors (AACTs) if a canonical spanning set were found. AACTs are 4-tensors that have only the first two properties above. As a result of this added generality there has never been a definition of a canonical AACT.

The goal of this paper is to define the notion of a canonical AACT by finding spanning sets of the affine algebraic curvature tensors. This is significant because having these sets will allow researchers generalize the results previously reserved only for ACTs, and it will disentangle the algebraic study of curvature tensors from the Levi-Civita connection. As such, mathematicians will be able to analyze the broader subject of affine geometry through an algebraic perspective of AACTs.

In Section 2 we will review the necessary differential geometry to understand the differences between affine and classic differential geometry. This will set the stage for the following sections and highlight why the generalization of ACTs to AACTs is not immediate.

In Section 3 we will geometrically derive the symmetric spanning set which will provide us with the first notion of a canonical AACT. Specifically, we will highlight how the canonical R_φ's require the Levi-Civita connection, while the canonical AACTs do no need metric compatibility.

In Sections 4 and 5 we will algebraically prove that the two spanning set we create are actually spanning sets of the AACTs.

Then, in Section 6 we will prove that every AACT on a two dimensional hypersurface can be represented by a single canonical AACT in both the symmetric and anti-symmetric build. These final sections will act as a jumping off point for further study of the canonical representations of AACTs.

2 Preliminaries

This section will follow the standard submanifolds and connections theory as laid out in [4]. With that being said, one of the fundamental objects of study for this paper are connections on manifolds. We lay out the definition we will be using below.

Definition 1. *Let $\pi : E \to M$ be a vector bundle on the manifold M, $\mathcal{T}(M)$ be the tangent bundle of M, and $\mathcal{E}(M)$ be the space of smooth sections of E. A connection in E is a map*

$$\nabla : \mathcal{T}(M) \times \mathcal{E}(M) \to \mathcal{E}(M),$$

written as $(X, Y) \to \nabla_X Y$ that has the following properties:

1. *$\nabla_X Y$ is $C^\infty(M)$-linear over X. That is for $f, g \in C^\infty(M)$ we have that*

$$\nabla_{fX_1+gX_2} Y = f\nabla_{X_1} Y + g\nabla_{X_2} Y$$

2. *$\nabla_X Y$ is linear over $\mathbb{R}$ in Y. That is for $a, b \in \mathbb{R}$ we have that*

$$\nabla_X a_1Y_1 + a_2Y_2 = a_1\nabla_X Y_1 + a_2\nabla_X Y_2.$$

3. *∇ satisfies this product rule for $f \in C^\infty(M)$:*

$$\nabla_X (fY) = f\nabla_X Y + (Xf)Y.$$

We call $\nabla_X Y$ the covariant derivative of Y along X. It can be thought of as the derivative of Y in the direction of X. Next, we define a linear connection on M.

Definition 2. *An linear connection is a connection in TM. That is*

$$\nabla : \mathcal{T}(M) \times \mathcal{T}(M) \to \mathcal{T}(M).$$

A well known and important result is that every manifold admits a linear connection. A proof of this result can be found on page 52 of [4]. We will use this when deriving the symmetric build in Section 3.

Now we will consider the relationship between a connection and a manifold. Let M be a manifold embedded into $\mathbb{R}^n$. For any surface M embedded into a space with an ambient connection $\overline{\nabla}$ we can break it up into a tangential and perpendicular component.

Definition 3. *Let X and Y be vector fields on $\mathbb{R}^n$. Let $\mathbb{R}^n$ have the metric $g(\cdot, \cdot)$. We define $(\overline{\nabla})^\top$ and $(\overline{\nabla})^\perp$ to be the tangential and perpendicular components of $\overline{\nabla}$ respectively.*

Note that for points on M $(\overline{\nabla}_X Y)^\top$ is a linear connection on M, and $(\overline{\nabla}_X Y)^\perp : \mathcal{T}(M) \times \mathcal{T}(M) \to \mathcal{N}(M)$. We call $(\overline{\nabla}_X Y)^\perp$ the second fundamental form, and will denote it as $\eta(X, Y)$.

Proposition 1. *If $X, Y \in \mathcal{T}(M)$ are extended arbitrarily to vector fields on the ambient space, the following formula holds along M:*

$$\overline{\nabla}_X Y = (\overline{\nabla}_X Y)^\top + \eta(X, Y).$$

Note that this proof does not require metric compatibility of $\overline{\nabla}$. Also, a proof that the extension of X and Y does not change the result can be found on page 50 of [4].

Up until this point, there have been no differences between affine differential geometry and regular differential geometry. The next definition provides the first split in the two fields.

Definition 4. *Let M have the metric $\langle \cdot, \cdot \rangle$ and let X, Y, and Z be vector fields on M. The Levi-Civita connection (or Riemannian connection), denoted as ∇^{LC}, is the linear connection such that*

1. $\nabla^{LC}_X Y - \nabla^{LC}_Y X = [X, Y]$,
2. $X\langle Y, Z\rangle = \langle \nabla^{LC}_X Y, Z\rangle + \langle Y, \nabla^{LC}_X Z\rangle$.

We say ∇^{LC} is torsion-free and metric compatible respectively.

The fundamental fact about the Levi-Civita connection is that it is the unique connection on M such that both properties hold [4]. Affine geometry is the geometry of manifolds that are not necessarily equipped with the Levi-Civita connection. So, for our purposes we will only assume that the connections being used are torsion-free, symmetric, and flat. We define these last two terms below.

Definition 5. *A connection ∇ is said to be symmetric if*

$$\nabla_X Y = \nabla_Y X.$$

Definition 6. *Let X,Y, and Z be vector fields on M. Then we define the curvature operator on M with ∇ to be*

$$\mathcal{R}(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

We denote the curvature operator on $\mathbb{R}^n$ as $\overline{\mathcal{R}}(X,Y)Z$.

Definition 7. *A connection ∇ is said to be flat if*

$$\mathcal{R} \equiv 0.$$

We use this curvature operator to define the curvature on M and $\mathbb{R}^n$.

Definition 8. *Let X,Y, Z, and W be vector fields on M. Then we define the curvature tensor on M with ∇ to be*

$$R(X,Y,Z,W) := \langle \mathcal{R}(X,Y)Z, W \rangle.$$

We define $\overline{R}(X,Y,Z,W)$ to be the curvature on $\mathbb{R}^n$.

Now that we have defined these fundamental objects we can move onto the geometric derivation of a spanning set.

3 Geometric Derivation of the Spanning Set

For this section we will make four standing assumptions: (1) M^m is a manifold embedded into $\mathbb{R}^n$ with codimension k, (2) M has a torsion-free connection ∇, (3) $g(\cdot,\cdot)$ is a metric on $\mathbb{R}^n$, and (4) $\mathbb{R}^n$ has a flat, symmetric, and torsion-free connection $\overline{\nabla}$ such that $(\overline{\nabla})^\top = \nabla$.

Ultimately, the fact that we do not have metric compatibility manifests itself in the Weingarten equation.

Proposition 2. *Suppose $X, Y \in \mathcal{T}(M)$ and $N \in \mathcal{N}(M)$. When X, Y, N are extended arbitrarily to the ambient space, the following equation holds at points of M:*

$$g(\overline{\nabla}_X N, Y) = g(N, \eta(X,Y)).$$

We call this the Weingarten equation.

The Weingarten equation is critical in deriving the canonical ACTs, but it also requires $\overline{\nabla}$ to be metric compatible. As we will show, this is the primary reason that the set of canonical ACTs is not a complete set of canonical AACTs.

Proposition 3. *$(\overline{\nabla})^\top$ and η are torsion-free connections.*

Proof. First we will prove $(\overline{\nabla})^\top$ and η are connections and then that they are torsion-free.

1. $\eta(X,Y)$ and $(\overline{\nabla}_X Y)^\top$ are $C^\infty(M)$-linear over X because

$$\begin{aligned}\eta(fX_1+gX_2,Y) = (\overline{\nabla}_{fX_1+gX_2}Y)^\perp = f(\overline{\nabla}_{X_1}Y)^\perp + g(\overline{\nabla}_{X_2}Y)^\perp \\ = f\eta(X_1,Y)+gf\eta(X_2,Y),\end{aligned}$$

   and

$$(\overline{\nabla}_{fX_1+gX_2}Y)^\top = f(\overline{\nabla}_{X_1}Y)^\top + g(\overline{\nabla}_{X_2}Y)^\top$$

   since $\overline{\nabla}$ is a connection.

2. $\eta(X,Y)$ and $(\overline{\nabla}_X Y)^\top$ are linear over $\mathbb{R}$ over Y because for $a_1, a_2 \in \mathbb{R}$ we have

$$\begin{aligned}\eta(X,a_1Y_1+a_2Y_2) = (\overline{\nabla}_X a_1Y_1+a_2Y_2)^\perp = (\overline{\nabla}_X a_1Y_1)^\perp + (\overline{\nabla}_X a_2Y_2)^\perp \\ = \eta(X,a_1Y_1)+\eta(X,a_2Y_2)\end{aligned}$$

   and

$$(\overline{\nabla}_X a_1Y_1+a_2Y_2)^\top = (\overline{\nabla}_X a_1Y_1)^\top + (\overline{\nabla}_X a_2Y_2)^\top.$$

   This too is because $\overline{\nabla}$ is a connection.

3. Finally, $\eta(X,Y)$ and $(\overline{\nabla}_X Y)^\top$ follow the product rule. We see that for $f \in C^\infty(M)$

$$\eta(X,fY) = (\overline{\nabla}_X fY)^\perp = f(\overline{\nabla}_X Y)^\perp + ((Xf)Y)^\perp$$

   and

$$(\overline{\nabla}_X fY)^\top = f(\overline{\nabla}_X Y)^\top + ((Xf)Y)^\top.$$

So, $\eta(X,Y)$ and $(\overline{\nabla}_X Y)^\top$ are connections.

Next, they are torsion-free because

$$\overline{\nabla}_X Y - \overline{\nabla}_Y X = [X,Y].$$

That means that

$$\eta(X,Y) - \eta(X,Y) = (\overline{\nabla}_X Y)^\perp - (\overline{\nabla}_Y X)^\perp = [X,Y]^\perp$$

and

$$(\overline{\nabla}_X Y)^\top - (\overline{\nabla}_Y X)^\top = [X,Y]^\top.$$

□

We also will prove one last property of η.

Proposition 4. *$\eta(A,B)$ is symmetric.*

Proof. η is symmetric due to the symmetry of $\overline{\nabla}_X Y$ and the fact that two vectors are equal if and only if their perpendicular parts are equal. Thus,

$$\eta(X,Y) = (\overline{\nabla}_X Y)^\perp = (\overline{\nabla}_Y X)^\perp = \eta(Y,X).$$

□

We now begin to derive our basis for the AACTs geometrically. First of all, notice that for any vector fields A and B on M, $\eta(A, B)$ is in the normal bundle of M. Let $n_1, ..., n_k$ to be a basis for NM, and let $h_i(A, B)$ be smooth functions from M to $\mathbb{R}$. Then,

$$\eta(A, B) = \sum_{i=1}^{k} h_i(A, B) n_i.$$

Let us look at $\overline{R}$. Let X, Y, Z and W be vector fields on $\mathbb{R}^n$ in the tangent bundle of M. We see that by expanding $\overline{\nabla}$ into $\nabla + \eta$ we get that

$$\begin{aligned}
0 &= \overline{R}(X, Y, Z, W) = g(\overline{\mathcal{R}}(X, Y)Z, W) \\
&= g(\overline{\nabla}_X \overline{\nabla}_Y Z - \overline{\nabla}_Y \overline{\nabla}_X Z - \overline{\nabla}_{[X,Y]} Z, W) \\
&= g(\overline{\nabla}_X (\nabla_Y Z + \eta(Y, Z)) - \overline{\nabla}_Y (\nabla_X Z + \eta(X, Z)) - \overline{\nabla}_{[X,Y]} Z, W) \\
&= g(\overline{\nabla}_X (\nabla_Y Z) + \overline{\nabla}_X (\eta(Y, Z)) - \overline{\nabla}_Y (\nabla_X Z) - \overline{\nabla}_Y (\eta(X, Z)) - \overline{\nabla}_{[X,Y]} Z, W) \\
&= g(\nabla_X \nabla_Y Z + \eta(X, \nabla_Y Z), W) - g(\nabla_Y \nabla_X Z + \eta(Y, \nabla_X Z), W) \\
&- g(\nabla_{[X,Y]} Z + \eta([X, Y], Z), W) + g(\overline{\nabla}_X (\eta(Y, Z)), W) - g(\overline{\nabla}_Y (\eta(X, Z)), W).
\end{aligned}$$

Recall that W is a vector field in the tangent bundle of M and η exclusively maps into the tangent bundle. So, we can eliminate their inner products to get

$$\begin{aligned}
0 &= g(\nabla_X \nabla_Y Z, W) - g(\nabla_Y \nabla_X Z, W) - g(\nabla_{[X,Y]} Z, W) + g(\overline{\nabla}_X (\eta(Y, Z)), W) \\
&- g(\overline{\nabla}_Y (\eta(X, Z)), W) \\
&= R(X, Y, Z, W) + g(\overline{\nabla}_X (\eta(Y, Z)), W) - g(\overline{\nabla}_Y (\eta(X, Z)), W).
\end{aligned}$$

Then by breaking down η into its components we get

$$\begin{aligned}
\overline{R}(X, Y, Z, W) = R(X, Y, Z, W) &+ \sum_{i=1}^{k} g(\overline{\nabla}_X (h_i(Y, Z) k_i), W) \\
&- \sum_{j=1}^{k} g(\overline{\nabla}_Y (h_j(X, Z) k_j), W).
\end{aligned}$$

Using the product rule of connections gives us

$$\begin{aligned}
\overline{R}(X, Y, Z, W) = R(X, Y, Z, W) &+ \sum_{i=1}^{k} g((h_i(Y, Z) \overline{\nabla}_X k_i) + X(h_i(Y, Z)) k_i, W) \\
&- \sum_{j=1}^{k} g(h_j(X, Z) \overline{\nabla}_Y k_j + X(h_j(X, Z)) k_j, W).
\end{aligned}$$

Then, since $X(h_j(X, Z))k_j$ is in the normal bundle of M its inner product with W is 0. Hence we can simplify it down to

$$\overline{R}(X,Y,Z,W) = R(X,Y,Z,W) + \sum_{i=1}^{k} g((h_i(Y,Z)\overline{\nabla}_X k_i), W)$$
$$- \sum_{j=1}^{k} g(h_j(X,Z)\overline{\nabla}_Y(k_j), W).$$

Simply letting $\alpha_j(X,W) := -g(\overline{\nabla}_X k_j, W)$ gives us that

$$\overline{R}(X,Y,Z,W) = R(X,Y,Z,W) - \sum_{i=1}^{k} h_i(Y,Z)\alpha_i(X,W)$$
$$+ \sum_{j=1}^{k} h_j(X,Z)\alpha_j(Y,W).$$

But, $\overline{R} = 0$ because it is flat. So we have the result

$$R(X,Y,Z,W) = \sum_{i=1}^{k} \alpha_i(X,W)h_i(Y,Z) - \alpha_i(Y,W)h_i(X,Z).$$

This has a strikingly similar form to the R_φ's that generate the ACTs, so we suspect that these functions generate the AACTs. But, before we prove that, we must check that these functions are AACTs to begin with.

Proposition 5. *$R(X,Y,Z,W) = \sum_{i=1}^{k} \alpha_i(X,W)h_i(Y,Z) - \alpha_i(Y,W)h_i(X,Z)$ where $\alpha_i \in \bigotimes^2(V^*)$ and $h_i \in S^2(V^*)$ is an AACT.*

Proof. We first check the anti-symmetry in the first two spots. We get that

$$R(X,Y,Z,W) = \sum_{i=1}^{k} \alpha_i(X,W)h_i(Y,Z) - \alpha_i(Y,W)h_i(X,Z)$$
$$= -\left(\sum_{i=1}^{k} \alpha_i(Y,W)h_i(X,Z) - \alpha_i(X,W)h_i(Y,Z)\right) = -(R(Y,X,Z,W)).$$

Then, checking the Bianchi identity gives us that

$$\begin{aligned}
&R(X,Y,Z,W) + R(Y,Z,X,W) + R(Z,X,Y,W) \\
&\qquad = \sum_{i=1}^{k} [\alpha_i(X,W)h_i(Y,Z) - \alpha_i(Y,W)h_i(X,Z) \\
&\qquad + \alpha_i(Y,W)h_i(Z,X) - \alpha_i(Z,W)h_i(Y,X) \\
&\qquad + \alpha_i(Z,W)h_i(X,Y) - \alpha_i(X,W)h_i(Z,Y)] \\
&\qquad = 0.
\end{aligned}$$

□

Definition 9. *We define*

$$R_{\alpha,h}(X,Y,Z,W) := \sum_{i=1}^{k} \alpha_i(X,W)h_i(Y,Z) - \alpha_i(Y,W)h_i(X,Z)$$

where $\alpha_i \in \bigotimes^2(V^*)$ *and* $h_i \in S^2(V^*)$

So, we have that the $R_{\alpha,h}$'s are AACTs, and that they can be found in a similar way to the R_φ. Next, we will prove our conjecture that the $R_{\alpha,h}$'s really are a spanning set.

4 The Symmetric Build

While we define $R_{\alpha,h}(X,Y,Z,W) := \sum_{i=1}^{k} \alpha_i(X,W)(Y,Z) - \alpha_i(Y,W)h_i(X,Z)$, in Section 4 we will only be denoting it as $\alpha(X,W)h(Y,Z) - \alpha(Y,W)h(X,Z)$ for simplicity. The proofs can easily be adapted back to the sum definition simply by defining α_1 and h_1 to be tensors we pick below and letting all other α_i and h_i equal 0.

Definition 10. *The set* $A := \{R_{\alpha,h} : \alpha \in \bigotimes^2(V^*), h \in S^2(V^*)\}$ *where* $S^2(V)$ *is the set of symmetric 2-tensors.*

Theorem 1. *The set of affine algebraic curvature tensors on* V *is spanned by the set* A*.*

Proof. Let W be an arbitrary affine algebraic curvature tensor, and let $T_{ijkl} = e^i \otimes e^j \otimes e^k \otimes e^l - e^j \otimes e^i \otimes e^k \otimes e^l$ where e_i is a basis vector for V and e^i is a dual basis vector of V^*. From here, W can be broken up like this:

$$\begin{aligned} W = &\sum_{i,j \text{ distinct}} c_{ijij}T_{ijij} + c_{ijii}T_{ijii} \\ &+ \sum_{i,j,k \text{ distinct}} c_{ijki}T_{ijki} + c_{ijik}T_{ijik} + c_{ijkk}T_{ijkk} \\ &+ \sum_{i,j,k,l \text{ distinct}} c_{ijkl}T_{ijkl} \end{aligned}$$

where all indices go from 1 to n. As such, T_{ijkl} essentially encodes how W acts on (e_i, e_j, e_k, e_l). Also, note that all other possible T_{ijkl}'s can be obtained from swapping the first two indices. Then, we can see that proving each of these sums is in A will prove that $W \in A$, which proves the theorem. Following this logic, we will break this argument into cases dealing with these sums.

Case 1: T_{ijij}
We know thatT_{ijij} is a map that takes (e_i, e_j, e_k, e_l) to 1, (e_j, e_i, e_k, e_l) to -1, and all other basis vectors to 0. So, if there is a linear combination of $R_{\alpha,h}$'s that

agrees with T_{ijij} on the basis vectors, then $T_{ijij} \in A$ and the linear combination of T_{ijij}'s is in A as well. Now, for any arbitrary $\alpha \in \bigotimes^2(V), h \in S^2(V)$ we have that

$$R_{\alpha,h}(e_i, e_j, e_i, e_j) = \alpha(e_i, e_j)h(e_j, e_i) - \alpha(e_j, e_j)h(e_i, e_i).$$

If we pick α and h so that $\alpha(e_j, e_j) = -1, h(e_i, e_i) = 1$ and they map all other basis vectors to 0, then $T_{ijij}(e_i, e_j, e_i, e_j) = R_{\alpha,h}(e_i, e_j, e_i, e_j) = 1$. This in turn means that $T_{ijij}(e_j, e_i, e_i, e_j) = R_{\alpha,h}(e_j, e_i, e_i, e_j) = -1$. We also know that T_{ijij} sends all other combinations of basis vectors to 0, so we need to show that $R_{\alpha,h}$ does as well.

In order for an input to be non-zero for $R_{\alpha,h}$, we have to have e_i in the third position and e_j in the fourth position. Also, e_i and e_j must occupy either the first or second positions. Hence, the only non-zero inputs are (e_i, e_j, e_i, e_j) and (e_j, e_i, e_i, e_j) which is what was desired. Therefore $T_{ijij} = R_{\alpha,h} \in A$.

Case 2: T_{ijii}
Similarly to the last case, we want to find an $R_{\alpha,h}$ that equals T_{ijii}. Again, for an arbitrary $\alpha \in \bigotimes^2(V^*), h \in S^2(V^*)$ we have that

$$R_{\alpha,h}(e_i, e_j, e_i, e_i) = \alpha(e_i, e_i)h(e_j, e_i) - \alpha(e_j, e_i)h(e_i, e_i).$$

So, if we pick α and h so that $\alpha(e_j, e_i) = -1, h(e_i, e_i) = 1$ and they map all other basis vectors to 0, then we have that $T_{ijii} = R_{\alpha,h}$.

Case 3: T_{ijik}
Following the same logic as the previous cases, we see that for an arbitrary α and h we have that

$$R_{\alpha,h}(e_i, e_j, e_i, e_k) = \alpha(e_i, e_k)h(e_j, e_i) - \alpha(e_j, e_k)h(e_i, e_i).$$

So, picking α and h such that $\alpha(e_j, e_k) = -1, h(e_i, e_i) = 1$ and all other combinations of basis vectors are mapped to zero gives us that $T_{ijik} = R_{\alpha,h}$.

Case 4: T_{ijki} and T_{ijkk}
We start off in a similar manner to the last three cases, and see that

$$R_{\alpha,h}(e_i, e_j, e_k, e_i) = \alpha(e_i, e_i)h(e_j, e_k) - \alpha(e_j, e_i)h(e_i, e_k).$$

Picking α and h such that $\alpha(e_j, e_i) = -1, h(e_i, e_k) = 1$, and all other basis vectors get mapped to 0. Unlike in previous cases where we could ignore the symmetry of h by defining its only nonzero term to be (e_i, e_i), in this case we also have that $h(e_j, e_k) = h(e_k, e_j) = 1$. This means that

$$\begin{cases} R_{\alpha,h}(e_i, e_j, e_k, e_i) = 1 \\ R_{\alpha,h}(e_j, e_i, e_k, e_i) = -1 \\ R_{\alpha,h}(e_k, e_j, e_i, e_i) = 1 \\ R_{\alpha,h}(e_j, e_k, e_i, e_i) = -1 \end{cases}$$

So unlike in the previous cases $R_{\alpha,h} \neq T_{ijki}$. Instead, $R_{\alpha,h} = T_{ijki} + T_{kjii}$.

Now we will examine T_{kjii}. Consider another arbitrary $\tilde{\alpha} \in \bigotimes^2(V^*)$ and $\tilde{h} \in S^2(V^*)$. Then, we have that

$$R_{\tilde{\alpha},\tilde{h}}(e_k, e_j, e_i, e_i) = \tilde{\alpha}(e_k, e_i)\tilde{h}(e_j, e_i) - \tilde{\alpha}(e_j, e_i)\tilde{h}(e_k, e_i).$$

Pick $\tilde{\alpha}$ and $\tilde{h}$ so that $\tilde{\alpha}(e_k, e_i) = 1$ and $\tilde{h}(e_j, e_i) = \tilde{h}(e_i, e_j) = 1$. That means that

$$\begin{cases} R_{\tilde{\alpha},\tilde{h}}(e_k, e_j, e_i, e_i) = 1 \\ R_{\tilde{\alpha},\tilde{h}}(e_j, e_k, e_i, e_i) = -1 \\ R_{\tilde{\alpha},\tilde{h}}(e_k, e_i, e_j, e_i) = 1 \\ R_{\tilde{\alpha},\tilde{h}}(e_i, e_k, e_j, e_i) = -1 \end{cases}$$

So, $R_{\tilde{\alpha},\tilde{h}} = T_{kjii} + T_{kiji}$.

Let $W(e_i, e_j, e_k, e_l) = c_{ijkl}$ for any $i, j, k, l \in \{1, ..., n\}$. Since W satisfies the Bianchi identity, we can pick these constants so that $c_{ijkl} + c_{jkil} + c_{kijl} = 0$. We claim that all terms in the T_{ijki} and T_{kjii} sums can be expressed by a linear combination of $R_{\alpha,h}$ and $R_{\tilde{\alpha},\tilde{h}}$. We need

$$\begin{cases} R_{\alpha,h}(e_i, e_j, e_k, e_i) + yR_{\tilde{\alpha},\tilde{h}}(e_i, e_j, e_k, e_i) = c_{ijki} \\ R_{\alpha,h}(e_j, e_k, e_i, e_i) + yR_{\tilde{\alpha},\tilde{h}}(e_j, e_k, e_i, e_i) = c_{jkii} \\ R_{\alpha,h}(e_k, e_i, e_j, e_i) + yR_{\tilde{\alpha},\tilde{h}}(e_k, e_i, e_j, e_i) = c_{kiji} \end{cases}$$

If we let $x = c_{ijki}$ and $y = c_{kiji}$ we can check that we get

$$c_{ijki}R_{\alpha,h}(e_i, e_j, e_k, e_i) + c_{kiji}R_{\tilde{\alpha},\tilde{h}}(e_i, e_j, e_k, e_i) = c_{ijki} \cdot 1 + c_{kiji} \cdot 0 = c_{ijki}.$$

Similarly, for the input (e_k, e_i, e_j, e_i) this linear combination gives us

$$c_{ijki}R_{\alpha,h}(e_k, e_i, e_j, e_i) + c_{kiji}R_{\tilde{\alpha},\tilde{h}}(e_k, e_i, e_j, e_i) = c_{ijki} \cdot 0 + c_{kiji} \cdot 1 = c_{kiji}.$$

The final non-zero, independent input is (e_j, e_k, e_i, e_i). We have that

$$c_{ijki}R_{\alpha,h}(e_j, e_k, e_i, e_i) + c_{kiji}R_{\tilde{\alpha},\tilde{h}}(e_j, e_k, e_i, e_i) = c_{ijki} \cdot -1 + c_{kiji} \cdot -1 = c_{jkii}.$$

And, since

$$c_{ijkl}R_{\alpha,h} + c_{kiji}R_{\tilde{\alpha},\tilde{h}} = c_{ijkl}(T_{ijki} + T_{kjii}) + c_{kiji}(T_{kjii} + T_{kiji})$$

we see that all other combinations of basis vectors are mapped to 0.

Hence, $R_{\alpha,h} + R_{\tilde{\alpha},\tilde{h}}$ covers the T_{ijki} and T_{jkii} terms, and thus the $\sum c_{ijki}T_{ijki}$ and $\sum c_{ijkk}T_{ijkk}$ summations can be completely replicated through the summation of $(R_{\alpha,h} + R_{\tilde{\alpha},\tilde{h}})$'s.

Case 5: T_{ijkl}

The final case is very similar to the fourth one. We start by considering an arbitrary $R_{\alpha,h}$ for the input (e_i, e_j, e_k, e_l). This is

$$R_{\alpha,h}(e_i, e_j, e_k, e_l) = \alpha(e_i, e_l)h(e_j, e_k) - \alpha(e_j, e_l)h(e_i, e_k).$$

Let $\alpha(e_i, e_l) = 1 = h(e_j, e_k) = h(e_k, e_j)$. The non-zero, independent inputs are (e_i, e_j, e_k, e_l) and (e_i, e_k, e_j, e_l) both of which map to 1. Thus, $R_{\alpha,h} = T_{ijkl} + T_{ikjl}$.

We now shift to consider another arbitrary $R_{\tilde{\alpha},\tilde{h}}$ for the input (e_i, e_k, e_j, e_l). That gives us

$$R_{\tilde{\alpha},\tilde{h}}(e_i, e_k, e_j, e_l) = \tilde{\alpha}(e_i, e_l)\tilde{h}(e_k, e_j) - \tilde{\alpha}(e_k, e_l)\tilde{h}(e_i, e_j).$$

Let $\tilde{\alpha}(e_k, e_l) = -1$ and $\tilde{h}(e_i, e_j) = \tilde{h}(e_j, e_i) = 1$. Then the non-zero, independent inputs of $R_{\tilde{\alpha},\tilde{h}}$ are (e_i, e_k, e_j, e_l) and (e_j, e_k, e_i, e_l). Thus, $R_{\tilde{\alpha},\tilde{h}} = T_{ikjl} + T_{jkil}$.

As in the previous case, let $W(e_i, e_j, e_k, e_l) = c_{ijkl}$ for any $i, j, k, l \in \{1, ..., n\}$. Again, we can pick these constants so that $c_{ijkl} + c_{jkil} + c_{kijl} = 0$. Then, we can consider a linear combination $xR_{\alpha,h} + yR_{\tilde{\alpha},\tilde{h}}$. We need

$$\begin{cases} xR_{\alpha,h}(e_i, e_j, e_k, e_l) + yR_{\tilde{\alpha},\tilde{h}}(e_i, e_j, e_k, e_l) = c_{ijkl} \\ xR_{\alpha,h}(e_j, e_k, e_i, e_l) + yR_{\tilde{\alpha},\tilde{h}}(e_j, e_k, e_i, e_l) = c_{jkil} \\ xR_{\alpha,h}(e_k, e_i, e_j, e_l) + yR_{\tilde{\alpha},\tilde{h}}(e_k, e_i, e_j, e_l) = c_{kijl} \end{cases}$$

So, we find that $x = c_{ijkl}$ and $y = c_{jkil}$. Plugging these inputs into the linear combination shows us that

$$c_{ijkl}R_{\alpha,h}(e_i, e_j, e_k, e_l) + c_{jkil}R_{\tilde{\alpha},\tilde{h}}(e_i, e_j, e_k, e_l) = c_{ijkl} \cdot 1 + c_{jkil} \cdot 0 = c_{ijkl},$$
$$c_{ijkl}R_{\alpha,h}(e_j, e_k, e_i, e_l) + c_{jkil}R_{\tilde{\alpha},\tilde{h}}(e_j, e_k, e_i, e_l) = c_{ijkl} \cdot 0 + c_{jkil} \cdot 1 = c_{jkil},$$

and

$$\begin{aligned} c_{ijkl}R_{\alpha,h}(e_k, e_i, e_,j, e_l) + c_{jkil}R_{\tilde{\alpha},\tilde{h}}(e_k, e_i, e_j, e_l) &= c_{ijkl} \cdot -1 + c_{jkil} \cdot -1 \\ &= -(c_{ijkl} + c_{jkil}) = c_{kjil}. \end{aligned}$$

Notice that the last equality follows from the Bianchi identity. Finally, we know that

$$c_{ijkl}R_{\alpha,h} + c_{kijl}R_{\tilde{\alpha},\tilde{h}} = c_{ijkl}(T_{ijkl} + T_{ikjl}) + c_{kijl}(T_{ikjl} + T_{jkil}).$$

So, we can see that the linear combination is zero on all other inputs. As a result, a summation of $(R_{\alpha,h} + R_{\tilde{\alpha},\tilde{h}})$'s can give the same output as $\sum c_{ijkl}T_{ijkl}$. So, $\sum c_{ijkl}T_{ijkl} \in A$, and that proves that for any W an AACT

$$W \in A.$$

□

Not only can we span the AACTs on V with A, but we can also use anti-symmetric p's to create another spanning set.

5 The Anti-Symmetric Build

We derived the $R_{\alpha,h}$'s geometrically, but for the anti-symmetric case we obtain our spanning set through analogy to the canonical anti-symmetric ACTs, R_ψ's.

Definition 11. *We define*

$$R_{\alpha,p}(X,Y,Z,W) = \sum_{i=1}^{k} \alpha_i(X,W)p_i(Y,Z) - \alpha_i(Y,W)p_i(Z,Z) - 2\alpha_i(Z,W)p_i(X,Y)$$

where $\alpha_i \in \bigotimes^2(V^*)$ and $p_i \in \Lambda^2(V^*)$.

Proposition 6. *$R_{\alpha,p}$ is an AACT.*

Proof. Again, we check the anti-symmetry in the first two spots. We get that

$$\begin{aligned} R_{\alpha,p}(X,Y,Z,W) = \sum_{i=1}^{k} [&\alpha_i(X,W)p_i(Y,Z) - \alpha_i(Y,W)p_i(X,Z) \\ &- 2\alpha_i(Z,W)p_i(X,Y)] \\ = -\left(\sum_{i=1}^{k} \alpha_i(Y,W)p_i(X,Z) - \alpha_i(X,W)p_i(Y,Z) - 2\alpha_i(Z,W)p_i(X,Y)\right) \\ = -(R_{\alpha,p}(Y,X,Z,W)). \end{aligned}$$

Then, checking the Bianchi identity gives us that

$$\begin{aligned} &R_{\alpha,p}(X,Y,Z,W) + R_{\alpha,p}(Y,Z,X,W) + R_{\alpha,p}(Z,X,Y,W) \\ &= \sum_{i=1}^{k} [\alpha_i(X,W)p_i(Y,Z) - \alpha_i(Y,W)p_i(X,Z) - 2\alpha_i(Z,W)p_i(X,Y) \\ &+ \alpha_i(Y,W)p_i(Z,X) - \alpha_i(Z,W)p_i(Y,X) - 2\alpha_i(X,W)p_i(Y,Z) \\ &+ \alpha_i(Z,W)p_i(X,Y) - \alpha_i(X,W)p_i(Z,Y) - 2\alpha_i(Y,W)p_i(Z,X)] \\ &= 0. \end{aligned}$$

□

Much like in the $R_{\alpha,h}$-case we will omit the sums in our usage of $R_{\alpha,p}$'s.

Definition 12. *The set $Q := \{R_{\alpha,p} : \alpha \in \bigotimes^2(V^*), p \in \Lambda^2(V)\}$ where $\Lambda^2(V)$ is the set of anti-symmetric 2-tensors.*

Theorem 2. *The set of affine algebraic curvature tensors on V is spanned by the set Q.*

Proof. This proofs follows very similarly to the proof that A spans the AACTs on V. In fact, we define T_{ijkl} the same as in the previous theorem, and we break up an arbitrary AACT, W, in the same way. We again have

$$
\begin{aligned}
W = &\sum_{i,j \text{ distinct}} c_{ijij}T_{ijij} + c_{ijii}T_{ijii} \\
+ &\sum_{i,j,k \text{ distinct}} c_{ijki}T_{ijki} + c_{ijik}T_{ijik} + c_{ijkk}T_{ijkk} \\
+ &\sum_{i,j,k,l \text{ distinct}} c_{ijkl}T_{ijkl}
\end{aligned}
$$

Also, like we had done in the previous proof let $W(e_i, e_j, e_k, e_l) = c_{ijkl}$ for any $i, j, k, l \in \{1, ..., n\}$, and pick these c's so that they satisfy the Bianchi identity.

Case 1: T_{ijij}
We again consider an arbitrary $R_{\alpha,h}$ with the input (e_i, e_j, e_i, e_j). We see that

$$R_{\alpha,p}(e_i, e_j, e_i, e_j) = \alpha(e_i, e_j)p(e_j, e_i) - \alpha(e_j, e_j)p(e_i, e_i) - 2\alpha(e_i, e_j)p(e_i, e_j).$$

Let $\alpha(e_i, e_j) = 1$, $p(e_j, e_i) = 1$, and all other independent combinations of basis vectors be sent to 0. Then, we have that $R_{\alpha,p}(e_i, e_j, e_i, e_j) = 3$ and $R_{\alpha,p}(e_j, e_i, e_i, e_j) = -3$. As such, we need to show that $R_{\alpha,p}$ is zero on all other independent combinations of basis vectors.

In order for $R_{\alpha,p}$ to be non-zero, its input must have an e_j in its fourth slot. But, then all other combinations of basis vectors will be (e_i, e_j, e_i, e_j), (e_j, e_i, e_i, e_j), or (e_i, e_i, e_j, e_j). And, the last of these is 0 for all AACTs. So, we have an R_α, p such that $\frac{1}{3}R_{\alpha,p} = T_{ijij} \in Q$.

Case 2: T_{ijii}
Similarly, computing $R_{\alpha,p}$ with the input (e_i, e_j, e_i, e_i) gives us that

$$R_{\alpha,p}(e_i, e_j, e_i, e_j) = \alpha(e_i, e_j)p(e_j, e_i) - \alpha(e_j, e_j)p(e_i, e_i) - 2\alpha(e_i, e_j)p(e_i, e_j).$$

Let $\alpha(e_i, e_j) = 1$ $p(e_j, e_i) = 1$, and all other combinations of basis vectors be mapped to 0. Then the only non-zero, independent input is (e_i, e_j, e_i, e_i) which is mapped to 3. So, $\frac{1}{3}R_{\alpha,p} = T_{ijii} \in Q$.

Case 3: T_{ijik}
We start in the same way as before by computing $R_{\alpha,p}(e_i, e_j, e_i, e_k)$. If we let $\alpha(e_i, e_k) = 1$, $p(e_j, e_i) = 1$, and all other combinations of basis vectors be mapped to 0 we get that

$$R_{\alpha,p}(e_i, e_j, e_i, e_k) = \alpha(e_i, e_k)p(e_j, e_i) - \alpha(e_j, e_k)p(e_i, e_i) - 2\alpha(e_i, e_k)p(e_i, e_j) = 3.$$

As in the first two cases, this is the only non-zero, independent input, and thus we get that $\frac{1}{3}R_{\alpha,p} = T_{ijik} \in Q$.

Case 4: T_{ijki} and T_{jkii}
We let $\alpha(e_j, e_i) = 1$, $p(e_i, e_k) = -1$, and compute that $R_{\alpha,p}(e_i, e_j, e_k, e_i) = 1$. This $R_{\alpha,p}$ has two other distinct inputs, namely (e_j, e_k, e_i, e_i) which is also

mapped to 1 and (e_k, e_i, e_j, e_i) which is mapped to -2. So, $R_{\alpha,p} = T_{ijki} + T_{jkii} - 2T_{kiji}$.

We now look at the T_{jkii} case. Let $\tilde{\alpha}(e_k, e_i) = 1$, $\tilde{p}(e_j, e_i) = -1$, and compute that

$$R_{\tilde{\alpha},\tilde{p}}(e_j, e_k, e_i, e_i) = \tilde{\alpha}(e_j, e_i)\tilde{p}(e_k, e_i) - \tilde{\alpha}(e_k, e_i)\tilde{p}(e_j, e_i) - 2\tilde{\alpha}(e_i, e_i)\tilde{p}(e_j, e_k) = 1.$$

We also see that $R_{\tilde{\alpha},\tilde{p}}$ maps (e_i, e_j, e_k, e_i) to -2 and (e_k, e_i, e_j, e_i) to 1. So, $R_{\tilde{\alpha},\tilde{p}} = T_{jkii} + T_{kiji} - 2T_{ijki}$.

Much like in case 4 of the symmetric build's proof, we solve the following systems of equations:

$$\begin{cases} xR_{\alpha,p}(e_i, e_j, e_k, e_i) + yR_{\tilde{\alpha},\tilde{p}}(e_i, e_j, e_k, e_i) = c_{ijki}, \\ xR_{\alpha,p}(e_j, e_k, e_i, e_i) + yR_{\tilde{\alpha},\tilde{p}}(e_j, e_k, e_i, e_i) = c_{jkii}, \\ xR_{\alpha,p}(e_k, e_i, e_j, e_i) + yR_{\tilde{\alpha},\tilde{p}}(e_k, e_i, e_j, e_i) = c_{kiji}. \end{cases}$$

Solving this ultimately gives us that $x = \frac{2c_{jkii}+c_{ijki}}{3}$ and $y = \frac{c_{jkii}-c_{ijki}}{3}$. Finally, we see that if we plug in x and y into the above linear combination and evaluate sum at (e_i, e_j, e_k, e_i), (e_j, e_k, e_i, e_i), or (e_k, e_i, e_j, e_i) we get the desired result. Moreover, since

$$\begin{aligned} \frac{2c_{jkii} + c_{ijki}}{3} R_{\alpha,p} + \frac{c_{jkii} - c_{ijki}}{3} R_{\tilde{\alpha},\tilde{p}} = \frac{2c_{jkii} + c_{ijki}}{3} (T_{ijki} + T_{jkii} - 2T_{kiji}) \\ + \frac{c_{jkii} - c_{ijki}}{3} (T_{jkii} + T_{kiji} - 2T_{ijki}) \end{aligned}$$

these are the only independent inputs that do not map to 0. Hence, we have that both $\sum c_{ijki}T_{ijki}$ and $\sum c_{jkii}T_{jkii}$ can be expressed as a sum of $R_{\alpha,p}$'s.

Case 5: T_{ijkl}
As in the last few cases, we let $\alpha(e_i, e_l) = 1$, $p(e_j, e_k) = 1$, and all other combinations of basis vectors be mapped to 0. Then, we get that

$$R_{\alpha,p}(e_i, e_j, e_k, e_l) = \alpha(e_i, e_l)p(e_j, e_k) - \alpha(e_j, e_l)p(e_i, e_k) - 2\alpha(e_k, e_l)p(e_i, e_j) = 1.$$

This has two other non-zero, independent inputs: (e_k, e_i, e_j, e_l) which is mapped to 1 and (e_j, e_k, e_i, e_l) which is mapped to -2. So, $R_{\alpha,p} = T_{ijkl} + T_{kijl} - 2T_{jkil}$.

Now let $\tilde{\alpha}(e_k, e_l) = 1$, $\tilde{p}(e_i, e_j) = 1$, and compute

$$R_{\tilde{\alpha},\tilde{p}}(e_k, e_i, e_j, e_l) = \tilde{\alpha}(e_k, e_l)\tilde{p}(e_i, e_j) - \tilde{\alpha}(e_i, e_l)\tilde{p}(e_k, e_j) - 2\tilde{\alpha}(e_j, e_l)\tilde{p}(e_k, e_i) = 1.$$

This also has two other non-zero, independent inputs outside its kernel, (e_j, e_k, e_i, e_l) which maps to 1 and (e_i, e_j, e_k, e_l) which maps to -2. So, $R_{\tilde{\alpha},\tilde{p}} = T_{kijl} + T_{jkil} - 2T_{ijkl}$.

Much like in Case 4, we now solve the following system of equations:

$$\begin{cases} xR_{\alpha,p}(e_i, e_j, e_k, e_l) + yR_{\tilde{\alpha},\tilde{p}}(e_i, e_j, e_k, e_l) = c_{ijkl}, \\ xR_{\alpha,p}(e_k, e_i, e_j, e_l) + yR_{\tilde{\alpha},\tilde{p}}(e_k, e_i, e_j, e_l) = c_{kijl}, \\ xR_{\alpha,p}(e_j, e_k, e_i, e_l) + yR_{\tilde{\alpha},\tilde{p}}(e_j, e_k, e_i, e_l) = c_{jkil}. \end{cases}$$

We get that $x = \frac{2c_{kijl}+c_{ijkl}}{3}$ and $y = \frac{c_{kijl}-c_{ijkl}}{3}$. So, we can express $\sum c_{ijkl}T_{ijkl}$ as a sum of $(xR_{\alpha,p} + yR_{\tilde{\alpha},\tilde{p}})$'s. Hence, $\sum c_{ijkl}T_{ijkl} \in Q$.

As such, all of the sums in (2) are a linear combination of $R_{\alpha,p}$'s which implies that $W \in Q$. Therefore Q spans the AACTs.

□

6 $\kappa(2) = \sigma(2) = 1$

It is natural to ask the maximum number of $R_{\alpha,h}$'s required to represent any given AACT on an n-dimensional manifold. We will denote this number $\sigma(n)$. In this section we will prove the following result.

Theorem 3. $\sigma(2) = 1$.

Proof. Let A be an arbitrary AACT on a 2-dimensional manifold. A's behavior for any (x, y, z, w) is defined by its behavior on a set of basis vectors of M which we will call $\{e_1, e_2\}$. Then, by evaluating A on all the independent sets of basis vectors we can show that there is an $R_{\alpha,h} = A$. If there was such an $R_{\alpha,h}$ then we would need

$$\begin{cases} A_{1211} = \alpha(e_1, e_1)h(e_2, e_1) - \alpha(e_2, e_1)h(e_1, e_1), \\ A_{1212} = \alpha(e_1, e_2)h(e_2, e_1) - \alpha(e_2, e_2)h(e_1, e_1), \\ A_{1221} = \alpha(e_1, e_1)h(e_2, e_2) - \alpha(e_2, e_1)h(e_1, e_2), \\ A_{1222} = \alpha(e_1, e_2)h(e_2, e_2) - \alpha(e_2, e_2)h(e_1, e_2). \end{cases}$$

Then, simply letting $h(e_1, e_1) = 1$, $h_(e_2, e_2) = 1$, $-\alpha(e_2, e_1) = A_{1211}$, $-\alpha(e_2, e_2) = A_{1212}$, $\alpha(e_1, e_1) = A_{1221}$, and $\alpha(e_1, e_2) = A_{1222}$ gives us an $R_{\alpha,h} = A$.

□

Similarly, we define $\kappa(n)$ to be the maximum number of $R_{\alpha,p}$'s needed to represent any AACT on an n-dimensional manifold. This gives us a similar result the to the previous theorem.

Theorem 4. $\kappa(2) = 1$.

Proof. By the same logic as in the previous proof, we consider if there was an $R_{\alpha,p} = A$. Then we would need

$$\begin{cases} A_{1211} = \alpha(e_1, e_1)p(e_2, e_1) - \alpha(e_2, e_1)p(e_1, e_1) - 2\alpha(e_1, e_1)p(e_1, e_2), \\ A_{1212} = \alpha(e_1, e_2)p(e_2, e_1) - \alpha(e_2, e_2)p(e_1, e_1) - 2\alpha(e_1, e_2)p(e_1, e_2), \\ A_{1221} = \alpha(e_1, e_1)p(e_2, e_2) - \alpha(e_2, e_1)p(e_1, e_2) - 2\alpha(e_2, e_1)p(e_1, e_2), \\ A_{1222} = \alpha(e_1, e_2)p(e_2, e_2) - \alpha(e_2, e_2)p(e_1, e_2) - 2\alpha(e_2, e_2)p(e_1, e_2). \end{cases}$$

So, much like in Theorem 3, letting $p(e_1, e_1) = 1$, $p_(e_2, e_2) = 1$, $-\alpha(e_2, e_1) = A_{1211}$, $-\alpha(e_2, e_2) = A_{1212}$, $\alpha(e_1, e_1) = A_{1221}$, and $\alpha(e_1, e_2) = A_{1222}$ gives us an $R_{\alpha,p} = A$. □

7 Conclusion and Open Problems

Thus far we have constructed two spanning sets for the AACTs and discovered $\sigma(2)$ and $\kappa(2)$. As was mentioned in the introduction, this is only the starting point for questions concerning AACTs. Here are just a few interesting open problems:

1. What are upper bounds for $\sigma(n)$ and $\kappa(n)$? Are these bounds sharp?
2. Under what conditions can linear independence of multiple canoncial AACTs occur?
3. Is there a geometric proof that the symmetric build spans the AACTs? There is such a proof for the R_φ, but this proof is not able to be directly adapted for AACTs due to needing each $R_{\alpha,h}$ to be geometrically realized on a manifold with a connection such that $\nabla = (\overline{\nabla})^\top$ but $(\overline{\nabla})^\top$ is flat in $\mathbb{R}^n$.

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